Third-Order Differential Superordination Involving the Generalized Bessel Functions

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Abstract

There are many articles in the literature dealing with the first-order and the second-order differential subordination and differential superordination problems for analytic functions in the unit disk, but there are only a few articles dealing with the third-order differential subordination problems. The concept of third-order differential subordination in the unit disk was introduced by Antonino and Miller, and studied recently by Tang and Deniz. Let Ω be a set in the complex plane $\mathbb{C}$, let $\mathfrak{p}(z)$ be analytic in the unit disk $\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}$, and let $\psi : \mathbb{C}^4 \times \mathbb{U} \to \mathbb{C}$. In this paper, we investigate the problem of determining properties of functions $\mathfrak{p}(z)$ that satisfy the following third-order differential superordination:

$$\Omega \subset \left\{\psi(\mathfrak{p}(z), z\mathfrak{p}'(z), z^2\mathfrak{p}''(z), z^3\mathfrak{p}'''(z); z) : z \in \mathbb{U}\right\}.$$

As applications, we derive some third-order differential superordination results for analytic functions in $\mathbb{U}$, which are associated with a family of generalized Bessel functions. The results are obtained by considering suitable classes of admissible functions. New third-order differential sandwich-type results are also obtained.

2010 Mathematics Subject Classification. Primary 30C45, 33C10; Secondary 30C80.

Key Words and Phrases. Differential subordination; Differential superordination; Analytic functions; Univalent functions; Hadamard product (or convolution); Admissible functions; Generalized Bessel functions; Bessel and modified Bessel functions; Spherical Bessel function.

1. Introduction, Definitions and Preliminaries

Let $\mathcal{H}(\mathbb{U})$ be the class of functions which are analytic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

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For $n \in \mathbb{N} := \{1, 2, 3, \cdots\}$ and $a \in \mathbb{C}$, let

$$\mathcal{H}[a, n] = \{f : f \in \mathcal{H}(\mathbb{U}) \quad \text{and} \quad f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots\}$$

and suppose that $\mathcal{H}_0 = \mathcal{H}[0, 1]$. We denote by $\mathcal{A}$ the class of all normalized analytic functions in $\mathbb{U}$ of the form:

$$f(z) = z + \sum_{n=1}^{\infty} a_{n+1} z^{n+1} \qquad (z \in \mathbb{U}). \tag{1.1}$$

Let f and F be members of the analytic function class $\mathcal{H}(\mathbb{U})$. The function f is said to be subordinate to F, or F is superordinate to f, if there exists a Schwarz function $\mathfrak{w}(z)$, analytic in $\mathbb{U}$ with

$$\mathfrak{w}(0) = 0 \qquad \text{and} \qquad |\mathfrak{w}(z)| < 1 \qquad (z \in \mathbb{U}),$$

such that

$$f(z) = F\big(\mathfrak{w}(z)\big) \qquad (z \in \mathbb{U}).$$

In such a case, we write

$$f \prec F \qquad \text{or} \qquad f(z) \prec F(z) \qquad (z \in \mathbb{U}).$$

Furthermore, if the function F is univalent in $\mathbb{U}$, then we have the following equivalence (see, for details, [21]; see also [12, 19, 35]):

$$f(z) \prec F(z) \quad (z \in \mathbb{U}) \iff f(0) = F(0) \qquad \text{and} \qquad f(\mathbb{U}) \subset F(\mathbb{U}).$$

Let $f, g \in \mathcal{A}$, where f is given by (1.1) and g is defined by

$$g(z) = z + \sum_{n=1}^{\infty} b_{n+1} z^{n+1}.$$

Then the Hadamard product (or convolution) $f * g$ of the functions f and g is defined by

$$(f * g)(z) := z + \sum_{n=1}^{\infty} a_{n+1} b_{n+1} z^{n+1} =: (g * f)(z).$$

We next consider the following second-order homogeneous linear differential equation (see, for details, [9])

$$z^2 \omega''(z) + bz\omega'(z) + [cz^2 - p^2 + (1 - b)p]\omega(z) = 0 \quad (b, c, p \in \mathbb{C}). \tag{1.2}$$

The function $\omega_{p,b,c}(z)$, which is called a generalized Bessel function of the first kind of order p, is defined as a particular solution of (1.2). Furthermore, the function $\omega_{p,b,c}(z)$ has the familiar representation as follows:

$$\omega_{p,b,c}(z) = \sum_{n=0}^{\infty} \frac{(-c)^n}{n!\Gamma\left(p + n + \frac{b+1}{2}\right)} \left(\frac{z}{2}\right)^{2n+p} \qquad (z \in \mathbb{C}), \tag{1.3}$$

where Γ stands for the Euler's Gamma function.

The series in (1.3) permits the study of the Bessel function $J_\nu(z)$, the modified Bessel function $I_\nu(z)$ and the spherical Bessel function $j_\nu(z)$ in a unified manner. In terms of the Bessel function $J_\nu(z)$ of order ν defined by (see [34] and [9])

$$J_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(\nu + n + 1)} \left(\frac{z}{2}\right)^{2n+\nu} \qquad (z \in \mathbb{C}),$$

the definition (1.3) immediately yields the following relationship:

$$\omega_{p,b,c}(z) = c^{\frac{2p+b-1}{4}} \left(\frac{z}{2}\right)^{-\frac{b-1}{2}} J_{p+\frac{b-1}{2}}\left(z\sqrt{c}\right).$$

We also note each of the following special cases of the function $\omega_{p,b,c}(z)$ defined by (1.3):
1. For $b = c = 1$ in (1.3), we have the familiar Bessel function of the first kind of order p, that is,

$$\omega_{p,1,1}(z) = J_p(z),$$

which follows also from the above-mentioned relationship.

2. For $b = 1$ and $c = -1$ in (1.3), we obtain

$$\omega_{p,1,-1}(z) = I_p(z),$$

where the modified Bessel function $I_\nu(z)$ of the first kind of order ν is defined by (see [34] and [9])

$$I_\nu(z) = \sum_{n=0}^{\infty} \frac{1}{n!\Gamma(\nu+n+1)} \left(\frac{z}{2}\right)^{2n+\nu} \qquad (z \in \mathbb{C}).$$

3. For $b = 2$ and $c = 1$ in (1.3), we have

$$\omega_{p,2,1}(z) = \sqrt{\frac{2}{\pi}}\, j_p(z),$$

where $j_\nu(z)$ denotes the spherical Bessel function of the first kind of order ν defined by (see [34] and [9])

$$j_\nu(z) = \sqrt{\frac{\pi}{2}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma\left(\nu+n+\frac{3}{2}\right)} \left(\frac{z}{2}\right)^{2n+\nu} \qquad (z \in \mathbb{C}).$$

Recently, Deniz *et al.* [16] and Deniz [15] (see also [8] to [11], [23] and [30]) considered the function $\varphi_{p,b,c}(z)$ defined, in terms of the generalized Bessel function $\omega_{p,b,c}(z)$ in (1.3), by the following transformation:

$$\varphi_{p,b,c}(z) = 2^p \Gamma\left(p + \frac{b+1}{2}\right) z^{1-\frac{p}{2}}\, \omega_{p,b,c}(\sqrt{z}). \tag{1.4}$$

By using the general Pochhammer symbol (or the shifted factorial) $(\lambda)_\nu$ defined, for $\lambda, \nu \in \mathbb{C}$ and in terms of Euler's Γ-function, by

$$(\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu = 0;\ \lambda \in \mathbb{C} \setminus \{0\}) \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (\nu = n \in \mathbb{N};\ \lambda \in \mathbb{C}), \end{cases}$$

it being understood *conventionally* that $(0)_0 := 1$ and assumed *tacitly* that the Γ-quotient exists, we can obtain the following series representation for the function $\varphi_{p,b,c}(z)$ given by (1.4):

$$\varphi_{p,b,c}(z) = z + \sum_{n=1}^{\infty} \frac{(-c)^n}{4^n(\kappa)_n} \frac{z^{n+1}}{n!} \qquad \left(\kappa = p + \frac{b+1}{2} \in \mathbb{C} \setminus \mathbb{Z}_0^-\right), \tag{1.5}$$

where

$$\mathbb{Z}_0^- = \{0, -1, -2, \cdots\} = \mathbb{Z}^- \cup \{0\}.$$

For simplicity, we write

$$\varphi_{\kappa,c}(z) = \varphi_{p,b,c}(z).$$

Baricz *et al.* [10] (see also [33]) introduced a new operator $B_\kappa^c : \mathcal{A} \to \mathcal{A}$, which is defined by means of the Hadamard product (or convolution) as follows:

$$B_\kappa^c f(z) := \varphi_{\kappa,c}(z) * f(z) = z + \sum_{n=1}^{\infty} \frac{(-c)^n a_{n+1}}{4^n (\kappa)_n} \frac{z^{n+1}}{n!} \tag{1.6}$$

in terms of the Taylor-Maclaurin coefficients a_{n+1} involved in (1.1). It is easy to verify from the definition (1.6) that

$$z\big(B_{\kappa+1}^c f(z)\big)' = \kappa B_\kappa^c f(z) - (\kappa - 1) B_{\kappa+1}^c f(z), \tag{1.7}$$

where

$$\kappa = p + \frac{b+1}{2} \in \mathbb{C} \setminus \mathbb{Z}_0^-.$$

In fact, the function $B_\kappa^c f(z)$ is an elementary transform of the generalized hypergeometric function defined by (see [20], [24], [25], [27] to [29]; see also [17] and [18])

$${}_qF_s(\alpha_1, \cdots, \alpha_q; \beta_1, \cdots, \beta_s; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_q)_n}{(\beta_1)_n \cdots (\beta_s)_n} \frac{z^n}{n!}$$

$$\big(\alpha_i \in \mathbb{C} \ \ (i = 1, \cdots, q); \ \beta_j \in \mathbb{C} \setminus \mathbb{Z}_0^- \ \ (j = 1, \cdots, s); \ q \leqq s + 1; \ q, s \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}\big).$$

For example, we have

$$B_\kappa^c f(z) = z \, {}_0F_1\left(\kappa; -\frac{c}{4} z\right) * f(z).$$

For suitable choices of the parameters b and c, we obtain several other (presumably new) operators as follows:

(i) Putting $b = c = 1$ in (1.6), we have the operator $\mathcal{J}_p : \mathcal{A} \to \mathcal{A}$ related with the Bessel function, which is defined by

$$\mathcal{J}_p f(z) = \varphi_{p,1,1}(z) * f(z) = \left[2^p \Gamma(p+1) z^{1-p/2} J_p(\sqrt{z})\right] * f(z) = z + \sum_{n=1}^{\infty} \frac{(-1)^n a_{n+1}}{4^n (p+1)_n} \frac{z^{n+1}}{n!}. \tag{1.8}$$

(ii) Setting $b = 1$ and $c = -1$ in (1.6), we obtain the operator $\mathcal{I}_p : \mathcal{A} \to \mathcal{A}$ related with the modified Bessel function, which is defined by

$$\mathcal{I}_p f(z) = \varphi_{p,1,-1}(z) * f(z) = \left[2^p \Gamma(p+1) z^{1-p/2} I_p(\sqrt{z})\right] * f(z) = z + \sum_{n=1}^{\infty} \frac{a_{n+1}}{4^n (p+1)_n} \frac{z^{n+1}}{n!}. \tag{1.9}$$

(iii) Taking $b = 2$ and $c = 1$ in (1.6), we get the operator $\mathcal{S}_p : \mathcal{A} \to \mathcal{A}$ related with the spherical Bessel function, which is defined by

$$\mathcal{S}_p f(z) = \left[\pi^{-1/2} 2^{p+\frac{1}{2}} \Gamma\left(p + \frac{3}{2}\right) z^{1-\frac{p}{2}} j_p(\sqrt{z})\right] * f(z) = z + \sum_{n=1}^{\infty} \frac{(-1)^n a_{n+1}}{4^n \left(p + \frac{3}{2}\right)_n} \frac{z^{n+1}}{n!}. \tag{1.10}$$

Let Ω be any set in $\mathbb{C}$, let $\mathfrak{p}$ be analytic in $\mathbb{U}$, and let $\psi : \mathbb{C}^4 \times \mathbb{U} \to \mathbb{C}$. Antonino and Miller [7] have extended the theory of second-order differential subordinations in $\mathbb{U}$ introduced by Miller and Mocanu [21] to the third-order case. They determined properties of functions $\mathfrak{p}$ that satisfy the following third-order differential subordination:

$$\left\{\psi(\mathfrak{p}(z), z\mathfrak{p}'(z), z^2\mathfrak{p}''(z), z^3\mathfrak{p}'''(z); z) : z \in \mathbb{U}\right\} \subset \Omega.$$

Recently, Tang and Deniz [33] have considered the applications of these results to third-order differential subordination for analytic functions in $\mathbb{U}$.

In the following, we will list some definitions and theorem due to Antonino and Miller [7], which are required in our next investigations.

Definition 1 (see [7, p. 440, Definition 1]). Let $\psi : \mathbb{C}^4 \times \mathbb{U} \to \mathbb{C}$ and $h(z)$ be univalent in $\mathbb{U}$. If $\mathfrak{p}(z)$ is analytic in $\mathbb{U}$ and satisfies the following third-order differential subordination:

$$\psi\left(\mathfrak{p}(z), z\mathfrak{p}'(z), z^2\mathfrak{p}''(z), z^3\mathfrak{p}'''(z); z\right) \prec h(z) \qquad (z \in \mathbb{U}), \tag{1.11}$$

then $\mathfrak{p}(z)$ is called a solution of the differential subordination. A univalent function $\mathfrak{q}(z)$ is called a dominant of the solutions of the differential subordination or, more simply, a dominant if $\mathfrak{p}(z) \prec \mathfrak{q}(z)$ for all $\mathfrak{p}(z)$ satisfying (1.11). A dominant $\widetilde{\mathfrak{q}}(z)$ that satisfies $\widetilde{\mathfrak{q}}(z) \prec \mathfrak{q}(z)$ for all dominants $\mathfrak{q}(z)$ of (1.11) is said to be the best dominant.

Definition 2 (see [7, p. 441, Definition 2]). Let $\mathcal{Q}$ denote the set of functions $\mathfrak{q}$ that are analytic and univalent on the set $\overline{\mathbb{U}} \setminus E(\mathfrak{q})$, where

$$E(\mathfrak{q}) = \{\xi : \xi \in \partial\mathbb{U} \quad \text{and} \quad \lim_{z\to\xi} \mathfrak{q}(z) = \infty\},$$

and are such that

$$\min |\mathfrak{q}'(\xi)| = \rho > 0$$

for $\xi \in \partial\mathbb{U} \setminus E(\mathfrak{q})$. Further, let the subclass of $\mathcal{Q}$ for which $\mathfrak{q}(0) = a$ be denoted by $\mathcal{Q}(a)$ and

$$\mathcal{Q}(0) = \mathcal{Q}_0.$$

Definition 3 (see [7, p. 449, Definition 3]). Let Ω be a set in $\mathbb{C}$, $\mathfrak{q} \in \mathcal{Q}$ and $n \in \mathbb{N} \setminus \{1\}$. The class of admissible functions $\Psi_n[\Omega, \mathfrak{q}]$ consists of those functions $\psi : \mathbb{C}^4 \times \mathbb{U} \to \mathbb{C}$ that satisfy the following admissibility condition:

$$\psi(r, s, t, u; z) \notin \Omega,$$

whenever

$$r = \mathfrak{q}(\xi), \qquad s = k\xi\mathfrak{q}'(\xi), \qquad \Re\left(\frac{t}{s} + 1\right) \geqq k\Re\left(\frac{\xi\mathfrak{q}''(\xi)}{\mathfrak{q}'(\xi)} + 1\right)$$

and

$$\Re\left(\frac{u}{s}\right) \geqq k^2\Re\left(\frac{\xi^2\mathfrak{q}'''(\xi)}{\mathfrak{q}'(\xi)}\right),$$

where $z \in \mathbb{U}$, $\xi \in \partial\mathbb{U} \setminus E(\mathfrak{q})$ and $k \geqq n$.

Theorem 1 (see [7, p. 449, Theorem 1]). Let $\mathfrak{p} \in \mathcal{H}[a, n]$ with $n \geqq 2$. Also let $\mathfrak{q} \in \mathcal{Q}(a)$ and satisfy the following conditions:

$$\Re\left(\frac{\xi\mathfrak{q}''(\xi)}{\mathfrak{q}'(\xi)}\right) \geqq 0 \qquad \text{and} \qquad \left|\frac{z\mathfrak{p}'(z)}{\mathfrak{q}'(\xi)}\right| \leqq k, \tag{1.12}$$

where $z \in \mathbb{U}$, $\xi \in \partial\mathbb{U} \setminus E(\mathfrak{q})$ and $k \geqq n$. If Ω is a set in $\mathbb{C}$, $\psi \in \Psi_n[\Omega, \mathfrak{q}]$ and

$$\psi\left(\mathfrak{p}(z), z\mathfrak{p}'(z), z^2\mathfrak{p}''(z), z^3\mathfrak{p}'''(z); z\right) \in \Omega,$$

then

$$\mathfrak{p}(z) \prec \mathfrak{q}(z) \qquad (z \in \mathbb{U}).$$

In this article, following the theory of second-order differential superordinations in $\mathbb{U}$ introduced by Miller and Mocanu [22], we consider the dual problem of determining properties of functions $\mathfrak{p}(z)$ that satisfy the following third-order differential superordination:

$$\Omega \subset \left\{\psi(\mathfrak{p}(z), z\mathfrak{p}'(z), z^2\mathfrak{p}''(z), z^3\mathfrak{p}'''(z); z) : z \in \mathbb{U}\right\}.$$

In other words, we determine conditions on $\Omega,\ \Delta$ and ψ for which the following implication holds:

$$\Omega \subset \left\{\psi(\mathfrak{p}(z), z\mathfrak{p}'(z), z^2\mathfrak{p}''(z), z^3\mathfrak{p}'''(z); z) : z \in \mathbb{U}\right\} \Longrightarrow \Delta \subset \mathfrak{p}(\mathbb{U}), \tag{1.13}$$

where Δ is any set in $\mathbb{C}$.

If either Ω or Δ is a simply connected domain, then (1.13) can be rephrased in terms of superordination. If $\mathfrak{p}(z)$ is univalent in $\mathbb{U}$, and if Δ is a simply connected domain with $\Delta \neq \mathbb{C}$, then there is a conformal mapping $\mathfrak{q}(z)$ of $\mathbb{U}$ onto Δ such that $\mathfrak{q}(0) = \mathfrak{p}(0)$. In this case, (1.13) can be rewritten as

$$\Omega \subset \left\{\psi(\mathfrak{p}(z), z\mathfrak{p}'(z), z^2\mathfrak{p}''(z), z^3\mathfrak{p}'''(z); z) : z \in \mathbb{U}\right\} \Longrightarrow \mathfrak{q}(z) \prec \mathfrak{p}(z). \tag{1.14}$$

If Ω is also a simply connected domain with $\Omega \neq \mathbb{C}$, then there is a conformal mapping h of $\mathbb{U}$ onto Ω such that $h(0) = \psi(\mathfrak{p}(0), 0, 0, 0; 0)$. In addition, if the function

$$\psi(\mathfrak{p}(z), z\mathfrak{p}'(z), z^2\mathfrak{p}''(z), z^3\mathfrak{p}'''(z); z)$$

is univalent in $\mathbb{U}$, then (1.14) can be rewritten as

$$h(z) \prec \psi(\mathfrak{p}(z), z\mathfrak{p}'(z), z^2\mathfrak{p}''(z), z^3\mathfrak{p}'''(z); z) \Longrightarrow q(z) \prec \mathfrak{p}(z).$$

There are three key ingredients in the implication relationship (1.14): the differential operator ψ, the set Ω and the"dominating" function $\mathfrak{q}(z)$. If two of these entities were given, one would hope to find conditions on the third so that (1.14) would be satisfied. In this article, we start with a given set Ω and a given function $\mathfrak{q}(z)$, and determine a set of "admissible" operators ψ so that (1.14) holds true.

We first introduce the following definition.

Definition 4. *Let $\psi : \mathbb{C}^4 \times \mathbb{U} \to \mathbb{C}$ and $h(z)$ be analytic in $\mathbb{U}$. If $\mathfrak{p}(z)$ and*

$$\psi(\mathfrak{p}(z), z\mathfrak{p}'(z), z^2\mathfrak{p}''(z), z^3\mathfrak{p}'''(z); z)$$

are univalent in $\mathbb{U}$ and satisfy the following third-order differential superordination:

$$h(z) \prec \psi(\mathfrak{p}(z), z\mathfrak{p}'(z), z^2\mathfrak{p}''(z), z^3\mathfrak{p}'''(z); z), \tag{1.15}$$

then $\mathfrak{p}(z)$ is called a solution of the differential superordination. An analytic function $\mathfrak{q}(z)$ is called a subordinant of the solutions of the differential superordination, or more simply a subordinant if $\mathfrak{q}(z) \prec \mathfrak{p}(z)$ for $\mathfrak{p}(z)$ satisfying (1.15). *A univalent subordinant $\widetilde{\mathfrak{q}}(z)$ that satisfies $\mathfrak{q}(z) \prec \widetilde{\mathfrak{q}}(z)$ for all subordinants $\mathfrak{q}(z)$ of* (1.15) *is said to be the best subordinant. Note that the best subordinant is unique up to a rotation of $\mathbb{U}$.*

For Ω a set in $\mathbb{C}$, with ψ and $\mathfrak{p}(z)$ as given in Definition 4, we suppose that (1.15) is replaced by

$$\Omega \subset \left\{\psi(\mathfrak{p}(z), z\mathfrak{p}'(z), z^2\mathfrak{p}''(z), z^3\mathfrak{p}'''(z); z) : z \in \mathbb{U}\right\}.$$

Although this more general situation is a "differential containment", we also refer to it as a differential superordination, and the definitions of solution, subordinant and best subordinant as given above can be extended to this more general case.

We will use the following lemma [7, p. 445, Lemma D] from the theory of third-order differential subordinations in $\mathbb{U}$ to determine subordinants of third-order differential superordinations.

Lemma 1 (see [7]). Let $\mathfrak{p} \in \mathcal{Q}(a)$, and let $\mathfrak{q}(z) = a + a_n z^n + \cdots$ be analytic in $\mathbb{U}$ with $\mathfrak{q}(z) \neq a$ and $n \geqq 2$. If $\mathfrak{q}$ is not subordinate to $\mathfrak{p}$, then there exists points $z_0 = r_0 e^{i\theta_0} \in \mathbb{U}$ and $\xi_0 \in \partial\mathbb{U} \backslash E(\mathfrak{p})$, and an $m \geqq n$ for which $\mathfrak{q}(\mathbb{U}_{r_0}) \subset \mathfrak{p}(\mathbb{U})$,

(i) $\mathfrak{q}(z_0) = \mathfrak{p}(\xi_0)$,

(ii) $\Re\left(\frac{\xi_0 \mathfrak{p}''(\xi_0)}{\mathfrak{p}'(\xi_0)}\right) \geqq 0$ and $\left|\frac{z\mathfrak{q}'(z)}{\mathfrak{p}'(\xi_0)}\right| \leqq m$,

(iii) $z_0 \mathfrak{q}'(z_0) = m \xi_0 \mathfrak{p}'(\xi_0)$,

(iv) $\Re\left(1 + \frac{z_0 \mathfrak{q}''(z_0)}{\mathfrak{q}'(z_0)}\right) \geqq m \Re\left(1 + \frac{\xi_0 \mathfrak{p}''(\xi_0)}{\mathfrak{p}'(\xi_0)}\right)$, and

(v) $\Re\left(\frac{z_0^2 \mathfrak{q}'''(z_0)}{\mathfrak{q}'(z_0)}\right) \geqq m^2 \Re\left(\frac{\xi_0^2 \mathfrak{p}'''(\xi_0)}{\mathfrak{p}'(\xi_0)}\right)$.

2. Admissible functions and a fundamental result

We next define the class of admissible functions referred to in Section 1.

Definition 5. *Let Ω be a set in $\mathbb{C}$, $\mathfrak{q} \in \mathcal{H}[a,n]$ and $\mathfrak{q}'(z) \neq 0$. The class of admissible functions $\Psi'_n[\Omega, \mathfrak{q}]$ consists of those functions $\psi : \mathbb{C}^4 \times \overline{\mathbb{U}} \to \mathbb{C}$ that satisfy the admissibility condition*

$$\psi(r, s, t, u; \xi) \in \Omega,$$

whenever

$$r = \mathfrak{q}(z), \quad s = \frac{z\mathfrak{q}'(z)}{m}, \quad \Re\left(\frac{t}{s} + 1\right) \leqq \frac{1}{m} \Re\left(\frac{z\mathfrak{q}''(z)}{\mathfrak{q}'(z)} + 1\right) \tag{2.1}$$

and

$$\Re\left(\frac{u}{s}\right) \leqq \frac{1}{m^2} \Re\left(\frac{z^2 \mathfrak{q}'''(z)}{\mathfrak{q}'(z)}\right),$$

where $z \in \mathbb{U}$, $\xi \in \partial\mathbb{U}$ and $m \geqq n \geqq 2$.

If $\psi : \mathbb{C}^2 \times \overline{\mathbb{U}} \to \mathbb{C}$ and $\mathfrak{q} \in \mathcal{H}[a,n]$, then the admissibility condition (2.1) reduces to

$$\psi\left(\mathfrak{q}(z), \frac{z\mathfrak{q}'(z)}{m}; \xi\right) \in \Omega \quad (z \in \mathbb{U}; \xi \in \partial\mathbb{U}; m \geqq n \geqq 2).$$

If $\psi : \mathbb{C}^3 \times \overline{\mathbb{U}} \to \mathbb{C}$ and $\mathfrak{q} \in \mathcal{H}[a,n]$ with $\mathfrak{q}'(z) \neq 0$, then the admissibility condition (2.1) reduces to

$$\psi(r, s, t; \xi) \in \Omega,$$

whenever $r = \mathfrak{q}(z)$, $\quad s = \frac{z\mathfrak{q}'(z)}{m}$ $\quad$ and

$$\Re\left(\frac{t}{s} + 1\right) \leqq \frac{1}{m} \Re\left(\frac{z\mathfrak{q}''(z)}{\mathfrak{q}'(z)} + 1\right) \quad (z \in \mathbb{U}; \xi \in \partial\mathbb{U}; m \geqq n \geqq 2).$$

The next theorem is a foundation result in the theory of third-order differential superordinations.

Theorem 2. *Let $\psi \in \Psi'_n[\Omega, \mathfrak{q}]$. If $\psi(\mathfrak{p}(z), z\mathfrak{p}'(z), z^2\mathfrak{p}''(z), z^3\mathfrak{p}'''(z); z)$ is univalent in $\mathbb{U}$, $\mathfrak{p} \in \mathcal{Q}(a)$ and $\mathfrak{q} \in \mathcal{H}[a,n]$ satisfy the following condition:*

$$\Re\left(\frac{z\mathfrak{q}''(z)}{\mathfrak{q}'(z)}\right) \geqq 0 \quad \text{and} \quad \left|\frac{z\mathfrak{p}'(z)}{\mathfrak{q}'(z)}\right| \leqq m \quad (z \in \mathbb{U};\ m \geqq n \geqq 2), \tag{2.2}$$

then

$$\Omega \subset \left\{\psi(\mathfrak{p}(z), z\mathfrak{p}'(z), z^2\mathfrak{p}''(z), z^3\mathfrak{p}'''(z); z) : z \in \mathbb{U}\right\} \tag{2.3}$$

implies that

$$\mathfrak{q}(z) \prec \mathfrak{p}(z) \quad (z \in \mathbb{U}).$$

Proof. Suppose that $\mathfrak{q} \not\prec \mathfrak{p}$. By Lemma 1, there exists points $z_0 = r_0 e^{i\theta_0} \in \mathbb{U}$ and $\xi_0 \in \partial\mathbb{U}\backslash E(\mathfrak{p})$, and an $m \geqq n \geqq 2$ that satisfy the conditions (i)-(v) of Lemma 1. Using these conditions with $r = \mathfrak{p}(\xi_0)$, $s = \xi_0\mathfrak{p}'(\xi_0)$, $t = \xi_0^2\mathfrak{p}''(\xi_0)$, $u = \xi_0^3\mathfrak{p}'''(\xi_0)$ and $\xi = \xi_0$ in Definition 5, we obtain

$$\psi\left(\mathfrak{p}(\xi_0), \xi_0\mathfrak{p}'(\xi_0), \xi_0^2\mathfrak{p}''(\xi_0), \xi_0^3\mathfrak{p}'''(\xi_0); \xi_0\right) \in \Omega,$$

which contradicts (2.3), so we have

$$\mathfrak{q}(z) \prec \mathfrak{p}(z) \quad (z \in \mathbb{U}).$$

□

In the special case when $\Omega \neq \mathbb{C}$ is a simply connected domain and h is a conformal mapping of $\mathbb{U}$ onto Ω, we denote this class $\Psi'_n[h(\mathbb{U}), \mathfrak{q}]$ by $\Psi'_n[h, \mathfrak{q}]$. The following result is an immediate consequence of Theorem 2.

Theorem 3. *Let h be analytic in $\mathbb{U}$ and let $\psi \in \Psi'_n[h, \mathfrak{q}]$. If $\mathfrak{p} \in \mathcal{Q}(a)$ and $\mathfrak{q} \in \mathcal{H}[a, n]$ satisfy the condition* (2.2) *and $\psi(\mathfrak{p}(z), z\mathfrak{p}'(z), z^2\mathfrak{p}''(z), z^3\mathfrak{p}'''(z); z)$ is univalent in $\mathbb{U}$, then*

$$h(z) \prec \psi(\mathfrak{p}(z), z\mathfrak{p}'(z), z^2\mathfrak{p}''(z), z^3\mathfrak{p}'''(z); z) \tag{2.4}$$

implies that

$$\mathfrak{q}(z) \prec \mathfrak{p}(z) \quad (z \in \mathbb{U}).$$

Theorems 2 and 3 can only be used to obtain subordinants of third-order differential superordination of the form (2.3) or (2.4).

Theorem 4. *Let h be analytic in $\mathbb{U}$ and let $\psi : \mathbb{C}^4 \times \overline{\mathbb{U}} \to \mathbb{C}$. Suppose that the following differential equation*:

$$\psi\left(\mathfrak{q}(z), z\mathfrak{q}'(z), z^2\mathfrak{q}''(z), z^3\mathfrak{q}'''(z); z\right) = h(z) \tag{2.5}$$

has a solution $\mathfrak{q} \in \mathcal{Q}(a)$. If $\psi \in \Psi'_n[h, \mathfrak{q}]$, $\mathfrak{p} \in \mathcal{Q}(a)$ and $\psi(\mathfrak{p}(z), z\mathfrak{p}'(z), z^2\mathfrak{p}''(z), z^3\mathfrak{p}'''(z); z)$ is univalent in $\mathbb{U}$, then (2.4) *implies that*

$$\mathfrak{q}(z) \prec \mathfrak{p}(z) \quad (z \in \mathbb{U})$$

and $\mathfrak{q}(z)$ is the best subordinant.

Proof. Since $\psi \in \Psi'_n[h, \mathfrak{q}]$, by applying Theorem 3, we deduce that $\mathfrak{q}$ is a subordinant of (2.4). Since $\mathfrak{q}$ satisfies (2.5), it is also a solution of the differential superordination (2.4) and therefore all subordinants of (2.4) will be subordinate to $\mathfrak{q}$. Hence, $\mathfrak{q}$ will be the best subordinant of (2.4). □

Next, by making use of the third-order differential superordination results obtained in Section 2 (see, for details, Theorems 2, 3 and 4), we determine certain appropriate classes of admissible functions and investigate some third-order differential superordination properties of analytic functions associated with the operator B_κ^c defined by (1.6). New third-order differential sandwich-type results for the operator B_κ^c are also obtained. It should be remarked in passing that, in recent years, several authors obtained many interesting results involving various linear and nonlinear operators associated with (second-order) differential subordination and superordination, the interested reader may refer to, for example, (see [1] to [6], [12] to [14], [31] and [32]).

3. Third-order differential superordination and sandwich-type results

In this section, we obtain some third-order differential superordination and sandwich-type results for functions associated with the operator B_κ^c defined by (1.6). For this aim, the class of admissible functions is given in the following definition.

Definition 6. *Let Ω be a set in $\mathbb{C}$ and $\mathfrak{q} \in \mathcal{H}_0$ with $\mathfrak{q}'(z) \neq 0$. The class of admissible functions $\Phi'_B[\Omega, \mathfrak{q}]$ consists of those functions $\phi : \mathbb{C}^4 \times \overline{\mathbb{U}} \to \mathbb{C}$ that satisfy the admissibility condition:*

$$\phi(\alpha, \beta, \gamma, \delta; \xi) \in \Omega$$

whenever

$$\alpha = \mathfrak{q}(z), \quad \beta = \frac{z\mathfrak{q}'(z) + m(\kappa - 1)\mathfrak{q}(z)}{m\kappa},$$

$$\Re\left(\frac{\kappa(\kappa-1)\gamma - (\kappa-1)(\kappa-2)\alpha}{\kappa\beta - (\kappa-1)\alpha} - (2\kappa - 3)\right) \leqq \frac{1}{m}\Re\left(\frac{z\mathfrak{q}''(z)}{\mathfrak{q}'(z)} + 1\right)$$

and

$$\Re\left(\frac{\kappa(\kappa-1)((1-\kappa)\alpha + 3\kappa\beta + (1-3\kappa)\gamma + (\kappa-2)\delta)}{\alpha + \kappa(\beta - \alpha)}\right) \leqq \frac{1}{m^2}\Re\left(\frac{z^2\mathfrak{q}'''(z)}{\mathfrak{q}'(z)}\right),$$

where $z \in \mathbb{U}$, $\kappa \in \mathbb{C}\backslash\{0, 1, 2\}$, $\xi \in \partial\mathbb{U}$ and $m \geqq 2$.

Theorem 5. *Let $\phi \in \Phi'_B[\Omega, \mathfrak{q}]$. If the functions $f \in \mathcal{A}$, $B_{\kappa+1}^c f(z) \in \mathcal{Q}_0$ and $\mathfrak{q} \in \mathcal{H}_0$ with $\mathfrak{q}'(z) \neq 0$ satisfy the following condition:*

$$\Re\left(\frac{z\mathfrak{q}''(z)}{\mathfrak{q}'(z)}\right) \geqq 0, \qquad \left|\frac{B_\kappa^c f(z)}{\mathfrak{q}'(z)}\right| \leqq m, \tag{3.1}$$

and

$$\phi(B_{\kappa+1}^c f(z), B_\kappa^c f(z), B_{\kappa-1}^c f(z), B_{\kappa-2}^c f(z); z)$$

is univalent in $\mathbb{U}$, then

$$\Omega \subset \left\{\phi(B_{\kappa+1}^c f(z), B_\kappa^c f(z), B_{\kappa-1}^c f(z), B_{\kappa-2}^c f(z); z) : z \in \mathbb{U}\right\} \tag{3.2}$$

implies that

$$\mathfrak{q}(z) \prec B_{\kappa+1}^c f(z) \quad (z \in \mathbb{U}).$$

Proof. Define the analytic function $\mathfrak{p}(z)$ in $\mathbb{U}$ by

$$\mathfrak{p}(z) = B_{\kappa+1}^c f(z). \tag{3.3}$$

Then, differentiating (3.3) with respect to z and using (1.7), we have

$$B_\kappa^c f(z) = \frac{z\mathfrak{p}'(z) + (\kappa-1)\mathfrak{p}(z)}{\kappa}. \tag{3.4}$$

Further computations show that

$$B_{\kappa-1}^c f(z) = \frac{z^2\mathfrak{p}''(z) + 2(\kappa-1)z\mathfrak{p}'(z) + (\kappa-1)(\kappa-2)\mathfrak{p}(z)}{\kappa(\kappa-1)}, \tag{3.5}$$

and

$$B_{\kappa-2}^c f(z) = \frac{z^3\mathfrak{p}'''(z) + 3(\kappa-1)z^2\mathfrak{p}''(z) + 3(\kappa-1)(\kappa-2)z\mathfrak{p}'(z) + (\kappa-1)(\kappa-2)(\kappa-3)\mathfrak{p}(z)}{\kappa(\kappa-1)(\kappa-2)}. \tag{3.6}$$

We now define the transformation from $\mathbb{C}^4$ to $\mathbb{C}$ by

$$\alpha(r,s,t,u)=r,\quad \beta(r,s,t,u)=\frac{s+(\kappa-1)r}{\kappa},\quad \gamma(r,s,t,u)=\frac{t+2(\kappa-1)s+(\kappa-1)(\kappa-2)r}{\kappa(\kappa-1)} \tag{3.7}$$

and

$$\delta(r,s,t,u)=\frac{u+3(\kappa-1)t+3(\kappa-1)(\kappa-2)s+(\kappa-1)(\kappa-2)(\kappa-3)r}{\kappa(\kappa-1)(\kappa-2)}. \tag{3.8}$$

Let

$$\begin{aligned}\psi(r,s,t,u;z)&=\phi(\alpha,\beta,\gamma,\delta;z)\\&=\phi\bigg(r,\frac{s+(\kappa-1)r}{\kappa},\frac{t+2(\kappa-1)s+(\kappa-1)(\kappa-2)r}{\kappa(\kappa-1)},\\&\qquad\frac{u+3(\kappa-1)t+3(\kappa-1)(\kappa-2)s+(\kappa-1)(\kappa-2)(\kappa-3)r}{\kappa(\kappa-1)(\kappa-2)};z\bigg).\end{aligned} \tag{3.9}$$

Using equations (3.3) to (3.6), we find from (3.9) that

$$\psi\left(\mathfrak{p}(z),z\mathfrak{p}'(z),z^2\mathfrak{p}''(z),z^3\mathfrak{p}'''(z);z\right)=\phi(B^c_{\kappa+1}f(z),B^c_{\kappa}f(z),B^c_{\kappa-1}f(z),B^c_{\kappa-2}f(z);z). \tag{3.10}$$

Since $\phi\in\Phi'_B[\Omega,\mathfrak{q}]$, (3.10) and (3.2) yield

$$\Omega\subset\left\{\psi(\mathfrak{p}(z),z\mathfrak{p}'(z),z^2\mathfrak{p}''(z),z^3\mathfrak{p}'''(z);z):z\in\mathbb{U}\right\}.$$

From (3.7) and (3.8), we see that the admissible condition for $\phi\in\Phi'_B[\Omega,\mathfrak{q}]$ in Definition 10-6 is equivalent to the admissible condition for ψ as given in Definition 5 with $n=2$. Hence $\psi\in\Psi'_2[\Omega,\mathfrak{q}]$, and by using (3.1) and Theorem 2, we have

$$\mathfrak{q}(z)\prec\mathfrak{p}(z)\quad(z\in\mathbb{U})$$

or equivalently,

$$\mathfrak{q}(z)\prec B^c_{\kappa+1}f(z)\quad(z\in\mathbb{U}),$$

which evidently completes the proof of Theorem 5. □

If $\Omega\neq\mathbb{C}$ is a simply connected domain and $\Omega=h(\mathbb{U})$ for some conformal mapping $h(z)$ of $\mathbb{U}$ onto Ω, then the class $\Phi'_B[h(\mathbb{U}),\mathfrak{q}]$ is written as $\Phi'_B[h,\mathfrak{q}]$. Proceedings similarly as in the previous section, the following result is an immediate consequence of Theorem 5.

Theorem 6. *Let $\phi\in\Phi'_B[h,\mathfrak{q}]$ and h be analytic in $\mathbb{U}$. If the functions $f\in\mathcal{A}$, $B^c_{\kappa+1}f(z)\in\mathcal{Q}_0$ and $\mathfrak{q}\in\mathcal{H}_0$ with $\mathfrak{q}'(z)\neq0$ satisfy the condition* (3.1) *and*

$$\phi(B^c_{\kappa+1}f(z),B^c_{\kappa}f(z),B^c_{\kappa-1}f(z),B^c_{\kappa-2}f(z);z)$$

is univalent in $\mathbb{U}$, then

$$h(z)\prec\phi(B^c_{\kappa+1}f(z),B^c_{\kappa}f(z),B^c_{\kappa-1}f(z),B^c_{\kappa-2}f(z);z) \tag{3.11}$$

implies that

$$\mathfrak{q}(z)\prec B^c_{\kappa+1}f(z)\quad(z\in\mathbb{U}).$$

Theorems 5 and 6 can only be used to obtain subordinations of third-order differential superordination of the form (3.2) or (3.11). The following theorem proves the existence of the best subordinant of (3.11) for a suitable ϕ.

Theorem 7. *Let h be analytic in $\mathbb{U}$, and let $\phi : \mathbb{C}^4 \times \overline{\mathbb{U}} \to \mathbb{C}$ and ψ be given by* (3.9)*. Suppose that the differential equation*

$$\psi(\mathfrak{q}(z), z\mathfrak{q}'(z), z^2\mathfrak{q}''(z), z^3\mathfrak{q}'''(z); z) = h(z)$$

has a solution $\mathfrak{q}(z) \in \mathcal{Q}_0$. If the functions $f \in \mathcal{A}$, $B^c_{\kappa+1}f(z) \in \mathcal{Q}_0$ and $\mathfrak{q} \in \mathcal{H}_0$ with $\mathfrak{q}'(z) \neq 0$ satisfy the condition (3.1) *and*

$$\phi(B^c_{\kappa+1}f(z), B^c_{\kappa}f(z), B^c_{\kappa-1}f(z), B^c_{\kappa-2}f(z); z)$$

is univalent in $\mathbb{U}$, then

$$h(z) \prec \phi(B^c_{\kappa+1}f(z), B^c_{\kappa}f(z), B^c_{\kappa-1}f(z), B^c_{\kappa-2}f(z); z)$$

implies that

$$\mathfrak{q}(z) \prec B^c_{\kappa+1}f(z) \quad (z \in \mathbb{U})$$

and $\mathfrak{q}(z)$ is the best subordinant.

Proof. The proof of Theorem 7 is similar to that of Theorem 2.3 in [33] and it therefore omitted here. □

Combining the above Theorem 6 and Theorem 2.2 in [33], we obtain the following sandwich-type result.

Corollary 1. *Let h_1 and $\mathfrak{q}_1$ be analytic functions in $\mathbb{U}$, h_2 be univalent function in $\mathbb{U}$, $\mathfrak{q}_2 \in \mathcal{Q}_0$ with $\mathfrak{q}_1(0) = \mathfrak{q}_2(0) = 0$ and $\phi \in \Phi_B[h_2, \mathfrak{q}_2] \cap \Phi'_B[h_1, \mathfrak{q}_1]$. If the functions $f \in \mathcal{A}$, $B^c_{\kappa+1}f(z) \in \mathcal{Q}_0 \cap \mathcal{H}_0$ and*

$$\phi(B^c_{\kappa+1}f(z), B^c_{\kappa}f(z), B^c_{\kappa-1}f(z), B^c_{\kappa-2}f(z); z)$$

is univalent in $\mathbb{U}$, and the condition (2.1) *in* [33]*, that is, that*

$$\Re\left(\frac{\xi\mathfrak{q}''(\xi)}{\mathfrak{q}'(\xi)}\right) \geqq 0, \qquad \left|\frac{B^c_{\kappa}f(z)}{\mathfrak{q}'(\xi)}\right| \leqq k$$

and the condition (3.1) *are satisfied, then*

$$h_1(z) \prec \phi(B^c_{\kappa+1}f(z), B^c_{\kappa}f(z), B^c_{\kappa-1}f(z), B^c_{\kappa-2}f(z); z) \prec h_2(z)$$

implies that

$$\mathfrak{q}_1(z) \prec B^c_{\kappa+1}f(z) \prec \mathfrak{q}_2(z) \quad (z \in \mathbb{U}).$$

Definition 7. *Let Ω be a set in $\mathbb{C}$ and $\mathfrak{q} \in \mathcal{H}_0$ with $\mathfrak{q}'(z) \neq 0$. The class of admissible functions $\Phi'_{B,1}[\Omega, \mathfrak{q}]$ consists of those functions $\phi : \mathbb{C}^4 \times \overline{\mathbb{U}} \to \mathbb{C}$ that satisfy the admissibility condition*:

$$\phi(\alpha, \beta, \gamma, \delta; \xi) \in \Omega$$

whenever

$$\alpha = \mathfrak{q}(z), \quad \beta = \frac{z\mathfrak{q}'(z) + m\kappa\mathfrak{q}(z)}{m\kappa},$$

$$\Re\left(\frac{(\kappa-1)(\gamma-\alpha)}{\beta-\alpha} + (1-2\kappa)\right) \leqq \frac{1}{m}\Re\left(\frac{z\mathfrak{q}''(z)}{\mathfrak{q}'(z)} + 1\right)$$

and

$$\Re\left(\frac{(\kappa-1)(\kappa-2)(\delta-\alpha) - 3\kappa(\kappa-1)(\gamma-2\alpha+\beta)}{\beta-\alpha} + 6\kappa^2\right) \leqq \frac{1}{m^2}\Re\left(\frac{z^2\mathfrak{q}'''(z)}{\mathfrak{q}'(z)}\right),$$

where $z \in \mathbb{U}$, $\kappa \in \mathbb{C}\backslash\{0,1,2\}$, $\xi \in \partial\mathbb{U}$ *and* $m \geqq 2$.

Theorem 8. *Let* $\phi \in \Phi'_{B,1}[\Omega, \mathfrak{q}]$. *If the functions* $f \in \mathcal{A}$, $\frac{B^c_{\kappa+1}f(z)}{z} \in \mathcal{Q}_0$ *and* $\mathfrak{q} \in \mathcal{H}_0$ *with* $\mathfrak{q}'(z) \neq 0$ *satisfy the following condition*:

$$\Re\left(\frac{z\mathfrak{q}''(z)}{\mathfrak{q}'(z)}\right) \geqq 0, \qquad \left|\frac{B^c_{\kappa}f(z)}{z\mathfrak{q}'(z)}\right| \leqq m, \tag{3.12}$$

and

$$\phi\left(\frac{B^c_{\kappa+1}f(z)}{z}, \frac{B^c_{\kappa}f(z)}{z}, \frac{B^c_{\kappa-1}f(z)}{z}, \frac{B^c_{\kappa-2}f(z)}{z}; z\right)$$

is univalent in $\mathbb{U}$, *then*

$$\Omega \subset \left\{\phi\left(\frac{B^c_{\kappa+1}f(z)}{z}, \frac{B^c_{\kappa}f(z)}{z}, \frac{B^c_{\kappa-1}f(z)}{z}, \frac{B^c_{\kappa-2}f(z)}{z}; z\right) : z \in \mathbb{U}\right\} \tag{3.13}$$

implies that

$$\mathfrak{q}(z) \prec \frac{B^c_{\kappa+1}f(z)}{z} \quad (z \in \mathbb{U}).$$

Proof. Define the analytic function $\mathfrak{p}(z)$ in $\mathbb{U}$ by

$$\mathfrak{p}(z) = \frac{B^c_{\kappa+1}f(z)}{z}. \tag{3.14}$$

By making use of (1.7) and (3.14), we get

$$\frac{B^c_{\kappa}f(z)}{z} = \frac{z\mathfrak{p}'(z) + \kappa\mathfrak{p}(z)}{\kappa}. \tag{3.15}$$

Further computations show that

$$\frac{B^c_{\kappa-1}f(z)}{z} = \frac{z^2\mathfrak{p}''(z) + 2\kappa z\mathfrak{p}'(z) + \kappa(\kappa-1)\mathfrak{p}(z)}{\kappa(\kappa-1)} \tag{3.16}$$

and

$$\frac{B^c_{\kappa-2}f(z)}{z} = \frac{z^3\mathfrak{p}'''(z) + 3\kappa z^2\mathfrak{p}''(z) + 3\kappa(\kappa-1)z\mathfrak{p}'(z) + \kappa(\kappa-1)(\kappa-2)\mathfrak{p}(z)}{\kappa(\kappa-1)(\kappa-2)}. \tag{3.17}$$

We next define the transformation from $\mathbb{C}^4$ to $\mathbb{C}$ by

$$\alpha(r,s,t,u) = r, \qquad \beta(r,s,t,u) = \frac{s+\kappa r}{\kappa}, \qquad \gamma(r,s,t,u) = \frac{t + 2\kappa s + \kappa(\kappa-1)r}{\kappa(\kappa-1)} \tag{3.18}$$

and

$$\delta(r,s,t,u) = \frac{u + 3\kappa t + 3\kappa(\kappa-1)s + \kappa(\kappa-1)(\kappa-2)r}{\kappa(\kappa-1)(\kappa-2)}. \tag{3.19}$$

Then, upon setting

$$\begin{aligned}\psi(r,s,t,u;z) &= \phi(\alpha,\beta,\gamma,\delta;z)\\ &= \phi\Bigg(r, \frac{s+\kappa r}{\kappa}, \frac{t+2\kappa s+\kappa(\kappa-1)r}{\kappa(\kappa-1)},\\ &\qquad\qquad \frac{u+3\kappa t+3\kappa(\kappa-1)s+\kappa(\kappa-1)(\kappa-2)r}{\kappa(\kappa-1)(\kappa-2)}; z\Bigg),\end{aligned} \tag{3.20}$$

if we use the equations (3.14) to (3.17), we find from (3.20) that

$$\psi\left(\mathfrak{p}(z), z\mathfrak{p}'(z), z^2\mathfrak{p}''(z), z^3\mathfrak{p}'''(z); z\right) = \phi\left(\frac{B^c_{\kappa+1}f(z)}{z}, \frac{B^c_{\kappa}f(z)}{z}, \frac{B^c_{\kappa-1}f(z)}{z}, \frac{B^c_{\kappa-2}f(z)}{z}; z\right). \quad (3.21)$$

Since $\phi \in \Phi'_{B,1}[\Omega, q]$, it follows from (3.21) and (3.13) that

$$\Omega \subset \left\{\psi(\mathfrak{p}(z), z\mathfrak{p}'(z), z^2\mathfrak{p}''(z), z^3\mathfrak{p}'''(z); z) : z \in \mathbb{U}\right\}.$$

From (3.18) and (3.19), we see that the admissible condition for $\phi \in \Phi'_{B,1}[\Omega, \mathfrak{q}]$ in Definition 7 is equivalent to the admissible condition for ψ as given in Definition 5 with $n = 2$. Hence $\psi \in \Psi'_2[\Omega, \mathfrak{q}]$, and by using (3.12) and Theorem 2, we get

$$\mathfrak{q}(z) \prec \mathfrak{p}(z) \quad (z \in \mathbb{U})$$

or, equivalently,

$$\mathfrak{q}(z) \prec \frac{B^c_{\kappa+1}f(z)}{z} \quad (z \in \mathbb{U}).$$

□

In the case $\Omega \neq \mathbb{C}$ is a simply-connected domain with $\Omega = h(\mathbb{U})$ for some conformal mapping $h(z)$ of $\mathbb{U}$ onto Ω, the class $\Phi'_{B,1}[h(\mathbb{U}), \mathfrak{q}]$ is written as $\Phi'_{B,1}[h, \mathfrak{q}]$. Proceedings similarly, the following result is an immediate consequence of Theorem 8.

Theorem 9. *Let $\phi \in \Phi'_{B,1}[h, \mathfrak{q}]$ and h be analytic in $\mathbb{U}$. If the functions $f \in \mathcal{A}$, $\frac{B^c_{\kappa+1}f(z)}{z} \in \mathcal{Q}_0$ and $\mathfrak{q} \in \mathcal{H}_0$ with $\mathfrak{q}'(z) \neq 0$ satisfy the condition* (3.12) *and*

$$\phi\left(\frac{B^c_{\kappa+1}f(z)}{z}, \frac{B^c_{\kappa}f(z)}{z}, \frac{B^c_{\kappa-1}f(z)}{z}, \frac{B^c_{\kappa-2}f(z)}{z}; z\right)$$

is univalent in $\mathbb{U}$, then

$$h(z) \prec \phi\left(\frac{B^c_{\kappa+1}f(z)}{z}, \frac{B^c_{\kappa}f(z)}{z}, \frac{B^c_{\kappa-1}f(z)}{z}, \frac{B^c_{\kappa-2}f(z)}{z}; z\right)$$

implies that

$$\mathfrak{q}(z) \prec \frac{B^c_{\kappa+1}f(z)}{z} \quad (z \in \mathbb{U}).$$

Combining the above Theorem 9 and Theorem 2.5 in [33], we have the following sandwich-type result.

Corollary 2. *Let h_1 and $\mathfrak{q}_1$ be analytic functions in $\mathbb{U}$, h_2 be univalent function in $\mathbb{U}$, $\mathfrak{q}_2 \in \mathcal{Q}_0$ with $\mathfrak{q}_1(0) = \mathfrak{q}_2(0) = 0$ and $\phi \in \Phi_{B,1}[h_2, \mathfrak{q}_2] \cap \Phi'_{B,1}[h_1, \mathfrak{q}_1]$. If the functions $f \in \mathcal{A}$, $\frac{B^c_{\kappa+1}f(z)}{z} \in \mathcal{Q}_0 \cap \mathcal{H}_0$ and*

$$\phi\left(\frac{B^c_{\kappa+1}f(z)}{z}, \frac{B^c_{\kappa}f(z)}{z}, \frac{B^c_{\kappa-1}f(z)}{z}, \frac{B^c_{\kappa-2}f(z)}{z}; z\right)$$

is univalent in $\mathbb{U}$, and the condition (2.12) *in* [33], *that is, that*

$$\Re\left(\frac{\xi\mathfrak{q}''(\xi)}{\mathfrak{q}'(\xi)}\right) \geqq 0, \qquad \left|\frac{B^c_{\kappa}f(z)}{z\mathfrak{q}'(\xi)}\right| \leqq k$$

and the condition (3.12) *are satisfied, then*

$$h_1(z) \prec \phi\left(\frac{B^c_{\kappa+1}f(z)}{z}, \frac{B^c_{\kappa}f(z)}{z}, \frac{B^c_{\kappa-1}f(z)}{z}, \frac{B^c_{\kappa-2}f(z)}{z}; z\right) \prec h_2(z)$$

implies that

$$\mathfrak{q}_1(z) \prec \frac{B^c_{\kappa+1}f(z)}{z} \prec \mathfrak{q}_2(z) \quad (z \in \mathbb{U}).$$

Remark 1. By suitably specializing the results presented in this paper, we can obtain the corresponding results for the *simpler* operators $\mathcal{J}_p f(z)$, $\mathcal{I}_p f(z)$ and $\mathcal{S}_p f(z)$, which are defined by (1.8), (1.9) and (1.10), respectively.

4. Concluding Remarks and Observations

In our present investigation, we have derived several third-order differential superordination results for analytic functions in the open unit disk $\mathbb{U}$ by using the operator B^c_κ which is defined by means of the convolution in (1.6) involving the normalized form of the three-parameter family $\omega_{p,b,c}(z)$ of the generalized Bessel functions of the first kind, which is defined by (1.3). Our results have been obtained by considering suitable classes of admissible functions. Furthermore, some third-order differential sandwich-type results for the operator B^c_κ have been obtained.

Acknowledgements

The research of the first-named author (Huo Tang) was partly supported by the Natural Science Foundation of the People's Republic of China under Grant 11271045, the Higher School Doctoral Foundation of the People's Republic of China under Grant 20100003110004, the Natural Science Foundation of Inner Mongolia of the People's Republic of China under Grant 2010MS0117 and the Higher School Foundation of Inner Mongolia of the People's Republic of China under Grant NJZY13298. The research of the third-named author (Erhan Deniz) was supported by the Commission for the Scientific Research Projects of Kafkas Univertsity (Project Number 2012-FEF-30). The authors would like to thank Professors Rosihan M. Ali and V. Ravichandran for their valuable suggestions and the referees for their careful reading of the paper and for their helpful comments to improve our paper.

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