Nonsmooth interval-valued optimization and saddle-point optimality criteria

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Abstract: In this article, we focus our attention on a nonsmooth interval-valued optimization problem and establish sufficient optimality conditions for a feasible solution to be a LU optimal solution under the invexity assumption. Appropriate duality theorems for Wolfe and Mond-Weir type duals are presented in order to relate the LU optimal solution of primal and dual programs. Moreover, saddle-point type optimality conditions are established in order to find relation between LU optimal solution of primal and saddle-point of Lagrangian function.

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1 Introduction

Convexity is one of the most frequently used hypotheses in optimization theory. In recent years, many extensions have been considered for classical convexity. Several classes of functions have been defined for the purpose of weakening the limitations of convexity. One of the most useful generalization of convexity was introduced by Hanson [5] for the differentiable functions. For more details Arana *et al.* [1] has been refereed. On the other hand by substituting invexity for convexity, many optimization problems can be solved

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for differentiable functions. But in nonsmooth programming the corresponding results cannot be formulated using the concept of invexity as a derivative term is required in the definition of invexity.

The theory of nonsmooth optimization using locally Lipschitz functions was introduced by Clarke [3]. He extended the properties of convex functions to the case of locally Lipschitz functions by suitably defining a generalized derivative and a subdifferential. Later on, the notion of invexity was extended to locally Lipschitz functions by Craven [4], by replacing the derivative with Clarkes generalized gradient. Kim and Lee [8] presented optimality conditions and duality relations for nonsmooth multiobjective programming problems involving locally Lipschitz functions. Recently, Jiao and Liu [7] introduced some generalized cone-invex functions called K-α-generalized invex, K-α-nonsmooth invex and presented several sufficient optimality conditions and duality results for nonsmooth vector optimization problem under the assumptions of the generalized cone invexity.

In general mathematical programming problems, the coefficients of the problems are always considered as deterministic values. This assumption is not satisfied by great majority of real-life engineering and economical problems. The introduction of imprecision and uncertainty in the modeling process is an important issue of the approaching real practical problems.

Uncertainty can be handled in various manners namely by a stochastic process and fuzzy numbers. However, sometimes it is hard to find an appropriate membership function or probability distribution with insufficiency of data. Interval-valued optimization programming is one of the approaches to tackle the uncertain optimization problem in which only the range of the coefficients are known.

Many solution concepts have been introduced for solving interval-valued programming problems. Urli and Nadeau [11] derived a process to solve the multi-objective linear programming problem with interval co-efficient. In [14], Wu presented a new solution concept in interval-valued optimization problem by imposing a partial ordering on the set of all closed intervals.

Several authors have been interested in deriving sufficient conditions and duality results with differentiable objective and constraint functions in interval-valued programming problem. Wu [12] derived Karush-Kuhn-Tucker optimality conditions. Later, Wu [13] formulated Wolfe type dual problem and presented duality theorems by using the concept of nondominated solution. Zhou *et al.* [18] derived sufficient optimality conditions and formulated mixed type duality under convexity assumption.

Recently, Jayswal *et al.* [6] derived sufficient optimality conditions and duality theorems for interval-valued optimization problems involving generalized convex functions. Bhurjee and Panda [2] proposed a new approach for existence of an efficient solution of an interval optimization problem. Very recently, Zhang *et al.* [17] established the KKT optimality conditions in a class of nonconvex optimization problems with an interval-valued objective function. A very little work has done on nondifferentiable interval-valued programming problem. Sun and Wang [9] first derived the optimality conditions and duality theorems for the nondifferentiable interval-valued programming problem. Very recently, Sun *et al.* [10] presented saddle point optimality conditions and established a relation between optimal solution of the primal and saddle point of the Lagrangian function.

Saddle-point is a fundamental concept which is used in many areas of science and economics. The saddle-point of the Lagrangian is always a global optimum of the problem and they are also equivalent under the convexity assumption and constraint qualification for inequality constrained mathematical programming problem. Zalmai [16] established necessary and sufficient saddle-point-type optimality conditions and Lagrangian-type duality relations for a class of state and control-constrained generalized fractional optimal control problems. Yang *et al.* [15] derived duality theorems and a saddle-point type optimality condition by using theorems of alternative.

In this paper, we consider a nonsmooth optimization problem in which objective function to be considered as interval-valued and constraints as real valued functions. Sufficient optimality conditions of the considered problem are derived for a feasible solution to be a LU optimal solution under the invexity assumption. Weak, strong and strict converse duality theorems for Wolfe and Mond-Weir type duals are also formulated in order to relate the LU optimal solution of primal and dual programs. Furthermore, saddle-point optimality conditions are presented under invexity assumption in order to find a relation between LU optimal solution of primal and saddle-point of Lagrangian function.

The paper is unfolded as follows. Section 2 is devoted to notation and definitions. In Section 3, we derive some sufficient optimality conditions. Weak, strong and strict converse duality theorems for Wolfe and Mond-Weir type duals are proved in Sections 4 and 5. In Section 6, we define saddle-point of Lagrangian functions and discuss saddle-point optimality conditions. Conclusion and further development are presented in Section 7.

2 Notation and preliminaries

We denote by I the set of all closed and bounded intervals in R. Suppose $A \in I$, then we write $A = [a^L, a^U]$, where a^L and a^U means the lower and upper bounds of A, respectively. Throughout this paper our intervals are considered to be bounded and closed. Let $A = [a^L, a^U], B = [b^L, b^U] \in I$, we have

(i) $A + B = \{a + b : a \in A \text{ and } b \in B\} = [a^L + b^L, a^U + b^U]$,

(ii) $-A = \{-a : a \in A\} = [-a^U, -a^L]$,

(iii) $A - B = A + (-B) = [a^L - b^U, a^U - b^L]$,

(iv) $k + A = \{k + a : a \in A\} = [k + a^L, k + a^U]$,

(v) $kA = \{ka : a \in A\} = \begin{cases} [ka^L, ka^U] & \text{if } k \geq 0, \\ [ka^U, ka^L] & \text{if } k < 0, \end{cases}$

where k is a real number.

Let R^n denotes the n-dimensional Euclidean space and X be a non-empty subset of R^n. The function $F : R^n \to I$ is called an interval-valued function. Then $F(x) = F(x_1, x_2, ..., x_n)$ is a closed interval in R for each $x \in R^n$. We can write the interval-valued function F as $F(x) = [F^L(x), F^U(x)]$, where $F^L(x), F^U(x)$ are real valued functions defined on R^n and satisfy the condition $F^L(x) \leq F^U(x)$ for each $x \in R^n$.

If $A = [a^L, a^U]$ and $B = [b^L, b^U]$ are two closed intervals, we write $A \leq_{LU} B$ if and only if $a^L \leq b^L$ and $a^U \leq b^U$. It is easy to see that $\leq_{LU}$ is a partial ordering on I. Also we can write $A <_{LU} B$ if and only if $A \leq_{LU} B$ and $A \neq B$.

Equivalently, $A <_{LU} B$ if and only if

$$\begin{aligned} & a^L < b^L, \qquad a^U < b^U, \\ \text{or,} \quad & a^L \leq b^L, \qquad a^U < b^U, \\ \text{or,} \quad & a^L < b^L, \qquad a^U \leq b^U. \end{aligned}$$

Definition 2.1. *[3] A function $f : X \to R$ is said to be Lipschitz at $x \in X$, if there exist a positive constant K and a neighbourhood N of x such that, for any $y, z \in N$,*

$$|f(y) - f(z)| \leq K \|y - z\|.$$

We say that $f : X \to R$ is locally Lipschitz on X if it is Lipschitz at any point of X.

Definition 2.2. *[3] If $f : X \to R$ is Lipschitz at $x \in X$, the generalized derivative (in the sense of Clarke) of f at $x \in X$ in the direction $v \in R^n$, denoted by $f^0(x; v)$, is given by*

$$f^0(x; v) = \lim_{\substack{y \to x \\ t \downarrow 0}} \sup \left[\frac{f(y + tv) - f(y)}{t} \right].$$

Definition 2.3. *[3] The Clarke's generalized gradient of f at $x \in X$, denoted by $\partial f(x)$, is defined as follows:*

$$\partial f(x) = \{\zeta \in R^n : f^0(x; v) \geq \zeta^T v, \quad \forall\, v \in R^n\},$$

where "T" signifies the transpose of a vector or matrix.

It follows that, for any $v \in R^n$

$$f^0(x; v) = max\{\zeta^T v : \zeta \in \partial f(x)\}.$$

Definition 2.4. *The locally Lipschitz function $f : X \to R$ is said to be invex with respect to $\eta : X \times X \to R^n$ at x^* if for all $x \in X$,*

$$f(x) - f(x^*) \geq \eta(x, x^*)^T \zeta, \quad \forall\, \zeta \in \partial f(x^*).$$

Definition 2.5. *The locally Lipschitz function $f : X \to R$ is said to be strictly-invex with respect to $\eta : X \times X \to R^n$ at x^* if for all $x \in X$,*

$$f(x) - f(x^*) > \eta(x, x^*)^T \zeta, \quad \forall\, \zeta \in \partial f(x^*) \quad and \quad x \neq x^*.$$

Definition 2.6. *The locally Lipschitz function $f : X \to R$ is said to be (strictly) pseudo-invex with respect to $\eta : X \times X \to R^n$ at x^* if for all $x \in X$,*

$$\eta(x, x^*)^T \zeta \geq 0 \Longrightarrow f(x) - f(x^*)(>) \geq 0, \quad \forall\, \zeta \in \partial f(x^*).$$

Definition 2.7. *The locally Lipschitz function $f : X \to R$ is said to be quasi-invex with respect to $\eta : X \times X \to R^n$ at x^* if for all $x \in X$,*

$$f(x) - f(x^*) \leq 0 \Longrightarrow \eta(x, x^*)^T \zeta \leq 0, \quad \forall\, \zeta \in \partial f(x^*).$$

Now, we turn our attention to invexity of locally Lipschitz interval-valued function.

Definition 2.8. *The locally Lipschitz interval-valued function $F : X \to I$ is said to be invex with respect to $\eta : X \times X \to R^n$ at $x^* \in X$ if the functions F^L and F^U both are invex with respect to same η at x^*.*

Definition 2.9. *The locally Lipschitz interval-valued function* $F : X \to I$ *is said to be strictly-invex with respect to* $\eta : X \times X \to R^n$ *at* $x^* \in X$ *if the functions* F^L *and* F^U *both are strictly-invex or at least one of* F^L *or* F^U *is strictly-invex with respect to same* η *at* x^*.

In this paper, we consider the following nonsmooth optimization problem with interval-valued objective function:

(IVP) $\quad \min \ F(x) = [F^L(x), F^U(x)]$

subject to

$$\Lambda = \{x \in X : g_j(x) \leq 0, j = 1, 2, ..., m\},$$

where $F : X \to I$ is an interval-valued function, $F^L(x), F^U(x)$ and $g_j : X \to R,\ j = 1, 2, ..., m$ are locally Lipschitz on X.

Definition 2.10. *[9] A point* $x^* \in \Lambda$ *is said to be a LU optimal solution to (IVP), if there exists no* $x_0 \in \Lambda$ *such that* $F(x_0) <_{LU} F(x^*)$.

In [9], Sun and Wang established the Karush-Kuhn-Tucker type conditions for nonsmooth interval programming problem. Motivated by [9], we restate them as the following theorem.

Theorem 2.1. *(Karush-Kuhn-Tucker type conditions). Assume that* x^* *is a LU optimal solution to (IVP) and the suitable constraint qualification is satisfied at* x^*. *Then there exist scalars* $0 < \xi^L,\ \xi^U \in R$ *and* $0 \leq \mu_j \in R,\ j = 1, 2, ..., m$ *such that*

$$0 \in \xi^L \partial F^L(x^*) + \xi^U \partial F^U(x^*) + \sum_{j=1}^{m} \mu_j \partial g_j(x^*), \tag{1}$$

$$\mu_j g_j(x^*) = 0,\ j = 1, 2, ..., m. \tag{2}$$

3 Sufficient optimality conditions

In this section, we shall establish the following sufficient optimality conditions for (IVP).

Theorem 3.1. *(Sufficiency). Let* $x^* \in \Lambda$ *be a feasible solution of (IVP). Assume that there exist scalars* $0 < \xi^L,\ \xi^U \in R$ *and* $0 \leq \mu_j \in R,\ j = 1, 2, ..., m$ *such that*

(i) $0 \in \xi^L \partial F^L(x^*) + \xi^U \partial F^U(x^*) + \sum_{j=1}^{m} \mu_j \partial g_j(x^*)$,

(ii) $\mu_j g_j(x^*) = 0,\ j = 1, 2, ..., m$,

(iii) F and $\sum_{j=1}^{m} \mu_j g_j$ are invex with respect to same η at x^*.
Then x^* is a LU optimal solution to (IVP).

Proof. By hypothesis (i) it is clear that there exist some $v^L \in \partial F^L(x^*)$, $v^U \in \partial F^U(x^*)$ and $w_j \in \partial g_j(x^*)$, for each $j = 1, 2, ..., m$ such that

$$\xi^L v^L + \xi^U v^U + \sum_{j=1}^{m} \mu_j w_j = 0. \tag{3}$$

Suppose, contrary to the result that x^* is not a LU optimal solution to (IVP), then there exists a feasible solution x_0, such that

$$F(x_0) <_{LU} F(x^*).$$

That is,

$$\begin{cases} F^L(x_0) < F^L(x^*) \\ F^U(x_0) < F^U(x^*) \end{cases}, \text{ or } \begin{cases} F^L(x_0) \leq F^L(x^*) \\ F^U(x_0) < F^U(x^*) \end{cases}, \text{ or } \begin{cases} F^L(x_0) < F^L(x^*) \\ F^U(x_0) \leq F^U(x^*) \end{cases}.$$

Since $\xi^L > 0,\ \xi^U > 0$, we can write the above inequalities as

$$\xi^L F^L(x_0) + \xi^U F^U(x_0) < \xi^L F^L(x^*) + \xi^U F^U(x^*). \tag{4}$$

From the assumption that F is invex with respect to η at x^*, we have

$$F^L(x_0) - F^L(x^*) \geq \eta(x_0, x^*)^T v^L, \quad \forall\ v^L \in \partial F^L(x^*),$$

$$F^U(x_0) - F^U(x^*) \geq \eta(x_0, x^*)^T v^U, \quad \forall\ v^U \in \partial F^U(x^*).$$

The above inequalities together with the positivity of ξ^L and ξ^U, gives

$$\xi^L F^L(x_0) - \xi^L F^L(x^*) \geq \eta(x_0, x^*)^T \xi^L v^L, \quad \forall\ v^L \in \partial F^L(x^*),$$

$$\xi^U F^U(x_0) - \xi^U F^U(x^*) \geq \eta(x_0, x^*)^T \xi^U v^U, \quad \forall\ v^U \in \partial F^U(x^*).$$

Combining the above two inequalities, we get

$$\begin{aligned}(\xi^L F^L(x_0) + \xi^U F^U(x_0)) - (\xi^L F^L(x^*) + \xi^U F^U(x^*)) \\ \geq \eta(x_0, x^*)^T (\xi^L v^L + \xi^U v^U),\ \ \forall\ v^L \in \partial F^L(x^*),\ \forall\ v^U \in \partial F^U(x^*),\end{aligned}$$

which in view of (4), yields

$$\eta(x_0, x^*)^T (\xi^L v^L + \xi^U v^U) < 0,\ \ \forall\ v^L \in \partial F^L(x^*),\ \forall\ v^U \in \partial F^U(x^*). \tag{5}$$

On the other hand, since $\mu_j \geq 0,\ j = 1, 2, ..., m$ from the feasibility of x_0 to (IVP) and hypothesis (ii), we have

$$\sum_{j=1}^{m} \mu_j g_j(x_0) \leq \sum_{j=1}^{m} \mu_j g_j(x^*). \tag{6}$$

From the assumption that $\sum\limits_{j=1}^{m} \mu_j g_j$ is invex with respect to η at x^*, we have

$$\sum_{j=1}^{m} \mu_j g_j(x_0) - \sum_{j=1}^{m} \mu_j g_j(x^*) \geq \eta(x_0, x^*)^T \sum_{j=1}^{m} \mu_j w_j, \quad \forall\ w_j \in \partial g_j(x^*),\ j = 1, 2, ..., m,$$

which together with the inequality (6), yields

$$\eta(x_0, x^*)^T \sum_{j=1}^{m} \mu_j w_j \leq 0. \tag{7}$$

On adding (5) and (7), we get

$$\eta(x_0, x^*)^T (\xi^L v^L + \xi^U v^U + \sum_{j=1}^{m} \mu_j w_j) < 0,$$

which contradicts (3). Therefore x^* is a LU optimal solution to (IVP). This completes the proof. □

Theorem 3.2. *(Sufficiency). Let $x^* \in \Lambda$ be a feasible solution of (IVP). Assume that there exist scalars $0 < \xi^L,\ \xi^U \in R$ and $0 \leq \mu_j \in R,\ j = 1, 2, ..., m$ such that*
(i) $0 \in \xi^L \partial F^L(x^*) + \xi^U \partial F^U(x^*) + \sum\limits_{j=1}^{m} \mu_j \partial g_j(x^*)$,
(ii) $\mu_j g_j(x^*) = 0,\ j = 1, 2, ..., m$,
(iii) $\xi^L F^L + \xi^U F^U$ is pseudo-invex and $\sum\limits_{j=1}^{m} \mu_j g_j$ is quasi-invex with respect to same η at x^*.
Then x^* is a LU optimal solution to (IVP).

Proof. By hypothesis (i) it is clear that there exist some $v^L \in \partial F^L(x^*),\ v^U \in \partial F^U(x^*)$ and $w_j \in \partial g_j(x^*)$, for each $j = 1, 2, ..., m$ such that

$$\xi^L v^L + \xi^U v^U + \sum_{j=1}^{m} \mu_j w_j = 0. \tag{8}$$

Suppose, contrary to the result that x^* is not a LU optimal solution to (IVP). Then there exists a feasible solution $x_0 \in \Lambda$, such that

$$F(x_0) <_{LU} F(x^*).$$

That is,

$$\begin{cases} F^L(x_0) < F^L(x^*) \\ F^U(x_0) < F^U(x^*) \end{cases}, \text{ or } \begin{cases} F^L(x_0) \leq F^L(x^*) \\ F^U(x_0) < F^U(x^*) \end{cases}, \text{ or } \begin{cases} F^L(x_0) < F^L(x^*) \\ F^U(x_0) \leq F^U(x^*) \end{cases}.$$

Since $\xi^L > 0,\ \xi^U > 0$, we can write the above inequalities as

$$\xi^L F^L(x_0) + \xi^U F^U(x_0) < \xi^L F^L(x^*) + \xi^U F^U(x^*),$$

which together the pseudo-invexity of $\xi^L F^L + \xi^U F^U$ with respect to η at x^*, gives

$$\eta(x_0, x^*)^T(\xi^L v^L + \xi^U v^U) < 0, \quad \forall\ v^L \in \partial F^L(x^*), \quad \forall\ v^U \in \partial F^U(x^*). \tag{9}$$

On the other hand, since $\mu_j \geq 0,\ j = 1, 2, ..., m$ from the feasibility of x_0 to (IVP) and hypothesis (ii), we have

$$\sum_{j=1}^{m} \mu_j g_j(x_0) \leq \sum_{j=1}^{m} \mu_j g_j(x^*).$$

The above inequality together with the assumption that $\sum\limits_{j=1}^{m} \mu_j g_j$ is quasi-invex with respect to η at x^*, yields

$$\eta(x_0, x^*)^T \sum_{j=1}^{m} \mu_j w_j \leq 0. \tag{10}$$

On adding (9) and (10), we get

$$\eta(x_0, x^*)^T(\xi^L v^L + \xi^U v^U + \sum_{j=1}^{m} \mu_j w_j) < 0,$$

which contradicts (8). Therefore x^* is a LU optimal solution to (IVP). This completes the proof. □

4 Wolfe-type duality

In this section, we consider the following Wolfe-type dual problem:

$$\text{(IWD)} \qquad \max \qquad F(y) + \sum_{j=1}^{m} \mu_j g_j(y)$$

subject to

$$0 \in \xi^L \partial F^L(y) + \xi^U \partial F^U(y) + \sum_{j=1}^{m} \mu_j \partial g_j(y), \tag{11}$$

$$\xi^L > 0,\ \xi^U > 0,\ \mu_j \geq 0, \ \ j = 1, 2, ..., m, \tag{12}$$

where $F(y) + \sum\limits_{j=1}^{m} \mu_j g_j(y) = \left[F^L(y) + \sum\limits_{j=1}^{m} \mu_j g_j(y),\ F^U(y) + \sum\limits_{j=1}^{m} \mu_j g_j(y) \right]$ is an interval-valued function.

Definition 4.1. *Let* $(y^*, \xi^{*L}, \xi^{*U}, \mu^*)$ *be a feasible solution of dual problem (IWD). We say that* $(y^*, \xi^{*L}, \xi^{*U}, \mu^*)$ *is a LU optimal solution of dual problem (IWD), if there exists no* $(y, \xi^{*L}, \xi^{*U}, \mu^*)$ *such that* $F(y^*) + \sum_{j=1}^{m} \mu_j^* g_j(y^*) <_{LU} F(y) + \sum_{j=1}^{m} \mu_j^* g_j(y)$.

Now, we establish the following weak, strong and strict converse duality results in order to relate the feasibility of (IVP) and (IWD).

Theorem 4.1. *(Weak duality). Let* x *and* (y, ξ^L, ξ^U, μ) *are the feasible solutions to (IVP) and (IWD), respectively. Assume that* F *and* $\sum_{j=1}^{m} \mu_j g_j$ *are invex with respect to same* η *at* y *with* $\xi^L + \xi^U = 1$. *Then*

$$F(x) \geq_{LU} F(y) + \sum_{j=1}^{m} \mu_j g_j(y).$$

Proof. From (11), it is clear that there exist $v^L \in \partial F^L(y),\ v^U \in \partial F^U(y)$ and $w_j \in \partial g_j(y)$, for each $j = 1, 2, ..., m$ such that

$$\xi^L v^L + \xi^U v^U + \sum_{j=1}^{m} \mu_j w_j = 0. \tag{13}$$

Now, suppose contrary to the result that

$$F(x) <_{LU} F(y) + \sum_{j=1}^{m} \mu_j g_j(y).$$

That is,

$$\begin{cases} F^L(x) < F^L(y) + \sum\limits_{j=1}^{m} \mu_j g_j(y) \\ F^U(x) < F^U(y) + \sum\limits_{j=1}^{m} \mu_j g_j(y) \end{cases}, \text{ or } \begin{cases} F^L(x) \leq F^L(y) + \sum\limits_{j=1}^{m} \mu_j g_j(y) \\ F^U(x) < F^U(y) + \sum\limits_{j=1}^{m} \mu_j g_j(y) \end{cases},$$

$$\text{or } \begin{cases} F^L(x) < F^L(y) + \sum\limits_{j=1}^{m} \mu_j g_j(y) \\ F^U(x) \leq F^U(y) + \sum\limits_{j=1}^{m} \mu_j g_j(y) \end{cases}.$$

Since $\xi^L > 0,\ \xi^U > 0$ and $\xi^L + \xi^U = 1$, the above inequalities together with the feasibility of x to (IVP) gives

$$\xi^L F^L(x) + \xi^U F^U(x) + \sum_{j=1}^{m} \mu_j g_j(x) < \xi^L F^L(y) + \xi^U F^U(y) + \sum_{j=1}^{m} \mu_j g_j(y). \tag{14}$$

From the assumption that F is invex with respect to η at y, we have

$$F^L(x) - F^L(y) \geq \eta(x, y)^T v^L, \quad \forall\ v^L \in \partial F^L(y),$$

$$F^U(x) - F^U(y) \geq \eta(x,y)^T v^U, \quad \forall\, v^U \in \partial F^U(y).$$

The above inequalities together with the positivity of ξ^L and ξ^U, gives

$$\xi^L F^L(x) - \xi^L F^L(y) \geq \eta(x,y)^T \xi^L v^L, \quad \forall\, v^L \in \partial F^L(y), \tag{15}$$

$$\xi^U F^U(x) - \xi^U F^U(y) \geq \eta(x,y)^T \xi^U v^U, \quad \forall\, v^U \in \partial F^U(y). \tag{16}$$

Further, using the invexity of $\sum_{j=1}^{m} \mu_j g_j$ with respect to η at y, we get

$$\sum_{j=1}^{m} \mu_j g_j(x) - \sum_{j=1}^{m} \mu_j g_j(y) \geq \eta(x,y)^T \sum_{j=1}^{m} \mu_j w_j, \quad \forall\, w_j \in \partial g_j(y),\ j = 1,2,...,m. \tag{17}$$

On adding (15), (16) and (17), we obtain

$$(\xi^L F^L(x) + \xi^U F^U(x) + \sum_{j=1}^{m} \mu_j g_j(x)) - (\xi^L F^L(y) + \xi^U F^U(y) + \sum_{j=1}^{m} \mu_j g_j(y))$$

$$\geq \eta(x,y)^T (\xi^L v^L + \xi^U v^U + \sum_{j=1}^{m} \mu_j w_j).$$

The above inequality together with (14), yields

$$\eta(x,y)^T (\xi^L v^L + \xi^U v^U + \sum_{j=1}^{m} \mu_j w_j) < 0,$$

which contradicts (13). This completes the proof. □

Theorem 4.2. *(Strong duality). Let x^* be a LU optimal solution to (IVP) and suitable constraint qualification is satisfied at x^*. Then there exist $\xi^{*L} > 0,\ \xi^{*U} > 0$ and $\mu^* \geq 0$, such that $(x^*, \xi^{*L}, \xi^{*U}, \mu^*)$ is a feasible solution to (IWD) and the two objective values are equal. Further, if the hypothesis of weak duality Theorem 4.1 holds for all feasible solutions $(y^*, \xi^{*L}, \xi^{*U}, \mu^*)$. Then $(x^*, \xi^{*L}, \xi^{*U}, \mu^*)$ is a LU optimal solution to (IWD).*

Proof. Since x^* is a LU optimal solution to (IVP) and suitable constraint qualification is satisfied at x^*, then by Theorem 2.1 there exist scalars $\xi^{*L} > 0,\ \xi^{*U} > 0, \mu_j^* \geq 0,\ j = 1,2,...,m$ such that

$$0 \in \xi^{*L} \partial F^L(x^*) + \xi^{*U} \partial F^U(x^*) + \sum_{j=1}^{m} \mu_j^* \partial g_j(x^*),$$

$$\mu_j^* g_j(x^*) = 0,\ j = 1,2,...,m,$$

which yields that $(x^*, \xi^{*L}, \xi^{*U}, \mu^*)$ is a feasible solution to (IWD) and corresponding objective values are equal. Let $(x^*, \xi^{*L}, \xi^{*U}, \mu^*)$ is not a LU optimal solution to (IWD), then there exist a feasible solution $(y^*, \xi^{*L}, \xi^{*U}, \mu^*)$ to (IWD) such that

$$F(x^*) <_{LU} F(y^*) + \sum_{j=1}^{m} \mu_j^* g_j(y^*),$$

which contradicts the weak duality Theorem 4.1. Hence $(x^*, {\xi^*}^L, {\xi^*}^U, \mu^*)$ is a LU optimal solution to (IWD). □

Theorem 4.3. *(Strict converse duality). Let x^* and $(y^*, {\xi^*}^L, {\xi^*}^U, \mu^*)$ are the feasible solutions to (IVP) and (IWD), respectively. Suppose that F is strictly-invex and $\sum_{j=1}^{m} \mu_j^* g_j$ is invex with respect to same η at y^* and*

$$ {\xi^*}^L F^L(x^*) + {\xi^*}^U F^U(x^*) + \sum_{j=1}^{m} \mu_j^* g_j(x^*) \leq {\xi^*}^L F^L(y^*) + {\xi^*}^U F^U(y^*) + \sum_{j=1}^{m} \mu_j^* g_j(y^*). \quad (18) $$

Then $x^* = y^*$.

Proof. From (11), it is clear that there exist some $v^L \in \partial F^L(y^*)$, $v^U \in \partial F^U(y^*)$ and $w_j \in \partial g_j(y^*)$, for each $j = 1, 2, ..., m$ such that

$$ {\xi^*}^L v^L + {\xi^*}^U v^U + \sum_{j=1}^{m} \mu_j^* w_j = 0. \quad (19) $$

Now we assume that $x^* \neq y^*$ and exhibit a contradiction. Using the strict invexity of F with respect to η at y^*, one of the following is satisfied

$$ \begin{cases} F^L(x^*) - F^L(y^*) > \eta(x^*, y^*)^T v^L \\ F^U(x^*) - F^U(y^*) > \eta(x^*, y^*)^T v^U \end{cases}, \text{ or } \begin{cases} F^L(x^*) - F^L(y^*) \geq \eta(x^*, y^*)^T v^L \\ F^U(x^*) - F^U(y^*) > \eta(x^*, y^*)^T v^U \end{cases}, $$

$$ \text{or } \begin{cases} F^L(x^*) - F^L(y^*) > \eta(x^*, y^*)^T v^L \\ F^U(x^*) - F^U(y^*) \geq \eta(x^*, y^*)^T v^U \end{cases}, \ \forall\ v^L \in \partial F^L(y^*),\ \forall\ v^U \in \partial F^U(y^*). $$

Since ${\xi^*}^L > 0$, ${\xi^*}^U > 0$, we can write the above inequalities as

$$ ({\xi^*}^L F^L(x^*) + {\xi^*}^U F^U(x^*)) - ({\xi^*}^L F^L(y^*) + {\xi^*}^U F^U(y^*)) > \eta(x^*, y^*)^T ({\xi^*}^L v^L + {\xi^*}^U v^U). $$

On the other hand, using the invexity of $\sum_{j=1}^{m} \mu_j^* g_j$ with respect to η at y^*, we get

$$ \sum_{j=1}^{m} \mu_j^* g_j(x^*) - \sum_{j=1}^{m} \mu_j^* g_j(y^*) \geq \eta(x^*, y^*)^T \sum_{j=1}^{m} \mu_j^* w_j,\ \forall\ w_j \in \partial g_j(y^*). $$

Combining the above two inequalities, we obtain

$$ ({\xi^*}^L F^L(x^*) + {\xi^*}^U F^U(x^*) + \sum_{j=1}^{m} \mu_j^* g_j(x^*)) - ({\xi^*}^L F^L(y^*) + {\xi^*}^U F^U(y^*) + \sum_{j=1}^{m} \mu_j^* g_j(y^*)) $$
$$ > \eta(x^*, y^*)^T ({\xi^*}^L v^L + {\xi^*}^U v^U + \sum_{j=1}^{m} \mu_j^* w_j). $$

The above inequality together with (19), gives

$$ {\xi^*}^L F^L(x^*) + {\xi^*}^U F^U(x^*) + \sum_{j=1}^{m} \mu_j^* g_j(x^*) > {\xi^*}^L F^L(y^*) + {\xi^*}^U F^U(y^*) + \sum_{j=1}^{m} \mu_j^* g_j(y^*), $$

which contradicts (18). This completes the proof. □

5 Mond-Weir type duality

In this section, we consider the following Mond-Weir type dual problem:

$$\text{(IMWD)} \qquad \max \qquad F(y) = [F^L(y),\ F^U(y)]$$

subject to

$$0 \in \xi^L \partial F^L(y) + \xi^U \partial F^U(y) + \sum_{j=1}^{m} \mu_j \partial g_j(y), \tag{20}$$

$$\sum_{j=1}^{m} \mu_j g_j(y) \geq 0, \tag{21}$$

$$\xi^L > 0,\ \xi^U > 0,\ \mu_j \geq 0, \quad j = 1, 2, ..., m. \tag{22}$$

Definition 5.1. *Let $(y^*, \xi^{*L}, \xi^{*U}, \mu^*)$ be a feasible solution of dual problem (IMWD). We say that $(y^*, \xi^{*L}, \xi^{*U}, \mu^*)$ is a LU optimal solution of dual problem (IMWD), if there exists no $(y, \xi^{*L}, \xi^{*U}, \mu^*)$ such that $F(y^*) <_{LU} F(y)$.*

Now, we establish the following weak, strong and strict converse duality results in order to relate the feasibility of (IVP) and (IMWD).

Theorem 5.1. *(Weak duality). Let x and (y, ξ^L, ξ^U, μ) are the feasible solutions to (IVP) and (IMWD), respectively. Assume that $\xi^L F^L + \xi^U F^U$ is pseudo-invex and $\sum_{j=1}^{m} \mu_j g_j$ is quasi-invex with respect to same η at y. Then*

$$F(x) \geq_{LU} F(y).$$

Proof. From (20), it is clear that there exist $v^L \in \partial F^L(y),\ v^U \in \partial F^U(y)$ and $w_j \in \partial g_j(y)$, for each $j = 1, 2, ..., m$ such that

$$\xi^L v^L + \xi^U v^U + \sum_{j=1}^{m} \mu_j w_j = 0. \tag{23}$$

Now, suppose contrary to the result that

$$F(x) <_{LU} F(y).$$

That is,

$$\begin{cases} F^L(x) < F^L(y) \\ F^U(x) < F^U(y) \end{cases}, \text{ or } \begin{cases} F^L(x) \leq F^L(y) \\ F^U(x) < F^U(y) \end{cases},$$

or $\begin{cases} F^L(x) < F^L(y) \\ F^U(x) \le F^U(y) \end{cases}$.

Since $\xi^L > 0,\ \xi^U > 0$ the above inequalities can be written as

$$\xi^L F^L(x) + \xi^U F^U(x) < \xi^L F^L(y) + \xi^U F^U(y).$$

The above inequality together with the assumption that $\xi^L F^L + \xi^U F^U$ is pseudo-invex with respect to η at y, gives

$$\eta(x,y)^T(\xi^L v^L + \xi^U v^U) < 0, \quad \forall\ v^L \in \partial F^L(y), \quad \forall\ v^U \in \partial F^U(y). \tag{24}$$

On the other hand, from the feasibility of x and (y, ξ^L, ξ^U, μ) to (IVP) and (IMWD), respectively, we obtain

$$\sum_{j=1}^{m} \mu_j g_j(x) \le \sum_{j=1}^{m} \mu_j g_j(y).$$

The quasi invexity of $\sum_{j=1}^{m} \mu_j g_j$ with respect to η at y together with the above inequality, yields

$$\eta(x,y)^T \sum_{j=1}^{m} \mu_j w_j \le 0,\ \forall\ w_j \in \partial g_j(y). \tag{25}$$

On adding (24) and (25), we obtain

$$\eta(x,y)^T(\xi^L v^L + \xi^U v^U + \sum_{j=1}^{m} \mu_j w_j) < 0,$$

which contradicts (23). This completes the proof. □

Theorem 5.2. *(Strong duality). Let x^* be a LU optimal solution to (IVP) and suitable constraint qualification is satisfied at x^*. Then there exist ${\xi^*}^L > 0,\ {\xi^*}^U > 0$ and $\mu^* \ge 0$, such that $(x^*, {\xi^*}^L, {\xi^*}^U, \mu^*)$ is a feasible solution to (IMWD) and the two objective values are equal. Further, if the hypothesis of weak duality Theorem 5.1 holds for all feasible solutions $(y^*, {\xi^*}^L, {\xi^*}^U, \mu^*)$. Then $(x^*, {\xi^*}^L, {\xi^*}^U, \mu^*)$ is a LU optimal solution to (IMWD).*

Proof. Since x^* is a LU optimal solution to (IVP) and suitable constraint qualification is satisfied at x^*, then by Theorem 2.1 there exist scalars ${\xi^*}^L > 0,\ {\xi^*}^U > 0,\ \mu_j^* \ge 0,\ j = 1, 2, ..., m$ such that

$$0 \in {\xi^*}^L \partial F^L(x^*) + {\xi^*}^U \partial F^U(x^*) + \sum_{j=1}^{m} \mu_j^* \partial g_j(x^*),$$

$$\mu_j^* g_j(x^*) = 0,\ j = 1, 2, ..., m,$$

which yields that $(x^*, \xi^{*L}, \xi^{*U}, \mu^*)$ is a feasible solution to (IMWD) and corresponding objective values are equal. Let $(x^*, \xi^{*L}, \xi^{*U}, \mu^*)$ is not a LU optimal solution to (IMWD), then there exist a feasible solution $(y^*, \xi^{*L}, \xi^{*U}, \mu^*)$ to (IMWD) such that

$$F(x^*) <_{LU} F(y^*),$$

which contradicts the weak duality Theorem 5.1. Hence $(x^*, \xi^{*L}, \xi^{*U}, \mu^*)$ is a LU optimal solution to (IMWD). □

Theorem 5.3. *(Strict converse duality). Let x^* and $(y^*, \xi^{*L}, \xi^{*U}, \mu^*)$ are the feasible solutions to (IVP) and (IMWD), respectively. Suppose that $\xi^{*L}F^L + \xi^{*U}F^U$ is strictly pseudo-invex and $\sum_{j=1}^{m} \mu_j^* g_j$ is quasi-invex with respect to same η at y^* and*

$$\xi^{*L}F^L(x^*) + \xi^{*U}F^U(x^*) \leq \xi^{*L}F^L(y^*) + \xi^{*U}F^U(y^*). \tag{26}$$

Then $x^* = y^*$.

Proof. From (20), it is clear that there exist some $v^L \in \partial F^L(y^*),\ v^U \in \partial F^U(y^*)$ and $w_j \in \partial g_j(y^*)$, for each $j = 1, 2, ..., m$ such that

$$\xi^{*L}v^L + \xi^{*U}v^U + \sum_{j=1}^{m} \mu_j^* w_j = 0. \tag{27}$$

Now we assume that $x^* \neq y^*$ and exhibit a contradiction. Since $\mu_j^* \geq 0,\ j = 1, 2, ..., m$ from the feasibility of x^* and $(y^*, \xi^{*L}, \xi^{*U}, \mu^*)$ to (IVP) and (IMWD), respectively we obtain

$$\sum_{j=1}^{m} \mu_j^* g_j(x^*) \leq \sum_{j=1}^{m} \mu_j^* g_j(y^*),$$

which together with the assumption that $\sum_{j=1}^{m} \mu_j^* g_j$ is quasi-invex with respect to η at y^*, yields

$$\eta(x^*, y^*)^T \sum_{j=1}^{m} \mu_j^* w_j \leq 0,\ \forall\ w_j \in \partial g_j(y^*).$$

By (27) and the above inequality, we get

$$\eta(x^*, y^*)^T(\xi^{*L}v^L + \xi^{*U}v^U) \geq 0,\ \forall\ v^L \in \partial F^L(y^*),\ \forall\ v^U \in \partial F^U(y^*).$$

The above inequality together with the assumption that $\xi^{*L}F^L + \xi^{*U}F^U$ is strictly pseudo-invex with respect to η at y^*, gives

$$\xi^{*L}F^L(x^*) + \xi^{*U}F^U(x^*) > \xi^{*L}F^L(y^*) + \xi^{*U}F^U(y^*),$$

which contradicts (26). This completes the proof. □

6 Lagrangian function and saddle-point analysis

In this section, we define the real-valued Lagrangian function for interval-valued optimization problem (IVP) as follows:

$$L(x, \xi^L, \xi^U, \mu) = \xi^L F^L(x) + \xi^U F^U(x) + \sum_{j=1}^{m} \mu_j g_j(x), \tag{28}$$

where $x \in X,\ \xi^L \geq 0,\ \xi^U \geq 0,\ \mu \in R^m_+$.

Now we define a saddle-point of $L(x, \xi^L, \xi^U, \mu)$ and study its relation to the problem (IVP).

Definition 6.1. *Let $\xi^{*L} \geq 0$ and $\xi^{*U} \geq 0$ be fixed. A point $(x^*, \xi^{*L}, \xi^{*U}, \mu^*) \in X \times R_+ \times R_+ \times R^m_+$ is said to be a saddle-point of the real-valued function $L(x, \xi^L, \xi^U, \mu)$ if it satisfies the following conditions*

$$L(x^*, \xi^{*L}, \xi^{*U}, \mu) \leq L(x^*, \xi^{*L}, \xi^{*U}, \mu^*) \leq L(x, \xi^{*L}, \xi^{*U}, \mu^*), \tag{29}$$

$$\forall\ x \in X,\ \forall\ \mu \in R^m_+.$$

Theorem 6.1. *Let $\xi^{*L} > 0,\ \xi^{*U} > 0$ be fixed and $(x^*, \xi^{*L}, \xi^{*U}, \mu^*)$ is a saddle-point of $L(x, \xi^L, \xi^U, \mu)$. Then x^* is a LU optimal solution to (IVP).*

Proof. Suppose, contrary to the result that x^* is not a LU optimal solution to (IVP). Then there exists a feasible solution $x \in \Lambda$, such that

$$F(x) <_{LU} F(x^*).$$

That is,

$$\begin{cases} F^L(x) < F^L(x^*) \\ F^U(x) < F^U(x^*) \end{cases}, \text{ or } \begin{cases} F^L(x) \leq F^L(x^*) \\ F^U(x) < F^U(x^*) \end{cases}, \text{ or } \begin{cases} F^L(x) < F^L(x^*) \\ F^U(x) \leq F^U(x^*) \end{cases}.$$

Since $\xi^{*L} > 0,\ \xi^{*U} > 0$, the above inequalities gives

$$\xi^{*L} F^L(x) + \xi^{*U} F^U(x) < \xi^{*L} F^L(x^*) + \xi^{*U} F^U(x^*). \tag{30}$$

As $(x^*, \xi^{*L}, \xi^{*U}, \mu^*)$ is a saddle-point of $L(x, \xi^L, \xi^U, \mu)$ from (29), we have

$$L(x^*, \xi^{*L}, \xi^{*U}, \mu) \leq L(x^*, \xi^{*L}, \xi^{*U}, \mu^*),$$

$$i.e.\quad \sum_{j=1}^{m} \mu_j g_j(x^*) \leq \sum_{j=1}^{m} \mu^*_j g_j(x^*). \tag{31}$$

Setting $(\mu_1, \mu_2, ..., \mu_{j-1}, \mu_j, \mu_{j+1}, ..., \mu_m) = (\mu_1^*, \mu_2^*, ..., \mu_{j-1}^*, {\mu^*}_j + 1, {\mu^*}_{j+1}, ..., \mu_m^*)$ in the inequality (31), we obtain

$$g_j(x^*) \leq 0, \quad j = 1, 2, ..., m,$$

which shows that x^* is a feasible solution to (IVP).

Since $\mu^* \in R_+^m$, from the above inequality we have

$$\mu_j^* g_j(x^*) \leq 0, \quad j = 1, 2, ..., m.$$

Again setting $\mu_j = 0,\ j = 1, 2, ..., m$ in the inequality (31), we obtain

$$\mu_j^* g_j(x^*) \geq 0, \quad j = 1, 2, ..., m.$$

Therefore from the above two inequalities, we conclude that

$$\mu_j^* g_j(x^*) = 0, \quad j = 1, 2, ..., m. \tag{32}$$

On the other hand, since $(x^*, {\xi^*}^L, {\xi^*}^U, \mu^*)$ is a saddle-point of $L(x, \xi^L, \xi^U, \mu)$ from (29) we also have

$$L(x^*, {\xi^*}^L, {\xi^*}^U, \mu^*) \leq L(x, {\xi^*}^L, {\xi^*}^U, \mu^*),$$

$$i.e.\ \ {\xi^*}^L F^L(x^*) + {\xi^*}^U F^U(x^*) + \sum_{j=1}^m \mu_j^* g_j(x^*) \leq {\xi^*}^L F^L(x) + {\xi^*}^U F^U(x) + \sum_{j=1}^m \mu_j^* g_j(x).$$

Now by using the feasibility of x to (IVP) and (32), we have from the above inequality

$${\xi^*}^L F^L(x^*) + {\xi^*}^U F^U(x^*) \leq {\xi^*}^L F^L(x) + {\xi^*}^U F^U(x),$$

which contradicts (30). This completes the proof. □

Theorem 6.2. *Let x^* is a LU optimal solution to (IVP). Assume that there exist scalars $0 < {\xi^*}^L,\ {\xi^*}^U \in R$ and $0 \leq \mu_j^* \in R,\ j = 1, 2, ..., m$ such that*

(i) $0 \in {\xi^*}^L \partial F^L(x^*) + {\xi^*}^U \partial F^U(x^*) + \sum_{j=1}^m \mu_j^* \partial g_j(x^*)$,

(ii) $\mu_j^* g_j(x^*) = 0,\ j = 1, 2, ..., m$,

(iii) F and $\sum_{j=1}^m \mu_j^* g_j$ are invex with respect to same η at x^*.

Then $(x^*, {\xi^*}^L, {\xi^*}^U, \mu^*)$ is a saddle-point of $L(x, \xi^L, \xi^U, \mu)$.

Proof. By hypothesis (i) it is clear that there exist some $v^L \in \partial F^L(x^*),\ v^U \in \partial F^U(x^*)$ and $w_j \in \partial g_j(x^*)$, for each $j = 1, 2, ..., m$ such that

$${\xi^*}^L v^L + {\xi^*}^U v^U + \sum_{j=1}^m \mu_j^* w_j = 0. \tag{33}$$

From the assumption that F is invex with respect to η at x^*, we have

$$F^L(x) - F^L(x^*) \geq \eta(x, x^*)^T v^L, \quad \forall\, v^L \in \partial F^L(x^*),$$

$$F^U(x) - F^U(x^*) \geq \eta(x, x^*)^T v^U, \quad \forall\, v^U \in \partial F^U(x^*).$$

Since $\xi^{*L} > 0$ and $\xi^{*U} > 0$, we get

$$\xi^{*L} F^L(x) - \xi^{*L} F^L(x^*) \geq \eta(x, x^*)^T \xi^{*L} v^L, \quad \forall\, v^L \in \partial F^L(x^*),$$

$$\xi^{*U} F^U(x) - \xi^{*U} F^U(x^*) \geq \eta(x, x^*)^T \xi^{*U} v^U, \quad \forall\, v^U \in \partial F^U(x^*).$$

From the assumption that $\sum_{j=1}^{m} \mu_j^* g_j$ is invex with respect to η at x^*, we have

$$\sum_{j=1}^{m} \mu_j^* g_j(x) - \sum_{j=1}^{m} \mu_j^* g_j(x^*) \geq \eta(x, x^*)^T \sum_{j=1}^{m} \mu_j^* w_j, \quad \forall\, w_j \in \partial g_j(x^*),\ j = 1, 2, ..., m.$$

Combining the above three inequalities, we get

$$\begin{aligned}(\xi^{*L} F^L(x) + \xi^{*U} F^U(x) + \sum_{j=1}^{m} \mu_j^* g_j(x)) - (\xi^{*L} F^L(x^*) + \xi^{*U} F^U(x^*) + \sum_{j=1}^{m} \mu_j^* g_j(x^*))\\ \geq \eta(x, x^*)^T (\xi^{*L} v^L + \xi^{*U} v^U + \sum_{j=1}^{m} \mu_j^* w_j),\end{aligned}$$

which together with (33), yields

$$\xi^{*L} F^L(x^*) + \xi^{*U} F^U(x^*) + \sum_{j=1}^{m} \mu_j^* g_j(x^*) \leq \xi^{*L} F^L(x) + \xi^{*U} F^U(x) + \sum_{j=1}^{m} \mu_j^* g_j(x).$$

Hence, we have

$$L(x^*, \xi^{*L}, \xi^{*U}, \mu^*) \leq L(x, \xi^{*L}, \xi^{*U}, \mu^*). \tag{34}$$

On the other hand, since $\mu \in R_+^m$ from the feasibility of x^* to (IVP) we get

$$\mu_j g_j(x^*) \leq 0,\ j = 1, 2, ..., m. \tag{35}$$

By using (35) and the hypothesis (ii), we obtain

$$L(x^*, \xi^{*L}, \xi^{*U}, \mu) \leq L(x^*, \xi^{*L}, \xi^{*U}, \mu^*). \tag{36}$$

The relation (34) together with (36) shows that, $(x^*, \xi^{*L}, \xi^{*U}, \mu^*)$ is a saddle-point of $L(x, \xi^L, \xi^U, \mu)$. □

Theorem 6.3. *Let x^* is a LU optimal solution to (IVP). Assume that there exist scalars $0 < \xi^{*L},\ \xi^{*U} \in R$ and $0 \leq \mu_j^* \in R,\ j = 1, 2, ..., m$ such that*

(i) $0 \in \xi^{*L} \partial F^L(x^*) + \xi^{*U} \partial F^U(x^*) + \sum_{j=1}^{m} \mu_j^* \partial g_j(x^*)$,

(ii) $\mu_j^* g_j(x^*) = 0, \; j = 1, 2, ..., m,$

(iii) $\xi^{*L} F^L + \xi^{*U} F^U + \sum_{j=1}^{m} \mu_j^* g_j$ *is invex with respect to* η *at* x^*.

Then $(x^*, \xi^{*L}, \xi^{*U}, \mu^*)$ *is a saddle-point of* $L(x, \xi^L, \xi^U, \mu)$.

Proof. By hypothesis (i) it is clear that there exist some $v^L \in \partial F^L(x^*), \; v^U \in \partial F^U(x^*)$ and $w_j \in \partial g_j(x^*)$, for each $j = 1, 2, ..., m$ such that

$$\xi^{*L} v^L + \xi^{*U} v^U + \sum_{j=1}^{m} \mu_j^* w_j = 0.$$

The above inequality together with the assumption that $\xi^{*L} F^L + \xi^{*U} F^U + \sum_{j=1}^{m} \mu_j^* g_j$ is invex with respect to η at x^*, gives

$$\xi^{*L} F^L(x^*) + \xi^{*U} F^U(x^*) + \sum_{j=1}^{m} \mu_j^* g_j(x^*) \le \xi^{*L} F^L(x) + \xi^{*U} F^U(x) + \sum_{j=1}^{m} \mu_j^* g_j(x).$$

Now the proof is similar to that of Theorem 6.2. This completes the proof. □

7 Conclusion

In this paper, we have discussed optimality conditions for nondifferentiable interval-valued programming problem under the invexity assumption. Weak, strong and strict converse duality theorems are presented for two types of the dual models. Moreover, we derived some saddle-point type optimality conditions. It will be interesting to obtain the optimality and duality theorems under generalized invexity assumptions. Furthermore, it will also be interesting to see whether the results for a class of nonsmooth interval-valued programming problems presented in this paper hold for mixed type dual. This will orient the future research of the authors.

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