Blow-up of Smooth Solution to the Compressible Navier-Stokes-Poisson Equations

Tong Tang [a] * Zujin Zhang [b] †
[a] School of Mathematical Sciences,
Nanjing Normal University, Nanjing 210023, P.R. China
[b] School of Mathematics and Computer Science,
Gannan Normal University, Ganzhou 341000, P.R. China

Abstract

In this paper, we show that the blow-up phenomenon of smooth solutions to the compressible Navier-Stokes-Poisson (N-S-P) equations in $\mathbb{R}^2$, under the assumption that the initial density has compact support. The proof is based on some useful physical quantities. In particular, our result is valid for both isentropic and isothermal case.

Key words. Blow-up, Compressible Navier-Stokes-Poisson equations.
2010 Mathematics Subject Classifications: 35Q35, 35M10.

1 Introduction

This paper is concerned with the blow-up phenomena of smooth solutions to the Cauchy problem for the following Navier-Stokes-Poisson (N-S-P) system in two dimensional case

$$\begin{cases} \partial_t\rho + \mathrm{div}(\rho u) = 0, \\ \partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) + \nabla P(\rho) = \mu\Delta u + (\mu+\lambda)\nabla \mathrm{div} u + \kappa\rho\nabla\Phi, (x,t) \in \mathbb{R}^2 \times \mathbb{R}_+ \\ -\Delta\Phi = \rho, \end{cases} \tag{1.1}$$

with initial data

$$\rho|_{t=0} = \rho_0(x), \ \ u|_{t=0} = u_0(x). \tag{1.2}$$

The unknown functions $\rho(x,t)$, $u(x,t)$, P and Φ denote the density, velocity field, pressure and potential of underlying force respectively. Generally speaking, the pressure P depends on the density and temperature of fluid. However, there are physically relevant situations that we assume the fluid flow is barotropic, i.e., the pressure depends only on the density. This is the case when either the temperature or the entropy is supposed to be constant. The typical expression is

$$P(\rho) = \rho^\gamma \quad (\gamma \geq 1), \tag{1.3}$$

*Corresponding author Email: tt0507010156@126.com
†Email: zhangzujin361@163.com

where $\gamma = 1$ stands for the isothermal case, and $\gamma > 1$ represents the adiabatic constant in the isentropic regime. We introduce the viscous stress tensor T as

$$\mathrm{div}T = \mu\Delta u + (\mu + \lambda)\nabla\mathrm{div}u.$$

The coefficient κ, which signifies the property of the forcing, is repulsive if $\kappa > 0$ and attractive if $\kappa < 0$. The coefficients μ and λ represent shear coefficient viscosity of the fluid and the second viscosity coefficient respectively. As the fluid is assumed to be Newtonian, the two Lamé viscosity coefficients satisfy

$$\mu > 0, \ \ \mu + \lambda \geq 0. \tag{1.4}$$

The compressible N-S-P system can be used to describe many models if we consider different potential force. For example, (1.1) is the self-gravitation model if Φ is the gravitational potential force, and the semiconductor model if Φ is the electrostatic potential force. In the literatures, there have been a lot of studies on the N-S-P by physicists and mathematicians because of its physical importance and mathematical challenges. For these results, please refer to [5, 11, 12] and reference therein.

The blow-up of smooth solutions to the evolutionary equations arising in the mathematical fluid mechanics has been the subject of many theoretical studies. More precisely, Sideris [7] showed the life span of C^1 solution to the compressible Euler equations was finite when the initial data is constant outside a bounded set and the initial flow velocity has compact supports. In 1998, Xin [9] used a different method to prove the blow-up result for the compressible Navier-Stokes equations, under two basic hypotheses: the support of the density grows sublinearly in time and the entropy is bounded below. In 2004, Cho [7] extended Xin's result to the case of fluids with positive heat conduction. Recently, Xin and Yan [10] introduced the concept of isolated mass group to remove the condition that the initial density has compact support and the smooth solution has finite total energy and obtained a new blow-up result. Inspired by the above pioneering work, many authors study the blow-up phenomena for the N-S-P equations. Jiang and Tan [6] obtained blow-up result of the compressible reactive self-gravitating gas with chemical kinetics equations in $\mathbb{R}^3$. Xie [8] showed blow-up result of smooth solutions to the full compressible N-S-P in $\mathbb{R}^3$.

However, all the previous work are concerned with the non-isentropic case, which the energy equation plays an important role. By virtue of the energy equation and the adiabatic exponent $\gamma > 1$, we can easily obtain that the density ρ is compactly supported all the time. While the methods used above cannot be applied to our N-S-P model straightforwardly. On the one hand, it is difficult to obtain the fact that compact support of the initial data will not change in time. On the other hand, it is hard to construct the function as the total energy has a negative part when $\kappa > 0$. To overcome the above mentioned difficulties, we use some useful physical quantities in the radially symmetric case, proving the blow-up for both isothermal (i.e.$\gamma = 1$) and isentropic (i.e.$\gamma > 1$) case whether $\kappa > 0$ or $\kappa < 0$. In some senses, we improve the corresponding previous result [3], which prove the isothermal compressible Naiver-Stokes equations for two-dimensional case. For simplicity of presentation, we introduce the following physical quantities:

$$m(t) = \int_{\mathbb{R}^2} \rho(x,t)dx, \tag{1.5}$$

$$M(t) = \int_{\mathbb{R}^2} \rho(x,t)|x|^2dx, \tag{1.6}$$

$$F(t) = \int_{\mathbb{R}^2} \rho(x,t)u(x,t)\cdot xdx, \tag{1.7}$$

which represent the total mass, second moment and radial component of momentum respectively.

Throughout this paper, we always assume $m(0) > 0$ and $F(0) > 0$. Moreover, we assume that the initial density ρ_0 has compact support, i.e., there exists a positive constant R such that

$$\text{supp}\rho_0 \subset B_R, \tag{1.8}$$

where B_R denotes the ball in $\mathbb{R}^2$ centered at origin with radius R.

Our main results can be summarized as follows.

Theorem 1.1. *Let (μ, λ) satisfy* (1.4), $\kappa \neq 0$, $\gamma \geq 1$ *and* $(\rho, u) \in C^1([0,T], H^m(\mathbb{R}^2))(m > 2)$ *is a spherically symmetric solution to the compressible N-S-P system* (1.1) *with initial data $\rho_0(x)$ has compact support. Assume the initial data are spherically symmetric, i.e.*

$$\rho_0(x) = \rho_0(|x|), u_0(x) = \frac{x}{|x|}u_0(|x|).$$

Then, the lifespan of the solution (ρ, u) is finite.

In the following theorem, we point out that the function constructed by Xin in [9] can be applied to the N-S-P model under the special coefficient κ.

Theorem 1.2. *Let (μ, λ) satisfy* (1.4), $\kappa = -1$, $\gamma > 2$ *and* $(\rho, u) \in C^1([0,T], H^m(\mathbb{R}^2))$ $(m > 2)$ *is a spherically symmetric solution to the compressible N-S-P system* (1.1) *with initial data $\rho_0(x)$ has compact support. The initial data are spherically symmetric as Theorem 1.1. Then, the lifespan of the solution (ρ, u) is finite.*

Remark 1.1. *If changing the condition on u, such as $u \in C^1([0,T], W^{m,1}(\mathbb{R}^N))(m > 2)$, we can extend our result to high dimension case. The reader can refer to the work of Bian and Guo [1] for more detail.*

2 Proof of Theorem 1.1

Before the proof of Theorem 1.1, we give the following key lemma which plays an important role in the proof.

Lemma 2.1. *Assume (μ, λ) satisfies* (1.4), *and* $(\rho, u) \in C^1([0,T], H^m(\mathbb{R}^2))(m > 2)$ *is a spherically symmetric solution to the compressible N-S-P system. Then*

$$u(x,t) \equiv 0, \quad x \in B_R^c. \tag{2.1}$$

Moreover, the support of solution will not change in time.

Proof. First, let $X(t,\alpha)$ denotes the particle path starting at α when $t=0$, i.e.,

$$\frac{d}{dt}X(t,\alpha) = u(X(t,\alpha),t), \quad X(0,\alpha)=\alpha. \tag{2.2}$$

And we denote by $\Omega(t)$ the closed region that is the image of B_R under the flow map (2.2):

$$\Omega(t) = \{x = X(t,\alpha), \alpha \in B_R\}. \tag{2.3}$$

In the sequel, we shall prove that $\Omega(t) = \Omega(0)$.

In fact, it follows from the continuity equation $(1.1)_1$ that the density is simply transported along particle paths, so that

$$\text{supp}_x \rho(x,t) \subset \Omega(t).$$

Consequently, from momentum equation, one has

$$\mu\Delta u + (\mu+\lambda)\nabla \text{div} u = 0, \quad \text{on } \{t\}\times\mathbb{R}^2\backslash\Omega(t). \tag{2.4}$$

Since $u(x,t) = \frac{x}{|x|}\overline{u}(|x|,t)$ for some radially symmetric function $\overline{u}$, we obtain from (2.4) that

$$\overline{u}_{rr} + (\frac{\overline{u}}{r})_r = 0 \quad \text{on } \{t\}\times\mathbb{R}^2\backslash\Omega(t).$$

Using the condition $u \in H^m(\mathbb{R}^2)$, we have

$$\overline{u}_r + \frac{\overline{u}}{r} = 0 \quad \text{on } \{t\}\times\mathbb{R}^2\backslash\Omega(t).$$

Solving the ODE gives:

$$\overline{u} = \frac{C(t)}{r},$$

where $C(t)$ is a constant dependent on t. Then using $u \in C([0,T], L^2(\mathbb{R}^2))$ and adopting the method in [3], we get

$$\overline{u} \equiv 0 \quad \text{on } \{t\}\times\mathbb{R}^2\backslash\Omega(t).$$

It follows from the definition of $\Omega(t)$ that if $\alpha \in \Omega^c(0)$, then

$$u(X(t,\alpha),t) = 0.$$

Thus

$$X(t,\alpha) = \alpha + \int_0^t u(X(s,\alpha),s)ds = \alpha,$$

and

$$\Omega(0) = \Omega(t) \quad \forall\, 0 \le t \le T.$$

This completes the proof of Lemma 2.1. $\square$

By virtue of Lemma 2.1, we are in a position to prove Theorem 1.1 in the following.

Proof of Theorem 1.1.
Firstly multiplying the momentum equation $(1.1)_2$ by x and integrating over $\mathbb{R}^2$, we have

$$\begin{aligned}\frac{d}{dt}F(t) &= \frac{d}{dt}\int_{\mathbb{R}^2}\rho u\cdot xdx = \int_{\mathbb{R}^2}(\rho u)_t\cdot xdx\\ &= -\int_{\mathbb{R}^2}\mathrm{div}(\rho u\otimes u)\cdot xdx - \int_{\mathbb{R}^2}\nabla\rho^\gamma\cdot xdx + \int_{\mathbb{R}^2}\mathrm{div}T\cdot xdx + \kappa\int_{\mathbb{R}^2}\rho\nabla\Phi\cdot xdx\\ &= \sum_{k=1}^{4} I_k. \end{aligned}\tag{2.5}$$

We calculate the integrals I_k one by one. Starting with I_1 and utilizing mass equation, we get

$$I_1 = \int_{\mathbb{R}^2}\rho|u|^2dx. \tag{2.6}$$

Similarly, we have

$$I_2 = 2\int_{\mathbb{R}^2}\rho^\gamma dx. \tag{2.7}$$

Using Lemma 2.1 and the definition of viscous stress tensor T, we can easily obtain

$$I_3 = 0. \tag{2.8}$$

It remains to simply $I_4(t)$. By the Poisson equation $(1.1)_3$, we get

$$\begin{aligned} I_4 &= \int_{\mathbb{R}^2}\rho\nabla\Phi\cdot xdx = \int_{\mathbb{R}^2}(-\Delta\Phi)\nabla\Phi\cdot xdx\\ &= -\sum_{i,j=1}^{2}\int_{\mathbb{R}^2}\partial_{ii}\Phi(\partial_j\Phi\cdot x_j)dx\\ &= \sum_{i,j=1}^{2}\int_{\mathbb{R}^2}\partial_i\Phi\partial_{ji}\Phi\cdot x_jdx + \sum_{i,j=1}^{2}\int_{\mathbb{R}^2}\partial_i\Phi\partial_j\Phi\cdot\partial_i x_jdx\\ &= \sum_{i,j=1}^{2}\frac{1}{2}\int_{\mathbb{R}^2}\partial_j|\partial_i\Phi|^2\cdot x_jdx + \sum_{i,j=1}^{2}\int_{\mathbb{R}^2}\partial_i\Phi\partial_j\Phi\delta_{ij}dx\\ &= -\int_{\mathbb{R}^2}|\nabla\Phi|^2dx + \int_{\mathbb{R}^2}|\nabla\Phi|^2dx = 0. \end{aligned}\tag{2.9}$$

Gathering the identities (2.5)-(2.9), we obtain

$$\frac{d}{dt}F(t) = \int_{\mathbb{R}^2}\rho|u|^2dx + 2\int_{\mathbb{R}^2}\rho^\gamma dx,$$

Integrating the above identity with respect to t, we get

$$F(t) = F(0) + \int_0^t\int_{\mathbb{R}^2}\rho|u|^2dxdt + 2\int_0^t\int_{\mathbb{R}^2}\rho^\gamma dxdt,$$

which implies

$$F(t) \geq F(0). \tag{2.10}$$

By virtue of the continuity equation and integrating by parts formula, we obtain

$$\frac{d}{dt}M(t) = \frac{d}{dt}\int_{\mathbb{R}^2} \rho|x|^2 dx = 2F(t). \tag{2.11}$$

Integrating (2.11), we get

$$M(t) = M(0) + 2\int_0^t F(s)ds. \tag{2.12}$$

From (2.10) and (2.12), we deduce

$$M(t) \geq M(0) + 2F(0)t. \tag{2.13}$$

Obviously, the right-hand side of (2.9) grows linearly in t. We shall show that the left hand side is bounded. On the other hand, from the continuity equation, we obtain

$$\frac{d}{dt}m(t) = \frac{d}{dt}\int_{\mathbb{R}^2} \rho dx = 0,$$

which implies

$$\int_{\mathbb{R}^2} \rho dx = \int_{\mathbb{R}^2} \rho_0 dx = m(0).$$

Thus, from (1.8) and mass conservation, we have

$$M(t) = \int_{\mathbb{R}^2} \rho|x|^2 dx = \int_{B_R} \rho|x|^2 dx \leq R^2 \int_{B_R} \rho dx = R^2 m(0). \tag{2.14}$$

Putting (2.13) and (2.14) together, we conclude that

$$R^2 m(0) \geq M(0) + 2F(0)t.$$

Hence, the lifespan of the classical solution of N-S-P is finite and we finish the proof of Theorem 1.1. □

3 Proof of Theorem 1.2

In this section, we will give the proof of Theorem 1.2 and show that the the blow-up result can be obtained by the nonlinear function introduced in [9], only when $\kappa = -1$ and $\gamma > 2$.

Proof of Theorem 1.2.
First, we introduce the function

$$I(t) = \int_{\mathbb{R}^2} x^2 \rho dx - 2(1+t)\int_{\mathbb{R}^2} xu\rho dx + (1+t)^2 \int_{\mathbb{R}^2} (\rho u^2 + \frac{2}{\gamma - 1}\rho^\gamma + |\nabla\Phi|^2)dx. \tag{3.1}$$

Lemma 2.1 is valid, since in the radially symmetric case

$$\Delta u = \nabla \text{div} u = \partial_r(\overline{u}_r + \frac{\overline{u}}{r})\frac{x}{r}.$$

Consequently, one has

$$\Omega(0) = \Omega(t).$$

Then

$$\begin{aligned}
I'(t) &= \int_{\mathbb{R}^2}(x^2\rho_t - 2x\rho u)dx - 2(1+t)\int_{\mathbb{R}^2}[x(\rho u)_t - \rho u^2 - \frac{2}{\gamma-1}\rho^\gamma - |\nabla\Phi|^2]dx \\
&+ (1+t)^2\int_{\mathbb{R}^2}(\rho u^2 + \frac{2}{\gamma-1}\rho^\gamma + |\nabla\Phi|^2)_t dx \\
&= I_1 + I_2 + I_3.
\end{aligned}$$

Direct calculation by using the mass equation and $u \in C^1([0,T], H^m(\mathbb{R}^2))(m>2)$ implies that $I_1 = 0$. As to the second term

$$\begin{aligned}
I_2 &= 2(1+t)\int_{\mathbb{R}^2} x\cdot[\text{div}(\rho u\otimes u) + \nabla\rho^\gamma - \mu\Delta u - (\mu+\lambda)\nabla\text{div}u + \rho\nabla\Phi]dx \\
&+ 2(1+t)\int_{\mathbb{R}^2}(\rho u^2 + \frac{2}{\gamma-1}\rho^\gamma + |\nabla\Phi|^2)dx \\
&= 2(1+t)[\int_{\mathbb{R}^2}(\frac{2(2-\gamma)}{\gamma-1}\rho^\gamma + |\nabla\Phi|^2)dx + (2\mu+\lambda)\int_0^\infty \partial_r(\frac{\overline{u}}{r} + \overline{u}_r)\frac{|x|^2}{r}rdr] \\
&= 2(1+t)\int_{\mathbb{R}^2}[\frac{2(2-\gamma)}{\gamma-1}\rho^\gamma + |\nabla\Phi|^2 + 2(2\mu+\lambda)(\frac{\overline{u}}{r} + \overline{u}_r)]dx,
\end{aligned}$$

and

$$\begin{aligned}
I_3 &= (1+t)^2\int_{\mathbb{R}^2} 2[-\nabla\rho^\gamma\cdot u + \mu\Delta u\cdot u + (\mu+\lambda)(\nabla\text{div}u)\cdot u]dx \\
&+ (1+t)^2\frac{2}{\gamma-1}\int_{\mathbb{R}^2}[-\text{div}(\rho^\gamma u) - \big((\rho^\gamma)'\rho - \rho\big)\text{div}u]dx \\
&= 2(1+t)^2\int_0^\infty (2\mu+\lambda)\partial_r(\frac{\overline{u}}{r} + \overline{u}_r)\frac{x}{r}\cdot(\overline{u}\frac{x}{r})rdr \\
&= -2(1+t)^2(2\mu+\lambda)\int_{\mathbb{R}^2}(\frac{\overline{u}}{r} + \overline{u}_r)^2 dx.
\end{aligned}$$

Combining all the identities, we get

$$\begin{aligned}
I'(t) &= 4(1+t)\frac{2-\gamma}{\gamma-1}\int_{\mathbb{R}^2}\rho^\gamma dx + 4(1+t)(2\mu+\lambda)\int_{\mathbb{R}^2}(\overline{u}_r + \frac{\overline{u}}{r})dx \\
&- 2(1+t)^2(2\mu+\lambda)\int_{\mathbb{R}^2}(\overline{u}_r + \frac{\overline{u}}{r})^2 dx + 2(1+t)\int_{\mathbb{R}^2}|\nabla\Phi|^2 dx \\
&\le 4(1+t)\frac{2-\gamma}{\gamma-1}\int_{\mathbb{R}^2}\rho^\gamma dx + 2(2\mu+\lambda)|\Omega(0)| + 2(1+t)\int_{\mathbb{R}^2}|\nabla\Phi|^2 dx \\
&\le \frac{6-2\gamma}{1+t}I(t) + 2(2\mu+\lambda)|\Omega(0)|. \qquad (3.2)
\end{aligned}$$

In the following, we adopt a akin method in [4].

Case 1: If $\gamma \geq 3$, we have from (3.2)

$$I'(t) \leq 2(2\mu + \lambda)|\Omega(0)|,$$

which implies

$$I(t) \leq I(0) + 2(2\mu + \lambda)|\Omega(0)|t. \tag{3.3}$$

From (3.1) and (3.3), we get

$$\int_{\mathbb{R}^2} \rho^\gamma dx \leq \frac{\gamma - 1}{2} I(0)(1+t)^{-2} + (2\mu + \lambda)(\gamma - 1)|\Omega(0)|(1+t)^{-1}. \tag{3.4}$$

By conservation of mass and the Hölder inequality, we have

$$\begin{aligned}
\int_{\Omega(0)} \rho_0 &= \int_{\Omega(t)} \rho dx \\
&\leq (\int_{\Omega(t)} \rho^\gamma dx)^{\frac{1}{\gamma}} (\Omega(t))^{\frac{\gamma-1}{\gamma}} \\
&\leq (\Omega(0))^{\frac{\gamma-1}{\gamma}} \big(\frac{\gamma - 1}{2} I(0)(1+t)^{-2} + (2\mu + \lambda)(\gamma - 1)|\Omega(0)|(1+t)^{-1}\big)^{\frac{1}{\gamma}},
\end{aligned}$$

which yields that T must be finite.

Case 2: If $2 < \gamma < 3$, we get

$$\big((1+t)^{2\gamma-6} I(t)\big)' \leq 2(2\mu + \lambda)|\Omega(0)|(1+t)^{2\gamma-6},$$

which gives

$$(1+t)^{2\gamma-6} I(t) \leq I(0) + 2(2\mu + \lambda)|\Omega(0)|H(t), \tag{3.5}$$

where

$$H(t) = \begin{cases} \frac{1}{2\gamma-5}(1+t)^{2\gamma-5} & \text{if} \quad 2\gamma - 5 \neq 0; \\ \ln(1+t) & \text{if} \quad 2\gamma - 5 = 0. \end{cases}$$

Thus we obtain that

$$\int_{\mathbb{R}^2} \rho^\gamma dx \leq \frac{\gamma - 1}{2} I(0)(1+t)^{4-2\gamma} + (2\mu + \lambda)(\gamma - 1)|\Omega(0)|(1+t)^{4-2\gamma} H(t). \tag{3.6}$$

Similar to the estimates (3.3) – (3.4), (3.6) also imply that T must be finite. So we finish the proof. □

4 Acknowledgments

The authors are grateful to the referees and the editor whose comments and suggestions greatly improved the presentation of this paper. Moreover, the authors would like to thank Doctors Yongfu, Yang and Xiaoxin, Zheng for their helpful discussions during the preparation of this work. Tong Tang is partially supported by China NSF Grant 11171158 and NSF Grant 11271192, and the innovation project for graduate education of Jiangsu province (No. CXZZ13_0388). Zujin Zhang is partially supported by the Youth Natural Science Foundation of Jiangxi Province (20132BAB211007), the National Natural Science Foundation of China (11326138).

References

[1] D. F. Bian and B. L. Guo, Blow-up of smooth solutions to the isentropic compressible MHD equations, to appear in *Appl. Anal.* 2013.

[2] Y. Cho and B. J. Jin, Blow-up of the viscous heat-conducting compressible flow, *J. Math. Anal. Appl.*, 320 (2006), 819-826.

[3] D. P. Du, J. Y. Li and K. J. Zhang, Blow-up of smooth solution to the Naiver-Stokes equations for compressible isothermal fluids, *Commun. Math. Sci*, 11 (2011), 541-546.

[4] B. Duan, Z. Luo and Y. X. Zheng, Local existence of classical solutions to shallow water equations with Cauchy data containing vacuum, *SIAM J. Math. Anal.*, 44 (2012), 541-567.

[5] L. Hsiao and H. L. Li, Compressible Navier-Stokes-Poisson equations, *Acta Math. Sci. Ser. B Engl. Ed.*, 30 (2010), 1937-1948.

[6] F. Jiang and Z. Tan, Blow-up of viscous compressible reactive self-gravitating gas, *Acta Math. Appl. Sin. Engl. Ser.*, 28 (2012), 401-408.

[7] T. C. Sideris, Formation of singularities in three-dimensional compressible fluids, *Comm. Math. Phys.*, 101 (1985), 475-485.

[8] H. Z. Xie, Blow-up of smooth solutions to the Navier-Stokes-Poisson equations, *Math. Methods Appl. Sci.*, 34 (2011), 242-248.

[9] Z. P. Xin, Blow-up of smooth solution to the compressible Naiver-Stokes equations with compact density, *Comm. Pure Appl. Math.*, 51 (1998), 229-240.

[10] Z. P. Xin and W. Yan, On blowup of classical solutions to the compressible Naiver-Stokes equations, *Comm. Math. Phys.*, 321 (2013), 529-541.

[11] J. P. Yin and Z. Tan, Local existence of the strong solutions for the full Navier-Stokes-Poisson equations, *Nonlinear Anal.*, 71 (2009), 2397-2415.

[12] X. X. Zheng, Global well-posedness for the compressible Navier-Stokes-Poisson system in the L^p framework, *Nonlinear Anal.*, 75 (2012), 4156-4175.