A QUASISTATIC ELECTRO-VISCOELASTIC CONTACT PROBLEM WITH ADHESION

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Abstract. The aim of this paper is to study the process of contact with adhesion between a piezoelectric body and an obstacle, the so-called foundation. The material's behavior is assumed to be electro-viscoelastic; the process is quasistatic, the contact is modeled by the *Signorini* condition. The adhesion process is modeled by a bonding field on the contact surface. We derive a variational formulation for the problem and then we prove the existence of a unique weak solution to the model.The proof is based on a general result on evolution equations with maximal monotone operators and fixed point arguments.

1. Introduction

A piezoelectric body is one that produces an electric charge when a mechanical stress is applied (the body is squeezed or stretched). Conversely, a mechanical deformation (the body shrinks or expands) is produced when an electric field is applied. This kind of materials appears usually in the industry as switches in radiotronics, electroacoustics or measuring equipments. Piezoelectric materials for which the mechanical properties are elastic are also called electro-elastic materials, and those for which the mechanical properties are viscoelastic are also called electro-viscoelastic materials. Different models have been developed to describe the interaction between the electrical and mechanical fields(see, e.g. [2, 14, 16, 17, 18, 19, 29, 30, 31] and the references therein). General models for elastic materials with piezoelectric effect, called electro-elastic materials, can be found in [2, 4, 14]. A static frictional contact problem for electric-elastic materials was considered in [1, 15], under the assumption that the foundation is insulated. Contact problems involving elasto-piezoelectric materials [1, 15, 28], viscoelastic piezoelectric materials [5, 25] have been studied.

Adhesion may take place between parts of the contacting surfaces. It may be intentional, when surfaces are bonded with glue, or unintentional, as a seizure between very clean surfaces. The adhesive contact is modeled by a bonding field on the contact surface, denoted in this paper by β; it describes the pointwise fractional density of active bonds on the contact surface, and sometimes referred to as the intensity of adhesion. Following [10], [11], the bonding field satisfies the restrictions $0 \le \beta \le 1$; when $\beta = 1$ at a point of the contact surface, the adhesion is complete and all the bonds are active; when $\beta = 0$ all the bonds are inactive, severed, and there is no

1991 *Mathematics Subject Classification.* 35J87, 74M10, 74M15, 49J40.

Key words and phrases. Electro-viscoelastic material; Signorini's condition; Adhesion; Quasi-variational inequality; Maximal monotone operator; Weak solution; Fixed point.

adhesion; when $0 < \beta < 1$ the adhesion is partial and only a fraction β of the bonds is active. Basic modelling can be found in [10, 11, 12]. Analysis of models for adhesive contact can be found in [7, 8] and in the monographs [24, 27]. An application of the theory of adhesive contact in the medical field of prosthetic limbs was considered in [22, 23]; there, the importance of the bonding between the bone-implant and the tissue was outlined, since debonding may lead to decrease in the persons ability to use the artificial limb or joint.

In this work we continue in this line of research, where we extend the result established in [3, 20] for contact problem described with the *Signorini* conditions into contact problem described with the *Signorini* conditions with adhesion where the obstacle is a perfect insulator and the resistance to tangential motion is generated by the glue, in comparison to which the frictional traction can be neglected. Therefore, the tangential contact traction depends only on the bonding field and the tangential displacement.

The paper is structured as follows. In Section 2 we present the electro-viscoelastic contact model with adhesion and provide comments on the contact boundary conditions. In Section 3 we list the assumptions on the data and derive the variational formulation. In Section 4, we present our main existence and uniqueness result, Theorem 4.1, which states the unique weak solvability of the *Signorini* adhesive contact problem. The proof of the theorem is provided in Section 5, where it is carried out in several steps and is based on a general result on evolution equations with maximal monotone operators and fixed point theorem.

2. **The model**

We consider a body made of a piezoelectric material which occupies the domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ with a smooth boundary $\partial\Omega = \Gamma$ and a unit outward normal ν. The body is acted upon by body forces of density f_0 and has volume free electric charges of density q_0 . It is also constrained mechanically and electrically on the boundary. To describe these constraints we assume a partition of Γ into three open disjoint parts Γ_1, Γ_2 and Γ_3, on the one hand, and a partition of $\Gamma_1\cup\ \Gamma_2$ into two open parts Γ_a and Γ_b, on the other hand. We assume that *meas* $\Gamma_1 > 0$ and *meas* $\Gamma_a > 0$; these conditions allow the use of coercivity arguments in the proof of the unique solvability of the model. The body is clamped on Γ_1 and, therefore, the displacement field vanishes there. Surface tractions of density f_2 act on Γ_2. We also assume that the electrical potential vanishes on Γ_a and a surface electrical charge of density q_2 is prescribed on Γ_b. On Γ_3 the body is in adhesive contact with an insulator obstacle, the so-called foundation. The contact is frictionless and, since the foundation is assume to be rigide, we model it with the *Signorini* condition.

We are interested in the deformation of the body on the time interval $[0\ T]$. The process is assumed to be quasistatic, i.e. the inertial effects in the equation of motion are neglected. We denote by $x \in \Omega \cup \Gamma$ and $t \in [0\ T]$ the spatial and the time variable, respectively, and, to simplify the notation, we do not indicate in what follows the dependence of various functions on x and t. Here and everywhere in this paper, i, j, k, $l = 1, ..., d$, summation over two repeated indices is implied, and the index that follows a comma represents the partial derivative with respect to the corresponding component of x. The dot above variable represents the time derivatives.

We denote by $\mathbb{S}^d$ the space of second-order symmetric tensors on $\mathbb{R}^d$ ($d = 2, 3$) and by "$.$", $\|.\|$ the inner product and the norm on $\mathbb{S}^d$ and $\mathbb{R}^d$, respectively, that is $u.\upsilon = u_i\, \upsilon_i$, $\|\upsilon\| = (\upsilon.\upsilon)^{1/2}$ for $u = (u_i)$, $\upsilon = (\upsilon_i) \in \mathbb{R}^d$, and $\sigma.\tau = \sigma_{ij}\, \tau_{ij}$, $\|\sigma\| = (\sigma.\sigma)^{1/2}$ for $\sigma = (\sigma_{ij})$, $\tau = (\tau_{ij}) \in \mathbb{S}^d$. We also use the usual notation for the normal components and the tangential parts of vectors and tensors, respectively, given by $\upsilon_\nu = \upsilon \cdot \nu$, $\upsilon_\tau = \upsilon - \upsilon_\nu \nu$, $\sigma_\nu = \sigma_{ij}\, \nu_i\, \nu_j$, and $\sigma_\tau = \sigma\nu - \sigma_\nu \nu$.

With these assumptions, the classical model for the process is the following.

Problem ($\mathcal{P}$). *Find a displacement field $u : \Omega \times [0, T] \to \mathbb{R}^d$, a stress field $\sigma : \Omega \times [0, T] \to \mathbb{S}^d$, an electric potential $\varphi : \Omega \times [0, T] \to \mathbb{R}$, an electric displacement field $D : \Omega \times [0, T] \to \mathbb{R}^d$ and a bonding field $\beta : \Omega \times [0, T] \to \mathbb{R}$ such that*

$$\sigma = \mathcal{A}\varepsilon(\dot{u}) + \mathcal{F}\varepsilon(u) - \mathcal{E}^*\mathbf{E}(\varphi) \quad \text{in } \Omega \times (0, T), \tag{2.1}$$

$$D = \mathcal{B}\mathbf{E}(\varphi) + \mathcal{E}\varepsilon(u) \quad \text{in } \Omega \times (0, T), \tag{2.2}$$

$$Div\sigma + f_0 = 0 \quad \text{in } \Omega \times (0, T), \tag{2.3}$$

$$divD = q_0 \quad \text{in } \Omega \times (0, T), \tag{2.4}$$

$$u = 0 \quad \text{on } \Gamma_1 \times (0, T), \tag{2.5}$$

$$\sigma\nu = f_2 \quad \text{on } \Gamma_2 \times (0, T), \tag{2.6}$$

$$\begin{cases} u_\nu \leq 0, \\ \sigma_\nu - \gamma_\nu \beta^2 R_\nu(u_\nu) \leq 0, \\ (\sigma_\nu - \gamma_\nu \beta^2 R_\nu(u_\nu))u_\nu = 0 \end{cases} \quad \text{on } \Gamma_3 \times (0, T), \tag{2.7}$$

$$-\sigma_\tau = p_\tau(\beta)R_\tau(u_\tau) \quad \text{on } \Gamma_3 \times (0, T), \tag{2.8}$$

$$\dot{\beta}(t) = -(\gamma_\nu \beta(t) R_\nu(u_\nu(t))^2 - \varepsilon_a)_+ \quad \text{on } \Gamma_3 \times (0, T), \tag{2.9}$$

$$\varphi = 0 \quad \text{on } \Gamma_a \times (0, T), \tag{2.10}$$

$$D.\nu = q_2 \quad \text{on } \Gamma_b \times (0, T), \tag{2.11}$$

$$D.\nu = 0 \quad \text{on } \Gamma_3 \times (0, T), \tag{2.12}$$

$$u(0) = u_0 \quad \text{in } \Omega, \tag{2.13}$$

$$\beta(0) = \beta_0 \quad \text{on } \Gamma_3. \tag{2.14}$$

We now provide some comments on equations and conditions (2.1)–(2.14).

First, equations (2.1) and (2.2) represent the electro-viscoelastic constitutive law in which $\sigma = (\sigma_{ij})$ is the stress tensor, $\varepsilon(u) = (\varepsilon_{ij}(u))$ denotes the linearized strain tensor, $\mathbf{E}(\varphi) = -\nabla\varphi$ is the electric field, $\mathcal{A}$ and $\mathcal{F}$ are the viscosity and elasticity operators, respectively, $\mathcal{E} = (e_{ijk})$ represents the third-order piezoelectric tensor, $\mathcal{E}^* = (e^*_{ijk})$, where $e^*_{ijk} = e_{kij}$, is its transpose, $\mathcal{B} =(\mathcal{B}_{ij})$ denotes the electric permittivity tensor and $D = (D_1, ..., D_d)$ is the electric displacement vector. Details on the constituve equations of the form (2.1) and (2.2) can be found, for instance, in [1, 2, 13, 21] and the references therein.

Next, equations (2.3) and (2.4) are the equilibrium equations for the stress and electric-displacement fields, respectively, in which "*Div*" and "*div*" denote the divergence operators for tensor and vector valued functions, respectively.

Conditions (2.5) and (2.6) are the displacement and traction boundary conditions, whereas (2.10) and (2.11) represent the electric boundary conditions. Note that we need to impose assumption (2.12) for physical reasons. Indeed, this condition models the case when the obstacle is a perfect insulator and was used in [1, 9, 15, 25, 26]. The evolution of the bonding field is governed by the differential equation (2.9) with given positive parameters γ_ν and ε_a where $r_+ = \max\{0, r\}$.

Condition (2.7) represents the *Signorini* contact condition with adhesion where u_ν is the normal displacement, σ_ν represents the normal stress, γ_ν denotes a given adhesion coefficient and R_ν is the truncation operator defined by

$$R_\nu(s) = \begin{cases} -L & \text{if } s < -L, \\ -s & \text{if } -L \le s \le 0, \\ 0 & \text{if } s > 0. \end{cases}$$

Here $L > 0$ is the characteristic length of the bond, beyond which it does not offer any additional traction (see [27]).

We assume that the resistance to tangential motion is generated only by the glue, and is assumed to depend on the adhesion field and on the tangential displacement, but, again, only up to the bond length L (see (2.8)), where the truncation operator R_τ is defined by

$$R_\tau(v) = \begin{cases} v & \text{if } \|v\| \le L, \\ L\dfrac{v}{\|v\|} & \text{if } \|v\| > L. \end{cases}$$

Then, $p_\tau(\beta)$ acts as the stiffness or spring constant, increasing with (β), and the traction is in the direction opposite to the displacement. The maximal modulus of the tangential traction is $p_\tau(1)L$.

Finally, (2.13) and (2.14) represent the initial conditions in which u_0 and β_0 are the prescribed initial displacement and bonding fields, respectively.

3. **Variational Formulation and Preliminairies**

In this section, we list the assumptions on the data and derive a variational formulation for the contact problem. To this end we need to introduce some notation and preliminaries.

Everywhere below, we use the classical notation for L^p and *Sobolev* spaces associated to Ω and Γ. Moreover, we use the notation $L^2(\Omega)^d$, $H^1(\Omega)^d$, $\mathcal{H}$ and $\mathcal{H}_1$ for the following spaces

$$L^2(\Omega)^d = \{\ v = (v_i) \mid v_i \in L^2(\Omega)\ \}, \quad H^1(\Omega)^d = \{\ v = (v_i) \mid v_i \in H^1(\Omega)\ \},$$
$$\mathcal{H} = \{\ \tau = (\tau_{ij}) \mid \tau_{ij} = \tau_{ji} \in L^2(\Omega)\ \}, \quad \mathcal{H}_1 = \{\ \tau \in \mathcal{H} \mid \tau_{ij,j} \in L^2(\Omega)\ \}.$$

The spaces $L^2(\Omega)^d$, $H^1(\Omega)^d$, $\mathcal{H}$ and $\mathcal{H}_1$ are real *Hilbert* spaces endowed with the canonical inner products given by

$$(u, v)_{L^2(\Omega)^d} = \int_\Omega u \cdot v\, dx, \quad (u, v)_{H^1(\Omega)^d} = \int_\Omega u \cdot v\, dx + \int_\Omega \nabla u \cdot \nabla v\, dx,$$

$$(\sigma,\tau)_{\mathcal{H}} = \int_\Omega \sigma\cdot\tau\,dx, \quad (\sigma,\tau)_{\mathcal{H}_1} = \int_\Omega \sigma\cdot\tau\,dx + \int_\Omega \operatorname{Div}\sigma\cdot\operatorname{Div}\tau\,dx,$$

and the associated norms $\|\cdot\|_{L^2(\Omega)^d}$, $\|\cdot\|_{H^1(\Omega)^d}$, $\|\cdot\|_{\mathcal{H}}$ and $\|\cdot\|_{\mathcal{H}_1}$, respectively. Here and below we use the notation

$$\nabla v = (v_{i,j}), \quad \varepsilon(v) = (\varepsilon_{ij}(v)), \quad \varepsilon_{ij}(v) = \tfrac{1}{2}(v_{i,j}+v_{j,i}) \quad \forall\, v \in H^1(\Omega)^d,$$
$$Div\,\tau = (\tau_{ij,j}) \quad \forall\,\tau \in \mathcal{H}_1.$$

For every element $v \in H^1(\Omega)^d$ we also write v for the trace of v on Γ and we denote by v_ν and v_τ the normal and tangential components of v on Γ.

We now list the assumptions on the problem's data. The viscosity operator $\mathcal{A}$ and the elasticity operator $\mathcal{F}$ are assumed to satisfy the conditions

$$\text{(3.1)} \quad \begin{cases} \text{(a) } \mathcal{A}:\ \Omega\times\mathbb{S}^d \longrightarrow \mathbb{S}^d. \\ \text{(b) } \mathcal{A}(x,\tau) = (a_{ijkl}(x)\tau_{kl}\ )\ \forall\tau\in S^d \ \text{ a.e. } x\in\Omega. \\ \text{(c) } a_{ijkl} = a_{klij} = a_{jikl} \in L^\infty(\Omega). \\ \text{(d) there exists } m_{\mathcal{A}} > 0 \text{ such that} \\ \quad a_{ijkl}\,\tau_{ij}\tau_{kl} \ge m_{\mathcal{A}}\|\tau\|^2,\ \forall\tau=(\tau_{ij})\in\mathbb{S}^d,\ \text{a.e. } x\in\ \Omega. \end{cases}$$

$$\text{(3.2)} \quad \begin{cases} \text{(a) } \mathcal{F}:\Omega\times\mathbb{S}^d\to\mathbb{S}^d. \\ \text{(b) There exists } L_{\mathcal{F}}>0 \text{ such that} \\ \quad \|\mathcal{F}(x,\tau_1)-\mathcal{F}(x,\tau_2)\| \le L_{\mathcal{F}}\|\tau_1-\tau_2\| \\ \quad \forall\,\tau_1,\tau_2\in\mathbb{S}^d,\ \text{ a.e. } \mathbf{x}\in\Omega. \\ \text{(c) The mapping } x\mapsto\mathcal{F}(x,\tau) \text{ is measurable on } \Omega, \\ \quad \text{for each } \tau\in\mathbb{S}^d. \\ \text{(d) The mapping } x\mapsto\mathcal{F}(x,\mathbf{0}) \text{ belongs to } \mathcal{H}. \end{cases}$$

The piezoelectric tensor $\mathcal{E}$ and the electric permittivity tensor $\mathcal{B}$ satisfy

$$\text{(3.3)} \quad \begin{cases} \text{(a) } \mathcal{E}:\Omega\times\mathbb{S}^d\to\mathbb{R}^d. \\ \text{(b) } \mathcal{E}(x,\tau) = (e_{ijk}(x)\tau_{jk}) \quad \forall\tau=(\tau_{ij})\in\mathbb{S}^d,\ \text{ a.e. } x\in\Omega. \\ \text{(c) } e_{ijk}=e_{ikj}\in L^\infty(\Omega). \end{cases}$$

$$\text{(3.4)} \quad \begin{cases} \text{(a) } \mathcal{B}:\Omega\times\mathbb{R}^d\to\mathbb{R}^d. \\ \text{(b) } \mathcal{B}(x,\mathbf{E}) = (\mathcal{B}_{ij}(x)E_j) \quad \forall\mathbf{E}=(E_i)\in\mathbb{R}^d,\ \text{ a.e. } \mathbf{x}\in\Omega. \\ \text{(c) } \mathcal{B}_{ij}=\mathcal{B}_{ji}\in L^\infty(\Omega). \\ \text{(d) There exists } m_{\mathcal{B}}>0 \text{ such that } \mathcal{B}_{ij}(x)E_iE_j \ge m_{\mathcal{B}}\|\mathbf{E}\|^2 \\ \quad \forall\mathbf{E}=(E_i)\ \in\mathbb{R}^d,\ \text{a.e. } x\in\Omega. \end{cases}$$

As in [8] we assume that the tangential contact function satisfies

$$\text{(3.5)} \quad \begin{cases} \text{(a) } \ p_\tau:\Gamma_3\times\mathbb{R}\longrightarrow\mathbb{R}_+. \\ \text{(b) There exists } \ L_\tau>0 \text{ such that} \\ \quad |\ p_\tau(x,\beta_1)-p_\tau(x,\beta_2)\ | \le L_\tau\ |\ \beta_1-\beta_2\ | \\ \quad \forall\beta_1,\ \beta_2\in\mathbb{R},\ \text{ a.e. } x\in\Gamma_3. \\ \text{(c) There exists } \ M_\tau>0 \text{ such that} \\ \quad |\ p_\tau(x,\beta)\ | \le M_\tau \quad \forall\beta\in\mathbb{R},\ \text{ a.e. } x\in\Gamma_3. \\ \text{(d) For any } \beta\in\mathbb{R},\ x\mapsto p_\tau(x,\beta) \text{ is measurable on } \Gamma_3. \\ \text{(e) The mapping } \ x\mapsto p_\tau(x,0) \text{ belongs to } L^2(\Gamma_3). \end{cases}$$

The forces, tractions, volume and surface free charge densities satisfy

$$\text{(3.6)} \qquad f_0\in W^{1,1}(0,T;L^2(\Omega)^d), \quad f_2\in W^{1,1}(0,T;L^2(\Gamma_2)^d),$$

$$\text{(3.7)} \qquad q_0\in W^{1,1}(0,T;L^2(\Omega)), \quad q_2\in W^{1,1}(0,T;L^2(\Gamma_b)).$$

The adhesion coefficient γ_ν and the limit bound ε_a satisfy the conditions

(3.8) $\quad \gamma_\nu \in L^\infty(\Gamma_3), \quad \epsilon_a \in L^2(\Gamma_3), \quad \gamma_\nu,\, \gamma_\tau,\, \epsilon_a \geq 0 \quad$ a.e. on Γ_3.

Also, we assume that the initial bonding field satisfies

(3.9) $\quad \beta_0 \in L^2(\Gamma_3), \quad 0 \leq \beta_0 \leq 1\;$ a.e. on Γ_3.

Moreover, the tensor $\mathcal{E}$ and its transpose $\mathcal{E}^*$ satisfy the equality

(3.10) $\quad \mathcal{E}\sigma.\upsilon = \sigma.\mathcal{E}^*\upsilon \quad \forall \sigma \in \mathbb{S}^d,\; \upsilon \in \mathbb{R}^d$

Let now consider the closed subspace of $H^1(\Omega)^d$ defined by

$$V = \{\ \upsilon \in H^1(\Omega)^d \mid \upsilon = 0 \text{ on } \Gamma_1\ \}.$$

Since $meas\,(\Gamma_1) > 0$ and the viscosity tensor satisfies assumption (3.1), it follows that V is a real *Hilbert* space endowed with the inner product

(3.11) $\quad (u, \upsilon)_V = (\mathcal{A}\varepsilon(u), \varepsilon(\upsilon))_{\mathcal{H}},$

and let $\|\cdot\|_V$ be the associated norm.

We also introduce the following spaces

$$W = \{\ \psi \in H^1(\Omega) \mid \psi = 0 \text{ on } \Gamma_a\ \},\ \mathcal{W} = \{\ D = (D_i) \mid D_i \in L^2(\Omega),\ div\ D \in L^2(\Omega)\ \}.$$

Since $meas\,(\Gamma_a) > 0$ it is well known that W is a real *Hilbert* space endowed with the inner product

$$(\varphi, \psi)_W = (\nabla\varphi, \nabla\psi)_{L^2(\Omega)^d}\,,$$

and the associated norm $\|\cdot\|_W$. Also we have the following *Friedrichs-Poincaré* inequality

(3.12) $\quad \|\nabla\psi\|_{L^2(\Omega)^d} \geq c_F\, \|\psi\|_{H^1(\Omega)} \quad \forall\, \psi \in W,$

where $c_F > 0$ is a constant which depends only on Ω and Γ_a. The space $\mathcal{W}$ is a real *Hilbert* space endowed with the inner product

$$(D, \mathbf{E})_{\mathcal{W}} = \int_\Omega D \cdot \mathbf{E}\, dx + \int_\Omega div\ D \cdot div\ \mathbf{E}\, dx,$$

and the associated norm $\|\cdot\|_{\mathcal{W}}$. Moreover, by the *Sobolev* trace theorem, there exist two positive constants c_0 and $\tilde{c}_0$ such that

(3.13) $\quad \|\upsilon\|_{L^2(\Gamma_3)^d} \leq c_0\|\upsilon\|_V \quad \forall \upsilon \in V, \quad \|\psi\|_{L^2(\Gamma_3)} \leq \tilde{c}_0\|\psi\|_W \quad \forall\, \psi \in W.$

Next, we define the two mappings $f : [0\ T] \longrightarrow V$ and $q : [0\ T] \longrightarrow W$, respectively, by

$$(f(t), \upsilon)_V = \int_\Omega f_0(t) \cdot \upsilon\, dx + \int_{\Gamma_2} f_2(t) \cdot \upsilon\, da, \tag{3.14}$$

$$(q(t), \psi)_W = \int_\Omega q_0(t)\psi\, dx - \int_{\Gamma_b} q_2(t)\psi\, da, \tag{3.15}$$

for all $\upsilon \in V$, $\psi \in W$ and $t \in [0, T]$. We note that the definitions of f and q are based on the *Riesz* representation theorem. Moreover, it follows from assumptions (3.6) and (3.7) that

$$f \in W^{1,1}(0, T; V), \tag{3.16}$$

$$q \in W^{1,1}(0, T; W). \tag{3.17}$$

For *Signorini* problem, we use the convex subset of admissible displacements fields given by

$$U_{ad} = \{v \in V/\ v_\nu \le 0 \text{ on } \Gamma_3\},$$

and we make the regularity assumption

(3.18) $u_0 \in U_{ad}$,

on the initial data. Also, we introduce the set

$$\mathcal{Q} = \{\, \beta \in L^\infty(0,T;L^2(\Gamma_3)) \mid 0 \le \beta(t) \le 1 \ \ \forall\, t \in [0,T],\ a.e.\ on\ \Gamma_3 \,\}.$$

Next, we define the functional $j : L^2(\Gamma_3) \times V \times V \longrightarrow \mathbb{R}$ by

(3.19) $$j(\beta,u,v) = \int_{\Gamma_3} -\gamma_\nu \beta^2 R_\nu(u_\nu) v_\nu da + \int_{\Gamma_3} p_\tau(\beta) R_\tau(u_\tau)\cdot v_\tau da.$$

It follows from assumptions (3.5)–(3.8) that the integrals in (3.14), (3.15) and (3.19) are well defined.

Using a standard procedure based on *Green's formulas* and equalities (3.14), (3.15), (3.19), it is easy to see that if $(u,\sigma,\varphi,\beta,D)$ are sufficiently regular functions which satisfy (2.3)–(2.12) then

(3.20)
$$u(t) \in U_{ad},\ \ (\sigma(t),\varepsilon(v)-\varepsilon(u(t)))_{\mathcal{H}} + j(\beta(t),u(t),v-u(t) \ge (f(t),v-u(t))_V,$$

(3.21) $(D(t),\nabla\psi)_{L^2(\Omega)^d} + (q(t),\psi)_W = 0,$

for all $v \in U_{ad}$, $\psi \in W$ and $t \in [0,T]$. We substitute (2.1) in (3.20), (2.2) in (3.21), keeping in mind that $\mathbf{E}(\varphi) = -\nabla\varphi$ and use the initial condition (2.13) to derive the following variational formulation of Problem $(\mathcal{P})$.

Problem $(\mathcal{P}^V)$. *Find a displacement field* $u : [0,T] \longrightarrow V$, *an electric potential field* $\varphi : [0,T] \to W$ *and a bonding field* $\beta : [0,T] \to L^2(\Gamma_3)$ *such that*

(3.22)
$$\begin{aligned} u(t) \in U_{ad},\ (\mathcal{A}\varepsilon(\dot u(t)),\varepsilon(v)-\varepsilon(u(t)))_{\mathcal{H}} &+ (\mathcal{F}\varepsilon(u(t)),\varepsilon(v)-\varepsilon(u(t)))_{\mathcal{H}} + \\ (\mathcal{E}^*\nabla\varphi(t),\varepsilon(v)-\varepsilon(u(t)))_{\mathcal{H}} &+ j(\beta(t),u(t),v-u(t)) \\ \ge (f(t),v-u(t))_V \qquad &\forall v \in U_{ad}\ ,\ \text{a.e. } t \in (0\ T), \end{aligned}$$

(3.23)
$$\begin{aligned} (\mathcal{B}\nabla\varphi(t),\nabla\psi)_{L^2(\Omega)^d} - (\mathcal{E}\varepsilon(u(t)),\nabla\psi)_{L^2(\Omega)^d} &= (q(t),\psi)_W \\ \forall\psi \in W,\ \ \forall\, t \in [0\ T]\,, \end{aligned}$$

(3.24) $\dot\beta(t) = -(\gamma_\nu \beta(t) R_\nu(u_\nu(t))^2 - \varepsilon_a)_+$ .a.e. $t \in (0\ T)$,

(3.25) $u(0) = u_0$,

(3.26) $\beta(0) = \beta_0$.

4. **Existence and Uniqueness Result**

Our main existence and uniqueness result is the following.

Theorem 4.1. *Assume that (3.1)–(3.9) and (3.18) hold. Then, there exists a unique solution* (u, φ, β) *to Problem* $(\mathcal{P}^V)$. *Moreover, the solution satisfies*

(4.1) $\quad \boldsymbol{u} \in W^{1,\infty}(0, T; V),$

(4.2) $\quad \varphi \in W^{1,\infty}(0, T; W).$

(4.3) $\quad \beta \in W^{1,\infty}(0, T; L^2(\Gamma_3)) \cap \mathcal{Q}.$

A "quintuple" of functions (u, σ, φ, D, β) which satisfies (2.1), (2.2) and (3.22)–(3.26) is called a *weak solution* of the contact Problem $(\mathcal{P})$. We conclude by Theorem 4.1 that, under the assumptions (3.1)–(3.9) and (3.18), there exists a unique weak solution of Problem $(\mathcal{P})$.

To precise the regularity of the weak solution we note that the constitutive relations (2.1) and (2.2), the assumptions (3.1)–(3.4) and the regularities (4.1), (4.2) imply that $\sigma \in L^\infty(0, T; \mathcal{H})$, $\boldsymbol{D} \in W^{1,\infty}(0, T; L^2(\Omega)^d)$. By taking $\upsilon = u(t) \pm \xi$, where $\xi \in C_0^\infty(\Omega)^d$, in (3.20) and $\psi \in C_0^\infty(\Omega)$ in (3.21) and using the notation (3.14), (3.15), (3.19) we find

$$Div\ \sigma(t) + f_0(t) = 0, \ \ div\ D(t) = q_0(t),$$

for all $t \in [0, T]$. It follows now from the regularities (3.6), (3.7) that Div $\sigma \in L^\infty(0, T; L^2(\Omega)^d)$ and div $D \in W^{1,\infty}(0, T; L^2(\Omega))$, which shows that

$$\sigma \in L^\infty(0, T; \mathcal{H}_1), \tag{4.4}$$

$$D \in W^{1,\infty}(0, T; \mathcal{W}). \tag{4.5}$$

We conclude that the weak solution $(u, \sigma, \varphi, D, \beta)$ of the piezoelectric contact problem $(\mathcal{P})$ has the regularity (4.1)–(4.5).

The proof of Theorem 4.1 will be carried out in several steps and is based on the following abstract result.

Let X be a real *Hilbert* space with the inner product $(\cdot, \cdot)_X$ and the associated norm $\| \cdot \|_X$, and let $A : D(A) \subset X \longrightarrow 2^X$ be a multivalued operator, where $D(A)$ is the domain of A given by

$$D(A) = \{x \in X : Ax \neq \varnothing\},$$

and 2^X represents the set of the subsets of X. The graph of A denoted by $Gr(A)$ is given by

$$Gr(A) = \{(x, y) \in X \times X : y \in Ax\}.$$

The operator $A : X \longrightarrow 2^X$ is called

(i) monotone if
$\forall(x_1, y_1) \in Gr(A), \ \forall(x_2, y_2) \in Gr(A) : (y_1 - y_2, x_1 - x_2)_X \geq 0.$

(ii) maximal monotone if A is monotone and there is no monotone operator $B : X \longrightarrow 2^X$such that $Gr(A)$ is a proper subset of $Gr(B)$, which is equivalent to the following implication
$[(y_1 - y_2, x_1 - x_2)_X \geq 0, \ \forall(x_1, y_1) \in Gr(A)] \Rightarrow (x_2, y_2) \in Gr(A).$

For a function $\phi : X \longrightarrow]-\infty, +\infty]$ we use the notation $D(\phi)$ and $\partial\phi$ for the effective domain and the subdifferential of ϕ, i.e.

$$D(\phi) = \{u \in X : \phi(u) < \infty\}, \tag{4.6}$$

(4.7) $\partial\phi(u) = \{f \in X : \phi(v) - \phi(u) \geq (f, v-u)_X \quad \forall v \in X\}, \quad \forall u \in X.$

Finally, let $\phi_K : X \to]-\infty, +\infty]$ denote the indicator function of the set K, i.e.

$$\phi_K(v) = \begin{cases} 0 & \text{if } v \in K, \\ \infty & \text{if } v \notin K. \end{cases}$$

It can be shown that the subdifferential of the indicator function $\partial\phi_K : X \longrightarrow 2^X$ of a closed convex K of the space X is a maximal monotone operator. We can also show that the sum of a maximal monotone operator and a single-valued monotone *Lipschitz* continuous operator is a maximal monotone operator.

Finally, we use the usual notation for the *Lebesgue* spaces $L^p(0,T;X)$ and *Sobolev* spaces $W^{k,p}(0,T;X)$ where $1 \leq p \leq \infty$ and $k \in \mathbb{N}$. We will need the following result for existence and uniqueness proofs.

Theorem 4.2. *Let X be a real Hilbert space and let $A : D(A) \subset X \longrightarrow 2^X$ be a multivalued operator such that the operator $A + \omega I_X$ is maximal monotone for some real ω. Then, for every $f \in W^{1,1}(0,T;X)$ and $u_0 \in D(A)$, there exists a unique function $\boldsymbol{u} \in W^{1,\infty}(0,T;X)$ which satisfies*

(4.8) $$\dot{u}(t) + Au(t) \ni f(t) \text{ a.e. } t \in (0\ T),$$

(4.9) $$u(0) = u_0.$$

A proof of Theorem 4.2 may be found in ([6], page 32). Here and below I_X is the identity map on X.

5. Proof of Theorem 4.1

We assume in the following that the conditions of Theorem 4.1 hold and below we denote by c a generic positive constant which is independent of time and whose value may change from place to place.

By the *Riesz* representation theorem we can define the following operators $\mathcal{G} : W \longrightarrow W$ and $\mathcal{R} : V \longrightarrow W$, respectively, by

(5.1) $$(\mathcal{G}\varphi, \psi)_W = (\mathcal{B}\nabla\varphi, \nabla\psi)_{L^2(\Omega)^d} \qquad \forall \varphi, \psi \in W,$$

(5.2) $$(\mathcal{R}v, \varphi)_W = (\mathcal{E}\varepsilon(v), \nabla\varphi)_{L^2(\Omega)^d} \qquad \forall \varphi \in W,\ v \in V.$$

We can show that $\mathcal{G}$ is a linearly continuous symmetric positive definite operator. Therefore, $\mathcal{G}$ is an invertible operator on W. We can also prove that $\mathcal{R}$ is a linear continuous operator on V. Let $\mathcal{R}^*$ the adjoint of $\mathcal{R}$. Thus, from (3.10) we can write

(5.3) $$(\mathcal{R}^*\varphi, v)_V = (\mathcal{E}^*\nabla\varphi, \varepsilon(v))_{\mathcal{H}} \qquad \forall \varphi \in W,\ v \in V.$$

Let $t \in [0\ T]$. By introducing (5.1), (5.2) in (3.23) we get

(5.4) $$(\mathcal{G}\varphi(t), \psi)_W = (\mathcal{R}u(t), \psi)_W + (q(t), \psi)_W \quad \forall \psi \in W,$$

where we obtain

$$\mathcal{G}\varphi(t) = \mathcal{R}u(t) + q(t),$$

for all $t \in [0, T]$. On the other hand, $\mathcal{G}$ is invertible where the previous equality gives us

(5.5) $$\varphi(t) = \mathcal{G}^{-1}\mathcal{R}u(t) + \mathcal{G}^{-1}q(t).$$

Now, using (5.3), (5.5) and (3.22) we obtain

(5.6)
$$\begin{aligned}&u(t)\in U_{ad},\ (\mathcal{A}\varepsilon(\dot u(t)),\varepsilon(v)-\varepsilon(u(t)))_{\mathcal{H}}+(\mathcal{F}\varepsilon(u(t)),\varepsilon(v)-\varepsilon(u(t)))_{\mathcal{H}}+\\&(\mathcal{R}^*\mathcal{G}^{-1}\mathcal{R}u(t),v-u(t))_V+j(\beta(t),u(t),v-u(t))_V\\&\geq(f(t)-\mathcal{R}^*\mathcal{G}^{-1}q(t),v-u(t))_V\qquad\forall v\in U_{ad}\quad\text{a.e. } t\in(0\ T).\end{aligned}$$

Let $\eta\in W^{1,\infty}(0,T;V)$ be given. In the first step we prove the following existence and uniqueness result for the displacement field.

Lemma 5.1. *There exists a unique function $u_\eta\in W^{1,\infty}(0,T;V)$ such that*

(5.7)
$$\begin{aligned}&u_\eta(t)\in U_{ad},\ (\mathcal{A}\varepsilon(\dot u_\eta(t)),\varepsilon(v)-\varepsilon(u_\eta(t)))_{\mathcal{H}}+(\mathcal{F}\varepsilon(u_\eta(t)),\varepsilon(v)-\varepsilon(u_\eta(t)))_{\mathcal{H}}+\\&(\mathcal{R}^*\mathcal{G}^{-1}\mathcal{R}u_\eta(t),v-u_\eta(t))_V+(\eta(t),v-u_\eta(t))_V\\&\geq(f(t)-\mathcal{R}^*\mathcal{G}^{-1}q(t),v-u_\eta(t))_V\qquad\forall v\in U_{ad}\quad a.e.\ t\in(0\ T),\end{aligned}$$

(5.8) $u_\eta(0)=u_0.$

Proof. Let now the operator $L:V\to V$ defined by

(5.9) $L(v)=\mathcal{R}^*\mathcal{G}^{-1}\mathcal{R}(v),\ \forall v\in V.$

Using the properties of the operators $\mathcal{G}$, $\mathcal{R}$ and $\mathcal{R}^*$ we deduce that L is a continuous linear operator on V. Thus we have

(5.10) $\|Lu_1-Lu_2\|_V\leq\|L\|\|u_1-u_2\|_V\quad\forall u_1,\ u_2\in V.$

By the *Riesz* representation theorem we can define an operator $\mathbf{G}:V\longrightarrow V$ by

(5.11) $(\mathbf{G}u,v)_V=(\mathcal{F}\varepsilon(u),\varepsilon(v))_{\mathcal{H}}+(Lu,v)_V\quad\forall u,\ v\in V.$

Now, taking into account (3.1), (3.2), (3.11) and (5.11) it follows

(5.12) $\|\mathbf{G}u_1-\mathbf{G}u_2\|_V\leq(\frac{L_{\mathcal{F}}}{m_{\mathcal{A}}}+\|L\|)\|u_1-u_2\|_V\qquad\forall u_1,\ u_2\in V,$

that is, $\mathbf{G}$ is a *Lipschitz* continuous operator. Moreover, the operator

$$\mathbf{G}+(\frac{L_{\mathcal{F}}}{m_{\mathcal{A}}}+\|L\|)I_V:V\to V,$$

is a monotone *Lipschitz* continuous operator on V.

Let the function $\mathbf{f}:[0\ T]\longrightarrow V$ given by

(5.13) $\mathbf{f}(t)=f(t)-\mathcal{R}^*\mathcal{G}^{-1}q(t)-\eta(t),\ \ \forall t\in[0\ T]\,.$

Keeping in mind that $\eta\in W^{1,\infty}(0,T;V)$, using (3.16), (3.17) and the fact that $\mathcal{R}^*\mathcal{G}^{-1}$ is linearly continuous, it follows from (5.13) that

(5.14) $\mathbf{f}\in W^{1,1}(0,T;V).$

Let $\phi_{U_{ad}}:V\to\,]-\infty,+\infty]$ denote the indicator function of the set U_{ad} and let $\partial\phi_{U_{ad}}$ be the subdifferential of $\phi_{U_{ad}}$. Since U_{ad} is a nonempty, convex, closed part of V, it follows that $\partial\phi_{U_{ad}}$ is a maximal monotone operator on V and $D(\partial\phi_{U_{ad}})=U_{ad}$. Moreover, the sum

$$\partial\phi_{U_{ad}}+\mathbf{G}+(\frac{L_{\mathcal{F}}}{m_{\mathcal{A}}}+\|L\|)I_V:U_{ad}\subset V\to 2^V,$$

is a maximal monotone operator. Thus, conditions (3.18) and (5.14) allow us to apply Theorem 4.2 with $X = V$, $A = \partial\phi_{U_{ad}} + \mathbf{G} : D(A) = U_{ad} \subset V \to 2^V$, and $\omega = \frac{L_{\mathcal{F}}}{m_{\mathcal{A}}} + \|L\|$. We deduce that there exists a unique element $u_\eta \in W^{1,\infty}(0,T;V)$ such that

$$\dot{u}_\eta(t) + \partial\phi_{U_{ad}}(u_\eta(t)) + \mathbf{G}u_\eta(t) \ni \mathbf{f}(t) \qquad \text{a.e. } t \in (0\ T), \tag{5.15}$$

$$u_\eta(0) = u_0. \tag{5.16}$$

Since for any elements $u,\ g \in V$, the following equivalence holds

$$g \in \partial\phi_{U_{ad}}(u) \Leftrightarrow u \in U_{ad}, \quad (g, v-u)_V \leq 0 \quad \forall\ v \in U_{ad},$$

the differential inclusion (5.15) is equivalent to the following variational inequality

$$\begin{aligned} u_\eta(t) \in U_{ad}, \quad & (\dot{u}_\eta(t), v - u_\eta(t))_V + (\mathbf{G}u_\eta(t), v - u_\eta(t))_V \\ & \geq (\mathbf{f}(t), v - u_\eta(t))_V \quad \forall v \in U_{ad} \quad \text{a.e. } t \in (0\ T). \end{aligned} \tag{5.17}$$

We use now (5.17), (5.11), (3.11) to see that u_η satisfies the following inequality

$$\begin{aligned} & u_\eta(t) \in U_{ad},\ (\mathcal{A}\varepsilon(\dot{u}_\eta(t)), \varepsilon(v) - \varepsilon(u_\eta(t)))_{\mathcal{H}} + (\mathcal{F}\varepsilon(u_\eta(t)), \varepsilon(v) - \varepsilon(u_\eta(t)))_{\mathcal{H}} + \\ & (Lu_\eta(t), v - u_\eta(t))_V \geq (\mathbf{f}(t), v - u_\eta(t))_V \ \ \forall v \in U_{ad} \quad \text{a.e. } t \in (0\ T). \end{aligned} \tag{5.18}$$

It follows now from (5.18), (5.13), (5.9) and (5.16) that u_η satisfies (5.7) and (5.8), which concludes the proof of Lemma 5.1. □

In the second step we use the displacement field u_η obtained in Lemma 5.1 to obtain the following existence and uniqueness result for the electric potential field.

Lemma 5.2. *There exists a unique function $\varphi_\eta \in W^{1,\infty}(0,T;W)$ such that*

$$\begin{aligned} & (\mathcal{B}\nabla\varphi_\eta(t), \nabla\psi)_{L^2(\Omega)^d} - (\mathcal{E}\varepsilon(u_\eta(t)), \nabla\psi)_{L^2(\Omega)^d} = (q(t), \psi)_W \\ & \qquad \forall \psi \in W, \quad \forall\ t \in [0\ T], \end{aligned} \tag{5.19}$$

Proof. Let $u_\eta \in W^{1,\infty}(0,T;V)$ be the function defined in Lemma 5.1. Clearly, equality (5.19) holds from (5.4), (5.2) and (5.1). Moreover, since $u_\eta \in W^{1,\infty}(0,T;V)$ it follows from (5.5), (3.17) that $\varphi_\eta \in W^{1,\infty}(0,T;W)$. Now, using (5.5) we deduce that the uniqueness of φ_η follows from the uniqueness of the function u_η. □

In the third step, we use again the displacement field u_η obtained in Lemma 5.1 and we consider the following initial value problem.

Problem $\mathcal{P}^{\beta_\eta}$. *Find a bonding field $\beta_\eta : [0,T] \to L^2(\Gamma_3)$ such that*

$$\dot{\beta}_\eta(t) = -(\gamma_\nu \beta_\eta(t) R_\nu(u_{\eta\nu}(t))^2 - \varepsilon_a)_+ \text{ a.e. } t \in (0\ T), \tag{5.20}$$

$$\beta_\eta(0) = \beta_0. \tag{5.21}$$

We obtain the following result.

Lemma 5.3. *There exists a unique solution β_η to Problem $\mathcal{P}^{\beta_\eta}$ and it satisfies $\beta_\eta \in W^{1,\infty}(0,T,L^2(\Gamma_3)) \cap \mathcal{Q}$.*

Proof. Consider the mapping $F : [0,T] \times L^2(\Gamma_3) \to L^2(\Gamma_3)$ defined by

$$F(t,\beta_\eta) = -(\gamma_\nu \beta_\eta(t) R_\nu(u_{\eta\nu}(t))^2 - \varepsilon_a)_+, \tag{5.22}$$

for all $t \in [0,T]$ and $\beta_\eta \in L^2(\Gamma_3)$. It follows from the properties of the truncation operator R_ν that F is *Lipschitz* continuous with respect to the second argument, uniformly in time. Moreover, for any $\beta_\eta \in L^2(\Gamma_3)$, the mapping $t \mapsto F(t,\beta_\eta)$ belongs to $L^\infty(0,T;L^2(\Gamma_3))$. Using now a version of *Cauchy-Lipschitz* theorem (see, e.g., [27], page 48), we obtain the existence of a unique function $\beta_\eta \in W^{1,\infty}(0,T,L^2(\Gamma_3))$ which solves (5.20), (5.21). We note that the restriction $0 \le \beta_\eta \le 1$ is implicitly included in the *Cauchy* problem $\mathcal{P}^{\beta_\eta}$. Indeed, (5.20) and (5.21) guarantee that $\beta_\eta(t) \le \beta_0$ and, therefore, assumption (3.9) shows that $\beta_\eta(t) \le 1$ for $t \ge 0$, a.e. on Γ_3. On the other hand, if $\beta_\eta(t_0) = 0$ at $t = t_0$, then it follows from (5.20) and (5.21) that $\dot\beta_\eta(t) = 0$ for all $t \ge t_0$ and therefore, $\beta_\eta(t) = 0$ for all $t \ge t_0$, a.e. on Γ_3. We conclude that $0 \le \beta_\eta(t) \le 1$ for all $t \in [0,T]$, a.e. on Γ_3. Therefore, from the definition of the set $\mathcal{Q}$, we find that $\beta_\eta \in \mathcal{Q}$, which concludes the proof of Lemma 5.3. □

Now, for $\eta \in W^{1,\infty}(0,T;V)$ we denote by u_η and β_η the functions obtained in Lemmata 5.1 and 5.3, respectively. We use *Riesz*'s representation theorem to define the function $\Lambda\eta : [0,T] \longrightarrow V$ by

$$(\Lambda\eta(t), \upsilon)_V = j(\beta_\eta(t), u_\eta(t), \upsilon), \tag{5.23}$$

for all $\upsilon \in V$ and $t \in [0,T]$. We have the following result.

Lemma 5.4. *For all $\eta \in W^{1,\infty}(0,T;V)$ the function $\Lambda\eta$ belongs to $W^{1,\infty}(0,T;V)$. Moreover, there exists a unique element $\eta^* \in W^{1,\infty}(0,T;V)$ such that*

$$\Lambda\eta^* = \eta^*. \tag{5.24}$$

Proof. Let $\eta \in W^{1,\infty}(0,T;V)$ and let $t_1, t_2 \in [0,T]$. Using (5.23) and (3.19), we obtain

$$\begin{aligned}\| \Lambda\eta(t_1) - \Lambda\eta(t_2) \|_V \le c \| \beta_\eta^2(t_1) R_\nu(u_{\eta\nu}(t_1)) - \beta_\eta^2(t_2) R_\nu(u_{\eta\nu}(t_2)) \|_{L^2(\Gamma_3)} +\\ c \| p_\tau(\beta_\eta(t_1)) R_\tau(u_{\eta\tau}(t_1)) - p_\tau(\beta_\eta(t_2)) R_\tau(u_{\eta\tau}(t_2)) \|_{L^2(\Gamma_3)} .\end{aligned}$$

Now, keeping in mind (3.5), (3.13), the inequality $0 \le \beta_\eta(t) \le 1$ and the properties of the truncation operators R_ν and R_τ, we find that

$$\begin{aligned}\| \Lambda\eta(t_1) - \Lambda\eta(t_2) \|_V \le c \| u_\eta(t_1) - u_\eta(t_2) \|_V +\\ c \| \beta_\eta(t_1) - \beta_\eta(t_2) \|_{L^2(\Gamma_3)} .\end{aligned} \tag{5.25}$$

Since $u_\eta \in W^{1,\infty}(0,T;V)$ and $\beta_\eta \in W^{1,\infty}(0,T,L^2(\Gamma_3)) \cap \mathcal{Q}$, we deduce from inequality (5.25) that $\Lambda\eta \in W^{1,\infty}(0,T;V)$.

Let now $\eta_1, \eta_2 \in W^{1,\infty}(0,T;V)$ and let $u_i = u_{\eta_i}$, $\dot u_i = \dot u_{\eta_i}$, $\beta_i = \beta_{\eta_i}$ for $i = 1,2$. For $t \in [0,T]$ we integrate (5.20) with the initial conditions (5.21) to obtain

$$\beta_i(t) = \beta_0 - \int_0^t (\gamma_\nu \beta_i(s) R_\nu(u_{i\nu}(s))^2 - \varepsilon_a)_+ ds.$$

Using the definition of R_ν, the inequality $\mid R_\nu(u_\nu) \mid\leq L$, and writing $\beta_1 = \beta_1 - \beta_2 + \beta_2$, we get

$$\| \beta_1(t) - \beta_2(t) \|_{L^2(\Gamma_3)}\leq c\int_0^t \| \beta_1(s) - \beta_2(s) \|_{L^2(\Gamma_3)}\, ds + c\int_0^t \| u_{1\nu}(s) - u_{2\nu}(s) \|_{L^2(\Gamma_3)}\, ds.$$

By *Gronwall*'s inequality, it follows that

$$\| \beta_1(t) - \beta_2(t) \|_{L^2(\Gamma_3)}\leq c\int_0^t \| u_{1\nu}(s) - u_{2\nu}(s) \|_{L^2(\Gamma_3)}\, ds,$$

and, using (3.13) we obtain

$$\| \beta_1(t) - \beta_2(t) \|_{L^2(\Gamma_3)}\leq c\int_0^t \| u_1(s) - u_2(s) \|_V\, ds. \tag{5.26}$$

On the other hand, using arguments similar to those in the proof of (5.25), we find that

$$\| \Lambda\eta_1(t) - \Lambda\eta_2(t) \|_V\leq c \| u_1(t) - u_2(t) \|_V + c \| \beta_1(t) - \beta_2(t) \|_{L^2(\Gamma_3)}$$

Then, by (5.26) we have

$$\| \Lambda\eta_1(t)-\Lambda\eta_2(t) \|_V\leq c \| u_1(t)-u_2(t) \|_V + c\int_0^t \| u_1(s)-u_2(s) \|_V\, ds. \tag{5.27}$$

Next, we use (5.17) and (5.13) to find that

$$(\dot u_1(t) - \dot u_2(t), u_1(t) - u_2(t))_V \leq (\eta_2(t) - \eta_1(t), u_1(t) - u_2(t))_V + (\mathbf{G}u_2(t) - \mathbf{G}u_1(t), u_1(t) - u_2(t))_V,$$

using *Cauchy–Schwarz* inequality and (5.12) we obtain

$$(\dot u_1(t) - \dot u_2(t), u_1(t) - u_2(t))_V \leq \|\eta_1(t) - \eta_2(t)\|_V\|u_1(t) - u_2(t)\|_V + (\tfrac{L_{\mathcal{F}}}{m_{\mathcal{A}}} + \|L\|)\|u_1(t) - u_2(t)\|_V^2.$$

We integrate this inequality with respect to time and use the initial conditions $u_1\,(0) = u_2\,(0) = u_0$ to find that

$$\tfrac{1}{2} \| u_1(t) - u_2(t) \|_V^2\leq \int_0^t \|\eta_1(s) - \eta_2(s)\|_V\|u_1(s) - u_2(s)\|_V ds + (\tfrac{L_{\mathcal{F}}}{m_{\mathcal{A}}} + \|L\|)\int_0^t \|u_1(s) - u_2(s)\|_V^2 ds.$$

Applying the inequality

$$ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2 \quad a, b \in \mathbb{R},$$

we find that

$$\tfrac{1}{2} \| u_1(t) - u_2(t) \|_V^2\leq \tfrac{1}{2}\int_0^t \|\eta_1(s) - \eta_2(s)\|_V^2 ds + \tfrac{1}{2}\int_0^t \|u_1(s) - u_2(s)\|_V^2 ds + (\tfrac{L_{\mathcal{F}}}{m_{\mathcal{A}}} + \|L\|)\int_0^t \|u_1(s) - u_2(s)\|_V^2 ds,$$

where we obtain

$$\| u_1(t) - u_2(t) \|_V^2 \leq c \int_0^t \|\eta_1(s) - \eta_2(s)\|_V^2 ds + c \int_0^t \|u_1(s) - u_2(s)\|_V^2 ds,$$

and, after a *Gronwall* argument, we obtain

$$\| u_1(t) - u_2(t) \|_V^2 \leq c \int_0^t \|\eta_1(s) - \eta_2(s)\|_V^2 ds. \tag{5.28}$$

Using (5.27) we find that

$$\begin{aligned}
\| \Lambda\eta_1(t) - \Lambda\eta_2(t) \|_V^2 \leq\ & c^2 \| u_1(t) - u_2(t) \|_V^2 \ + \ c^2(\int_0^t \| u_1(s) - u_2(s) \|_V \, ds)^2 + \\
& 2c \| u_1(t) - u_2(t) \|_V \cdot c \int_0^t \| u_1(s) - u_2(s) \|_V \, ds, \\
\| \Lambda\eta_1(t) - \Lambda\eta_2(t) \|_V^2 \leq\ & c^2 \| u_1(t) - u_2(t) \|_V^2 + c^2(\int_0^t \| u_1(s) - u_2(s) \|_V \, ds)^2 + \\
& c^2 \| u_1(t) - u_2(t) \|_V^2 + c^2(\int_0^t \| u_1(s) - u_2(s) \|_V \, ds)^2, \\
\| \Lambda\eta_1(t) - \Lambda\eta_2(t) \|_V^2 \leq\ & 2c^2 \| u_1(t) - u_2(t) \|_V^2 \ + 2\, c^2(\int_0^t \| u_1(s) - u_2(s) \|_V \, ds)^2.
\end{aligned}$$

Using *Cauchy–Schwarz* inequality we find

$$\| \Lambda\eta_1(t) - \Lambda\eta_2(t) \|_V^2 \leq c \| u_1(t) - u_2(t) \|_V^2 + c \int_0^t \| u_1(s) - u_2(s) \|_V^2 \, ds \tag{5.29}$$

We combine now (5.28) and (5.29) to see that

$$\| \Lambda\eta_1(t) - \Lambda\eta_2(t) \|_V^2 \leq c \int_0^t \|\eta_1(s) - \eta_2(s)\|_V^2 ds \qquad \forall\, t \in [0\ T]\,. \tag{5.30}$$

Reiterating this inequality n times yields

$$\| \Lambda^n\eta_1(t) - \Lambda^n\eta_2(t) \|_{L^\infty(0,T;V)}^2 \leq \frac{c^n T^n}{n!} \|\eta_1(t) - \eta_2(t)\|_{L^\infty(0,T;V)}^2. \tag{5.31}$$

which implies that, for n sufficiently large, a power Λ^n of Λ is a contraction in the *Banach* space $L^\infty(0,T;V)$. Then, there exists a unique element $\eta^* \in L^\infty(0,T;V)$ such that $\Lambda^n\eta^* = \eta^*$ and η^* is also the unique fixed point of Λ, i.e $\Lambda\eta^* = \eta^*$. The regularity $\eta^* \in W^{1,\infty}(0,T;V)$ follows from the regularity $\Lambda\eta^* \in W^{1,\infty}(0,T;V)$, which concludes the proof. □

Now, we have all the ingredients necessary to prove Theorem 4.1.

Proof. of Theorem 4.1. **Existence**. Let $\eta^* \in W^{1,\infty}(0,T;V)$ be the fixed point of the operator Λ and let (u,φ,β) be the functions defined in Lemmata 5.1, 5.2 and 5.3, respectively, for $\eta = \eta^*$, i.e $u = u_{\eta^*}$, $\varphi = \varphi_{\eta^*}$, $\beta = \beta_{\eta^*}$. Clearly, equalities (3.23), (3.24) and (3.26) hold from Lemmata 5.2 and 5.3. Moreover, since $\Lambda\eta^* = \eta^*$ it follows from (5.7), (5.5), (5.3), (5.8) and (5.23) that (3.22) and (3.25) hold, too. The regularity of the solution expressed in (4.1), (4.2) and (4.3) follows from Lemmata 5.1, 5.2 and 5.3, respectively.

Uniqueness. The uniqueness of the solution follows from the uniqueness of the fixed point of Λ and the uniqueness part in Lemmata 5.1, 5.2 and 5.3. □

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