Qualitative analysis of a diffusive three species model with the Holling-Tanner scheme*

Guangping Hu†, Xiaoling Li, Shiping Lu

School of Mathematics and Statistics, Nanjing University of Information Science and Technology, Nanjing 210044, P. R. China

May 30, 2014

Abstract

In this paper, a diffusive three species predator-prey model with a Leslie-Gower term is considered. The stability of the unique positive constant equilibrium for the reaction-diffusion system is obtained. In particular, we establish the existence and non-existence of non-constant positive steady states of this system. The results indicate, under suitable assumptions, that the large diffusivity in predators is helpful for the appearance of the non-constant positive steady states (stationary patterns).

Keywords: Diffusion; Predator-prey system; Priori estimates; Non-constant positive steady state.

AMS Subject Classification (2010): 35J47, 35Q92, 92B05, 92D25.

1 Introduction

In general, a spatially homogeneous predator-prey system can be modeled [8] as

$$\begin{cases} \frac{\mathrm{d}u}{\mathrm{d}t} = ru\left(1 - \frac{u}{K}\right) - vp(u), \\ \frac{\mathrm{d}v}{\mathrm{d}t} = v\left(-m + ep(u)\right), \end{cases} \tag{1.1}$$

herein all the parameters are positive constants, u and v are the population densities of prey and predator respectively. Function $p(u)$, the functional response of predator to prey density, refers to the change in the density of prey attached per unit time per predator as the prey density changes. To model various different processes of energy transfer in ecology, many kinds of $p(u)$ have been developed, which were proposed by different backgrounds and have significant dynamics in mathematical theory. Moreover, there have been many

*This work was supported by NNSF of China (No.11271197).

†Corresponding author (E-mail: hugp@nuist.edu.cn).

results on the dynamics of predator-prey systems and part of these results implies the dynamical differences of different functional responses, we refer to [9, 14, 16].

Another interesting formulation for the predator dynamics given by [11] and discussed by Leslie and Gower [12] and Hsu and Huang [7] is the following predator-prey model:

$$\begin{cases} \frac{\mathrm{d}u}{\mathrm{d}t} = ru\left(1 - \frac{u}{K}\right) - vp(u), \\ \frac{\mathrm{d}v}{\mathrm{d}t} = v\left(\delta - \beta\frac{v}{u}\right), \end{cases} \tag{1.2}$$

in which the interaction between species v and its prey u has been modeled by the Leslie-Gower scheme, also known as the Holling-Tanner scheme. The predator consumes the prey according to the functional response $p(u)$ and grows logistically with intrinsic growth rate δ. The term $\beta v/\delta u$ is called the Leslie-Gower term. The carrying capacity of the predator is proportional to the population size of the prey.

A major trend in theoretical work on predator-prey dynamics has been launched so as to derive more realistic models [16]. These models had to be more consistent with real phenomena, trying to keep to maximum the unavoidable increase in complexity of their mathematics. In most cases, this effort has been concentrated mainly on the response function form of the predator species [1, 2, 3, 25, 26] and on taking into account the relationship between predators and preys (for example, food chain or one resource and two consumers) [6, 23].

Systems (1.1) and (1.2) share the same prey equation, but they possess different numerical responses for the predator. In [21], the authors investigated models (1.1) and (1.2) with type I functional responses. They found that the differing numerical responses cause the models to have different dynamic behavior.

Combining the above considerations, in this paper, we are interested in a predator-prey model with one resource and two consumers. It describes a prey population u_1, which serves as collective food for two predators u_2 and u_3. We assume that the predators consume the prey according to the Holling type II functional response. The interaction between species u_2 and its prey u_1 has been modeled by the Volterra scheme, the interaction between species u_3 and its prey u_1 has been modeled by the Holling-Tanner scheme. Then the model is a system of three differential equations of the form

$$\begin{cases} \frac{\mathrm{d}u_1}{\mathrm{d}t} = ru_1\left(1 - \frac{u_1}{K}\right) - \frac{au_1u_2}{1+bu_1} - \frac{Au_1u_3}{1+Bu_1}, \\ \frac{\mathrm{d}u_2}{\mathrm{d}t} = u_2\left(-m + \frac{eu_1}{1+bu_1}\right), \\ \frac{\mathrm{d}u_3}{\mathrm{d}t} = u_3\left(\delta - \beta\frac{u_3}{u_1}\right). \end{cases} \tag{1.3}$$

Using the scaling:

$$rt \mapsto t,\ \frac{u_1}{K} \mapsto u_1,\ \frac{a}{r} \mapsto a,\ \frac{A}{r} \mapsto A,\ bK \mapsto b,\ BK \mapsto B,\ \frac{a}{r} \mapsto a,\ \frac{m}{r} \mapsto m,$$
$$\frac{eK}{m} \mapsto e,\ \frac{\delta}{r} \mapsto \delta,\ \frac{\beta}{\delta K} \mapsto \beta,$$

then system (1.3) can be simplified as follows

$$\begin{cases} \frac{\mathrm{d}u_1}{\mathrm{d}t} = u_1(1-u_1) - \frac{au_1u_2}{1+bu_1} - \frac{Au_1u_3}{1+Bu_1}, \\ \frac{\mathrm{d}u_2}{\mathrm{d}t} = mu_2\left(-1 + \frac{eu_1}{1+bu_1}\right), \\ \frac{\mathrm{d}u_3}{\mathrm{d}t} = \delta u_3\left(1 - \beta\frac{u_3}{u_1}\right). \end{cases} \tag{1.4}$$

It is obvious that system (1.4) has a constant positive solution if and only if

$$e > b,\ \ \beta(e-b-1)(e-b+B) > A(e-b). \tag{1.5}$$

Moreover, if (1.5) holds, we have $e-b-1>0$, and the positive equilibrium $\tilde{\mathbf{u}} = (\tilde{u}_1, \tilde{u}_2, \tilde{u}_3)^{\mathsf{T}}$ is uniquely given by

$$\tilde{u}_1 = \frac{1}{e-b},\ \ \tilde{u}_2 = \frac{\beta e(e-b-1)(e-b+B) - Ae(e-b)}{a\beta(e-b)^2(e-b+B)},\ \ \tilde{u}_3 = \frac{1}{\beta(e-b)}.$$

Taking into account the inhomogeneous distribution of the predators and their prey in different spatial locations within a fixed bounded domain $\Omega \subset \mathbb{R}^N$ with smooth boundary at any given time, and natural tendency of each species to diffuse to areas of smaller population concentration, we are led to consider the following reaction-diffusion system:

$$\begin{cases} u_{1t} - d_1\Delta u_1 = u_1(1-u_1) - \frac{au_1u_2}{1+bu_1} - \frac{Au_1u_3}{1+Bu_1} \triangleq F_1(\mathbf{u}), & x \in \Omega, t > 0, \\ u_{2t} - d_2\Delta u_2 = mu_2\left(-1 + \frac{eu_1}{1+bu_1}\right) \triangleq F_2(\mathbf{u}), & x \in \Omega, t > 0, \\ u_{3t} - d_3\Delta u_3 = \delta u_3\left(1 - \beta\frac{u_3}{u_1}\right) \triangleq F_3(\mathbf{u}), & x \in \Omega, t > 0, \\ \frac{\partial u_1}{\partial \nu} = \frac{\partial u_2}{\partial \nu} = \frac{\partial u_3}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ u_1(x,0) > 0,\ \ u_i(x,0) \geq 0, i = 2,3, & x \in \Omega. \end{cases} \tag{1.6}$$

In the above, ν is the outward unit normal vector of the boundary $\partial\Omega$. The homogeneous Neumann boundary condition indicates that the system is self-contained with zero population flux across the boundary. The positive constants d_1, d_2 and d_3 are the diffusion coefficients, and the initial data $u_1(x,0) > 0$, $u_2(x,0) \geq 0$ and $u_3(x,0) \geq 0$ are continuous functions. For the sake of convenience, we also denote $\mathbf{u} = (u_1, u_2, u_3)^{\mathsf{T}}$.

The major objective of our paper is to study the existence of non-constant positive steady states of (1.6). In fact, the existence of positive steady states of reactive diffusion predator-prey system under homogeneous Neumann boundary condition has been studied in many works. In the case that the consumer $u_2 = 0$, system (1.6) reduces to a two species predator-prey model which has received extensive concerns. For example, in [17], the authors verified the existence and nonexistence of positive nonconstant steady. In [18], they further studied the local and global stability of the unique positive equilibrium of this model. In [10], Wonlyul Ko and Kimun Ryu have studied the similar system of (1.6) while $u_3 = 0$, they investigated the asymptotic behavior of spatially inhomogeneous solutions and the local existence of periodic solutions. For the more related works on positive steady states of diffusive predator-prey systems, we can refer to [9, 17, 19, 20, 22, 23, 24] and references therein.

For the convenience, throughout this paper we always assume that (1.5) holds. In Section 2, we will show that if the parameters a, b, A, B, e and β satisfy

$$\left[ABe^2 + ab(e-b+B)^2\right][(\beta e(e-b-1)(e-b+B) - Ae(e-b)] < a\beta e^2\,(e-b+B)^3\,. \tag{1.7}$$

Then the equilibrium of (1.4) and the constant positive steady state of (1.6) are all locally asymptotically stable. In Section 3, we first give a priori upper and lower bounds for positive solutions of (1.6), and then we deal with existence and non-existence of nonconstant positive solutions of (1.6).

2 Stability of the positive equilibrium for (1.4) and (1.6)

In this section, we study the local asymptotic stability of the constant positive steady state $\tilde{\mathbf{u}}$ for reaction-diffusion system (1.6). Firstly, we rewritten problem (1.4) as

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = \mathbf{F}(\mathbf{u}), \tag{2.1}$$

and the linearization of it at $\tilde{\mathbf{u}} = (\tilde{u}_1, \tilde{u}_2, \tilde{u}_3)$ is

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = \mathbf{F}_{\mathbf{u}}(\tilde{\mathbf{u}})\mathbf{u}. \tag{2.2}$$

Theorem 2.1 *If the parameters a, b, A, B, e and β satisfy* (1.7)*, then the equilibrium solution $\tilde{\mathbf{u}} = (\tilde{u}_1, \tilde{u}_2, \tilde{u}_3)^T$ of* (1.4) *is locally asymptotically stable.*

Proof. Let

$$\mathbf{F}_{\mathbf{u}}(\tilde{\mathbf{u}}) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix},$$

the direct calculation yields

$$\begin{cases} a_{11} = -\tilde{u}_1 + \frac{ab\tilde{u}_2}{e^2\tilde{u}_1} + \frac{AB\tilde{u}_1\tilde{u}_2}{(1+B\tilde{u}_1)^2} \\ \qquad = \frac{\left[ABe^2+ab(e-b+B)^2\right][(\beta e(e-b-1)(e-b+B)-Ae(e-b)]-a\beta e^2(e-b+B)^3}{a\beta e^2(e-b)(e-b+B)^3}, \\ a_{12} = -\frac{a\tilde{u}_1}{1+b\tilde{u}_1} < 0, \quad a_{13} = -\frac{A\tilde{u}_1}{1+B\tilde{u}_1} < 0, \quad a_{21} = \frac{me\tilde{u}_2}{(1+b\tilde{u}_1)^2} > 0, \quad a_{22} = a_{23} = 0, \\ a_{31} = \frac{\delta\beta\tilde{u}_3^2}{\tilde{u}^2} > 0, \quad a_{32} = 0, \quad a_{33} = \delta\left(1 - \frac{2\beta\tilde{u}_3}{\tilde{u}_1}\right) = -\delta < 0. \end{cases} \tag{2.3}$$

The characteristic polynomial of $\mathbf{F}_{\mathbf{u}}(\tilde{\mathbf{u}})$ can be written as

$$\varphi(\lambda) = \lambda^3 + A_1\lambda^2 + A_2\lambda + A_3.$$

Under the condition (1.7), it is easy to check that $a_{11} < 0$ since $e > b$, so one can calculate that

$$\begin{cases} A_1 = -(a_{11} + a_{33}) > 0, \\ A_2 = a_{11}a_{33} - a_{12}a_{21} - a_{31}a_{13} > 0, \\ A_3 = -\{\det \mathbf{F_u}(\tilde{\mathbf{u}})\} = a_{12}a_{21}a_{33} > 0. \end{cases} \tag{2.4}$$

By simple computations, it follows that

$$A_1A_2 - A_3 = -a_{11}^2 a_{33} + a_{11}a_{12}a_{21} + a_{11}a_{13}a_{31} - a_{11}a_{33}^2 + a_{13}a_{31}a_{33} > 0.$$

From the Routh-Hurwitz criteria, we can conclude that the characteristic polynomial of $\mathbf{F_u}(\tilde{\mathbf{u}})$ has only roots with negative real parts, and so $\tilde{\mathbf{u}}$ is local asymptotically stable. $\square$

Next, we discuss the local stability of the positive constant steady state $\tilde{\mathbf{u}} = (\tilde{u}_1, \tilde{u}_2, \tilde{u}_3)^\mathsf{T}$ of (1.6). Before developing our argument, let us set up the following notations.

Let $0 = \mu_0 < \mu_1 < \mu_2 < \cdots$ be the eigenvalues of the operator $-\Delta$ on Ω with the homogeneous Neumann boundary condition, and $E(\mu_i)$ be the eigenspace corresponding to μ_i in $C^1(\bar{\Omega})$. Let $\mathbf{X} = \{\mathbf{u} \in \left[C^1(\overline{\Omega})\right]^3 \mid \partial_n \mathbf{u} = \mathbf{0} \text{ on } \partial\Omega\}$, $\{\phi_{ij}; j = 1, \cdots, \dim E(\mu_i)\}$ be an orthonormal basis of $E(\mu_i)$, and $\mathbf{X}_{ij} = \{\mathbf{c}\phi_{ij} \mid \mathbf{c} \in \mathbb{R}^3\}$. Then,

$$\mathbf{X} = \oplus_{i=1}^{\infty} \mathbf{X}_i \quad \text{and} \quad \mathbf{X}_i = \oplus_{j=1}^{\dim E(\mu_i)} \mathbf{X}_{ij}. \tag{2.5}$$

Theorem 2.2 *If the parameters a, b, A, B, e and β satisfy* (1.7), *then the constant positive steady state $\tilde{\mathbf{u}}$ of* (1.6) *is uniformly asymptotically stable.*

Proof. The linearization of (1.6) at $\tilde{\mathbf{u}}$ can be expressed by

$$\mathbf{u}_t = \mathscr{L}\mathbf{u},$$

where $\mathscr{L} = \mathscr{D}\Delta + \mathbf{F_u}(\tilde{\mathbf{u}})$, $\mathscr{D} = \mathrm{diag}(d_1, d_2, d_3)$.

For each $i \geq 0$, $\mathbf{X}_i$ is invariant under the operator $\mathscr{L}$, and λ is an eigenvalue of $\mathscr{L}$ on $\mathbf{X}_i$ if and only if it is an eigenvalue of the matrix $-\mu_i\mathscr{D} + \mathbf{F_u}(\tilde{\mathbf{u}})$. The characteristic polynomial of $-\mu_i\mathscr{D} + \mathbf{F_u}(\tilde{\mathbf{u}})$ is given by

$$\psi_i(\lambda) = \lambda^3 + B_{1i}\lambda^2 + B_{2i}\lambda + B_{3i},$$

with

$$\begin{aligned} B_{1i} &= \mu_i(d_1 + d_2 + d_3) + A_1, \\ B_{2i} &= \mu_i^2(d_1d_2 + d_1d_3 + d_2d_3) - \mu_i[a_{33}d_1 + (a_{11} + a_{33})d_2 + a_{11}d_3] + A_2, \\ B_{3i} &= \mu_i^3 d_1d_2d_3 - \mu_i^2(d_1d_2a_{33} + d_2d_3a_{11}) + \mu_i[(a_{11}a_{33} - a_{13}a_{31})d_2 - a_{12}a_{21}d_3] + A_3, \end{aligned}$$

where a_{ij} and A_i are given in (2.3) and (2.4), respectively. Using (1.7) we see that $a_{11} < 0$, in view of (2.3) and (2.4), it follows that B_{1i}, B_{2i}, $B_{3i} > 0$. Through a series of calculation, we have that

$$B_{1i}B_{2i} - B_{3i} = M_1\mu_i^3 + M_2\mu_i^2 + M_3\mu_i + A_1A_2 - A_3,$$

in which

$$\begin{aligned}
M_1 &= (d_1d_2+d_1d_3+d_2d_3)(d_1+d_2+d_3)-d_1d_2d_3>0,\\
M_2 &= -(a_{11}+a_{33})(d_1d_2+d_1d_3+d_2d_3)-[a_{33}d_1^2+a_{33}d_1d_2+a_{33}d_1d_3+(a_{11}+a_{33})\\
&\quad (d_1d_2+d_2d_3)+(a_{11}+a_{33})d_2+a_{11}(d_1d_3+d_2d_3)+a_{11}d_3^2]+d_1d_2a_{33}+d_2d_3a_{11}\\
&= -(a_{11}+a_{33})(2d_1d_2+d_1d_3+2d_2d_3+d_2^2)-(a_{33}d_1+a_{11}d_3)(d_1+d_3)>0,\\
M_3 &= (d_1+d_2+d_3)(a_{11}a_{33}-a_{12}a_{21}-a_{13}a_{31})+(a_{11}+a_{33})[a_{33}d_1+(a_{11}+a_{33})d_2\\
&\quad +a_{11}d_3]-[(a_{11}a_{33}-a_{13}a_{31})d_2-a_{12}a_{21}d_3]\\
&= (2a_{11}a_{33}-a_{12}a_{21}+a_{33}^2a_{13}a_{31})d_1+(2a_{11}a_{33}-a_{12}a_{21}+a_{11}^2+a_{33}^2)d_2\\
&\quad +(2a_{11}a_{33}-a_{13}a_{31}+a_{11}^2)d_3>0.
\end{aligned}$$

In view of (2.3), and notice that $A_1A_2-A_3>0$, we conclude that $B_{1i}B_{2i}-B_{3i}>0$ for all $i\geq 0$. It thus follows from the Routh-Hurwitz criterion that, for each $i\geq 0$, the three roots $\lambda_{i,1},\lambda_{i,2},\lambda_{i,3}$ of $\psi_i(\lambda)=0$ all have negative real parts. Finally, Theorem 5.1.1 in Henry [5] concludes the results. $\square$

3 Non-constant positive steady states

The main aim of this article is to study the steady states problem of (1.6), that is to say, the existence and non-existence of non-constant positive solutions of the corresponding elliptic system:

$$\begin{cases}
-d_1\Delta u_1=F_1(\mathbf{u}), & x\in\Omega,\\
-d_2\Delta u_2=F_2(\mathbf{u}), & x\in\Omega,\\
-d_3\Delta u_3=F_3(\mathbf{u}), & x\in\Omega,\\
\frac{\partial u_1}{\partial\nu}=\frac{\partial u_2}{\partial\nu}=\frac{\partial u_3}{\partial\nu}=0, & x\in\partial\Omega.
\end{cases}\tag{3.1}$$

To do this, it is necessary to establish a priori positive upper and lower bounds for positive solutions of (3.1).

3.1 A priori estimates

We first cite two lemmas which are due to Lin, Ni and Takagi [13], and Lou and Ni [15], respectively.

Lemma 3.1 *(Harnack Inequality [13]). Assume that $c\in C(\overline{\Omega})$ and let $w\in C^2(\Omega)\bigcap C^1(\overline{\Omega})$ be a positive solution to $\Delta w(x)+c(x)w(x)=0$ in Ω, $\frac{\partial w}{\partial\nu}=0$ on $\partial\Omega$. Then there exists a positive constant $C_*=C_*(\|c\|_\infty)$ such that $\max\limits_{\overline{\Omega}} w\leq C_*\min\limits_{\overline{\Omega}} w$.*

Lemma 3.2 *(Maximum Principle [15]). Suppose that $g\in C(\overline{\Omega}\times\mathbb{R}^1)$.*

(i) Assume that $w \in C^2(\Omega)\bigcap C^1(\overline{\Omega})$ *satisfies* $\Delta w(x) + g(x, w(x)) \geq 0$ *in* Ω, $\frac{\partial w}{\partial \nu} \leq 0$ *on* $\partial\Omega$*. If* $w(x_0) = \max_{\overline{\Omega}} w$*, then* $g(x_0, w(x_0)) \geq 0$.

(ii) Assume that $w \in C^2(\Omega)\bigcap C^1(\overline{\Omega})$ *satisfies* $\Delta w(x) + g(x, w(x)) \leq 0$ *in* Ω, $\frac{\partial w}{\partial \nu} \geq 0$ *on* $\partial\Omega$*. If* $w(x_0) = \min_{\overline{\Omega}} w$*, then* $g(x_0, w(x_0)) \leq 0$.

Note that the positive solutions of (3.1) are contained in $C^2(\overline{\Omega})\times C^2(\overline{\Omega})$ by the standard regularity theory for elliptic equations [4], and so Lemmas 3.1 and 3.2 can be applied to system (3.1). For notational convenience, we write $\Lambda = \Lambda(a, b, A, B, m, e, \delta, \beta)$ in the sequel.

Theorem 3.3 *(Upper bounds). For any positive solution* (u_1, u_2, u_3) *of* (3.1)*,*

$$\max_{\overline{\Omega}} u_1 \leq 1, \quad \max_{\overline{\Omega}} u_2 \leq \frac{med_1 + ed_2}{ad_2}, \quad \max_{\overline{\Omega}} u_3 \leq \frac{1}{\beta}. \tag{3.2}$$

Proof. A direct application of Lemma 3.2 to the first equation of (3.1) yields the first inequality of (3.2). Since $0 < u_3 \leq \frac{1}{\beta}\|u_1\|_\infty \leq \frac{1}{\beta}$, we have $u_3 \leq \frac{1}{\beta}$ in $\overline{\Omega}$.

Let $w = med_1u_1 + ad_2u_2$, we can obtain

$$\begin{cases} -\Delta w = meu_1(1-u_1) - \frac{Ameu_1u_3}{1+Bu_1} - mau_2, & x \in \Omega, \\ \frac{\partial w}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases}$$

Let $w(x_0) = \max_{\overline{\Omega}} w(x)$. By the application of Lemma 3.2, we have

$$mau_2(x_0) \leq meu_1(1-u_1) - \frac{Ameu_1u_3}{1+Bu_1} \leq me.$$

Consequently,

$$\max_{\overline{\Omega}} u_2(x) \leq \frac{1}{ad_2}\max_{\overline{\Omega}} w(x) = \frac{w(x_0)}{ad_2} = \frac{med_1u_1(x_0) + ad_2u_2(x_0)}{ad_2} \leq \frac{med_1 + ed_2}{ad_2}.$$

The proof is completed. □

Theorem 3.4 *(Lower bounds). For any* $d > 0$*,* $d_2 > 0$ *and* $d_3 > 0$*, if*

$$d_1 \geq d \quad \text{and} \quad \frac{\beta - A}{\beta} > \frac{1}{e-b}.$$

Then there exist positive constant $c_i = c_i(d, \Lambda)$*,* $i = 1, 2, 3$*, such that any positive solution* (u_1, u_2, u_3) *of* (3.1) *satisfies*

$$\min_{\overline{\Omega}} u_i \geq c_i, \quad i = 1, 2, 3. \tag{3.3}$$

Proof. Let $c_1(x) = d_1^{-1}\left(1 - u_1 - \frac{au_2}{1+bu_1} - \frac{Au_3}{1+Bu_1}\right)$. In view of (3.2), there exists positive constants $\overline{C}(d,\Lambda)$ such that the inequalities $\|c_1(x)\|_\infty \le \overline{C}$ for any $d > 0$, if $d_1 \ge d$. The Harnack inequality in Lemma 3.1 shows that there exists a positive constant $C_* = C_*(d,\Lambda)$ such that

$$\max_{\overline{\Omega}} u_1 \le C_* \min_{\overline{\Omega}} u_1. \tag{3.4}$$

On the other hand, by integrating the second equation in (3.1), we have

$$\int_\Omega m u_2 \left(-1 + \frac{eu_1}{1+bu_1}\right) dx = 0,$$

which implies that there exists a point x_1 such that $-1 + \frac{eu_1(x_1)}{1+bu_1(x_1)} = 0$. So $u_1(x_1) \ge \frac{1}{e}$ and

$$\min_{\overline{\Omega}} u_1 \ge \frac{1}{C_*}\max_{\overline{\Omega}} u_1 \ge \frac{1}{C_*}u_1(x_1) \ge \frac{1}{C_* e} \triangleq c_1.$$

Let $u_3(x_2) = \min_{\overline{\Omega}} u_3(x)$, by Lemma 3.2, it is clear that $1 - \beta\frac{u_3(x_2)}{u_1(x_2)} \le 0$, and so

$$\min_{\overline{\Omega}} u_3 \ge \frac{1}{\beta}u_1(x_2) \ge \frac{1}{\beta}\min_{\overline{\Omega}} u_1 \ge \frac{1}{C_*\beta e} \triangleq c_3.$$

Now we need to estimate the positive lower bound of u_2. Suppose, on the contrary, that (3.3) does not hold for $i = 2$. Then there exists a sequence $\{d_{1n}, d_{2n}, d_{3n}\}_{n=1}^\infty$ with $(d_{1n}, d_{2n}, d_{3n}) \in [d,\infty) \times (0,\infty) \times (0,\infty)$ such that the corresponding positive solutions (u_{1n}, u_{2n}, u_{3n}) of (3.1) satisfy

$$\begin{cases} -d_{1n}\Delta u_{1n} = u_{1n}(1 - u_{1n}) - \frac{au_{1n}u_{2n}}{1+bu_{1n}} - \frac{Au_{1n}u_{3n}}{1+Bu_{1n}}, & x \in \Omega, \\ -d_{2n}\Delta u_{2n} = mu_{2n}\left(-1 + \frac{eu_{1n}}{1+bu_{1n}}\right), & x \in \Omega, \\ -d_{3n}\Delta u_{3n} = \delta u_{3n}\left(1 - \beta\frac{u_{3n}}{u_{1n}}\right), & x \in \Omega, \\ \partial_\nu u_{1n} = \partial_\nu u_{2n} = \partial_\nu u_{3n} = 0, & x \in \partial\Omega \end{cases} \tag{3.5}$$

and

$$\min_{\overline{\Omega}} u_{1n} \ge c_1, \ \ \min_{\overline{\Omega}} u_{2n} \to 0, \ \ \min_{\overline{\Omega}} u_{3n} \ge c_3 \ \text{ as } n \to \infty. \tag{3.6}$$

By (3.4), it is clear that

$$\max_{\overline{\Omega}} u_{2n} \to 0 \ \text{ as } n \to \infty. \tag{3.7}$$

For (3.5), the standard regularity theorem for the elliptic equations yields that there exists a subsequence of $\{(u_{1n}, u_{2n}, u_{3n})\}_{n=1}^\infty$, which we shall still denote by $\{(u_{1n}, u_{2n}, u_{3n})\}_{n=1}^\infty$, and three non-negative functions $u_1, u_2, u_3 \in C^2(\overline{\Omega})$, such that u_{in} converges uniformly to

u_i in $C^2(\overline{\Omega})$ as $n \to \infty$, $i = 1, 2, 3$. Thus, we may assume, by passing to a subsequence if necessary, that $(u_{1n}, u_{2n}, u_{3n}) \to (u_1, u_2, u_3)$ as $n \to \infty$. Moreover, we assume that $(d_{1n}, d_{2n}, d_{3n}) \to (\overline{d}_1, \overline{d}_2, \overline{d}_3) \in [d, \infty) \times (0, \infty) \times (0, \infty)$.

By (3.6) and (3.7), we note that $u_1 > 0$, $u_2 \equiv 0$, $u_3 > 0$.

Let $n \to \infty$ in the first equation of (3.5) we have

$$-\overline{d}_1 \Delta u_1 = u_1 \left[(1 - u_1) - \frac{Au_3}{1 + Bu_1} \right], \; x \in \Omega; \quad \partial_\nu u_1 = 0, \; x \in \partial\Omega. \tag{3.8}$$

Let $x_3 \in \overline{\Omega}$ be a point such that $u_1(x_3) = \min_{\overline{\Omega}} u_1(x)$. Applying Lemma 3.2 to (3.8), it yields

$$1 - u_1(x_3) - \frac{Au_3(x_3)}{1 + Bu_1(x_3)} \leq 0,$$

and then $u_1(x_3) \geq 1 - \frac{Au_3(x_3)}{1+Bu_1(x_3)} \geq 1 - Au_3(x_3) \geq 1 - A \max_{\overline{\Omega}} u_3 \geq \frac{\beta - A}{\beta}$.

As $\frac{\beta-A}{\beta} > \frac{1}{e-b}$, and hence $u_1 \geq \min_{\overline{\Omega}} u_1(x) > \frac{1}{e-b}$, so we can see that

$$-1 + \frac{eu_{1n}}{1 + bu_{1n}} > 0 \;\text{ on } \overline{\Omega}, \quad \text{for all } n \gg 1.$$

Integrating the differential equation for u_{2n} over Ω by parts, we have

$$0 = d_{2n} \int_{\partial\Omega} \partial_\nu u_{2n} ds = -d_{2n} \int_{\Omega} \Delta u_{2n} dx = \int_{\Omega} m u_{2n} \left(-1 + \frac{eu_{1n}}{1 + bu_{1n}} \right) dx > 0, \quad \text{for all } n \gg 1,$$

which is a contradiction. The proof is completed. $\square$

3.2 Non-existence of non-constant positive steady states

Theorem 3.5 *Let d_2^* and d_3^* are fixed positive constants and satisfy $\mu_1 d_2^* > \frac{m(e-b-1)}{1+b}$ and $\mu_1 d_3^* > \delta$. Then there exists a positive constant $D_1 = D_1(\Lambda, d_2^*, d_3^*)$ such that, when $d_1 \geq D_1$, $d_2 \geq d_2^*$ and $d_3 \geq d_3^*$, problem* (3.1) *has no non-constant positive solution.*

Proof. Assume that $\mathbf{u} = (u_1, u_2, u_3)$ is a positive solution of (3.1). Let $\bar{\varphi} = \frac{1}{|\Omega|} \int_\Omega \varphi dx$ for any $\varphi \in L^1(\Omega)$. Multiplying the differential equation (3.1) by $\mathbf{u} - \bar{\mathbf{u}}$, and then integrating over Ω by parts, we have

$$\begin{aligned}
&\sum_{i=1}^{3} \int_\Omega d_i |\nabla u_i|^2 dx = \sum_{i=1}^{3} \int_\Omega (F_i(\mathbf{u}) - F_i(\bar{\mathbf{u}}))(u_i - \bar{u}_i) dx \\
&= \int_\Omega \big\{ (u_1 - \bar{u}_1)^2 [1 - (u_1 + \bar{u}_1)] \\
&\quad - \frac{au_2(u_1 - \bar{u}_1)^2 + (a\bar{u}_1 + abu_1\bar{u}_1)(u_1 - \bar{u}_1)(u_2 - \bar{u}_2)}{(1 + bu_1)(1 + b\bar{u}_1)}
\end{aligned}$$

$$
\begin{aligned}
&-\frac{Au_3(u_1-\bar u_1)^2+(A\bar u_1+ABu_1\bar u_1)(u_1-\bar u_1)(u_3-\bar u_3)}{(1+Bu_1)(1+B\bar u_1)}\\
&-m(u_2-\bar u_2)^2+\frac{meu_2(u_1-\bar u_1)(u_2-\bar u_2)+me(bu_1\bar u_1+\bar u_1)(u_2-\bar u_2)^2}{(1+bu_1)(1+b\bar u_1)}\\
&+\frac{\delta\beta\bar u_3^2(u_1-\bar u_1)(u_3-\bar u_3)}{u_1\bar u_1}+\left(\delta-\frac{\delta\beta(u_3+\bar u_3)}{u_1}\right)(u_3-\bar u_3)^2\bigg\}\,dx\\
\le&\int_\Omega\left\{(1+C)(u_1-\bar u_1)^2+\left(\frac{m(e-b-1)}{1+b}+\varepsilon\right)(u_2-\bar u_2)^2+(\delta+\varepsilon)(u_3-\bar u_3)^2\right\}dx
\end{aligned}
\tag{3.9}
$$

for some positive constants $C=C(\Lambda,d_2^*,d_3^*,\varepsilon)$ and an arbitrary small positive constant ε.

In view of the Poincaré inequality $\mu_1\int_\Omega(f-\bar f)^2\le\int_\Omega\mid\nabla f\mid^2dx$, it follows from (3.9) that

$$
\begin{aligned}
\mu_1\sum_{i=1}^{3}\int_\Omega d_i(u_i-\bar u_i)^2dx\ \ \le&\ \ \int_\Omega\left\{(1+C)(u_1-\bar u_1)^2+\left(\frac{m(e-b-1)}{1+b}+\varepsilon\right)\right.\\
&\left.(u_2-\bar u_2)^2+(\delta+\varepsilon)(u_3-\bar u_3)^2\right\}dx.
\end{aligned}
\tag{3.10}
$$

Choose $\varepsilon>0$ very small such that $\mu_1d_2^*\ge\frac{m(e-b-1)}{1+b}+\varepsilon$, $\mu_1d_3^*\ge\delta+\varepsilon$. Let $D_1\triangleq\mu_1^{-1}(1+C)$, then we can conclude that $(u_1,u_2,u_3)=(\bar u_1,\bar u_2,\bar u_3)$, it is a constant solution of (3.1). This completes the proof . $\square$

3.3 Existence of non-constant positive steady states

In this subsection, we discuss the existence of non-constant positive solutions to (3.1) when the diffusion coefficient d_2 varies while the diffusion coefficients d_1 and d_3 are kept fixed by using the Leray-Schauder degree theory. In view of Theorem 2.2, we see that there might be no non-constant positive solutions of (3.1) if $a_{11}=a_{11}(\Lambda)<0$, and so we shall restrict this discuss to the case where the parameters Λ be fixed and satisfy $a_{11}>0$.

Let $\mathbf{X}$ be as in Section 2, and denote

$$
\begin{aligned}
&\mathbf{X}^+=\left\{\mathbf{u}\in\mathbf{X}\mid\mathbf{u}>\mathbf{0}\text{ on }\overline\Omega\right\},\\
&\mathscr{B}(c)=\left\{\mathbf{u}\in\mathbf{X}\mid c^{-1}<u_i<c\text{ on }\overline\Omega,\ i=1,2,3\right\},
\end{aligned}
$$

where c is a positive constant that is guaranteed to exist by Theorems 3.3 and 3.4.

The steady states problem (3.1) can be rewritten as

$$
\begin{cases}
-\mathscr{D}\Delta\mathbf{u}=\mathbf{F}(\mathbf{u}), & x\in\Omega,\\
\partial_\nu\mathbf{u}=\mathbf{0}, & x\in\partial\Omega.
\end{cases}
\tag{3.11}
$$

Thus $\mathbf{u}$ is a positive solution of (3.11) if and only if

$$
\mathbf{\Phi}(\mathbf{u})\triangleq\mathbf{u}-(\mathbf{I}-\Delta)^{-1}\left\{\mathscr{D}^{-1}\mathbf{F}(\mathbf{u})+\mathbf{u}\right\}=\mathbf{0}\ \text{ in }\mathbf{X}^+,
$$

where $(\mathbf{I}-\Delta)^{-1}$ is the inverse of $\mathbf{I}-\Delta$ in $\mathbf{X}$, subject to homogeneous Neumann boundary condition. Since $\mathbf{\Phi}(\cdot)$ is a compact perturbation of the identity operator, for any $\mathscr{B}=\mathscr{B}(c)$, the Leray-Schauder degree $\deg(\mathbf{\Phi}(\cdot),\mathbf{0},\mathscr{B})$ is well-defined if $\mathbf{\Phi}(\mathbf{u})\neq\mathbf{0}$ on $\partial\mathscr{B}$.

Further, we note that

$$D_{\mathbf{u}}\mathbf{\Phi}(\tilde{\mathbf{u}})=\mathbf{I}-(\mathbf{I}-\Delta)^{-1}\left\{\mathscr{D}^{-1}\mathbf{F}_{\mathbf{u}}(\tilde{\mathbf{u}})+\mathbf{I}\right\},$$

we recall that if $D_{\mathbf{u}}\mathbf{\Phi}(\tilde{\mathbf{u}})$ is invertible, the index of $\mathbf{\Phi}$ at $\tilde{\mathbf{u}}$ is defined as $\mathrm{index}(\mathbf{\Phi}(\cdot),\tilde{\mathbf{u}})=(-1)^{\gamma}$, where γ is the total number of eigenvalues of $D_{\mathbf{u}}\mathbf{\Phi}(\tilde{\mathbf{u}})$ with negative real parts (counting multiplicities). Then the degree $\deg(\mathbf{\Phi}(\cdot),\mathbf{0},\mathscr{B})$ is equal to the sum of the indexes over all isolated solutions to $\mathbf{\Phi}=\mathbf{0}$ in $\mathscr{B}(c)$, provided that $\mathbf{\Phi}\neq\mathbf{0}$ on $\partial\mathscr{B}$.

In order to calculate γ, we employ the eigenspaces of $-\Delta$. We refer to the decomposition (2.5) in our following discussion of the eigenvalues of $D_{\mathbf{u}}\mathbf{\Phi}(\tilde{\mathbf{u}})$. First, we know $\mathbf{X}_{ij}$ is invariant under $D_{\mathbf{u}}\mathbf{\Phi}(\tilde{\mathbf{u}})$ for each $i\in\mathbb{N}$ and each $j\in[1,\dim E(\mu_i)]\cap\mathbb{N}$, i.e., $D_{\mathbf{u}}\mathbf{\Phi}(\tilde{\mathbf{u}})\mathbf{u}\in\mathbf{X}_{ij}$ for any $\mathbf{u}\in\mathbf{X}_{ij}$. Hence, λ is an eigenvalue of $D_{\mathbf{u}}\mathbf{\Phi}(\tilde{\mathbf{u}})$ on $\mathbf{X}_{ij}$ if and only if it is an eigenvalue of the matrix

$$\mathbf{I}-\frac{1}{1+\mu_i}\left[\mathscr{D}^{-1}\mathbf{F}_{\mathbf{u}}(\tilde{\mathbf{u}})+\mathbf{I}\right]=\frac{1}{1+\mu_i}\left[\mu_i\mathbf{I}-\mathscr{D}^{-1}\mathbf{F}_{\mathbf{u}}(\tilde{\mathbf{u}})\right].$$

Thus, $D_{\mathbf{u}}\mathbf{\Phi}(\tilde{\mathbf{u}})$ is invertible if and only if, for all $i\geq 0$, the matrix $\mathbf{I}-\frac{1}{1+\mu_i}\left[\mathscr{D}^{-1}\mathbf{F}_{\mathbf{u}}(\tilde{\mathbf{u}})+\mathbf{I}\right]$ is non-singular. Denote

$$H(\mu)=H(\tilde{\mathbf{u}};\mu)\triangleq\det\left\{\mu\mathbf{I}-\mathscr{D}^{-1}\mathbf{F}_{\mathbf{u}}(\tilde{\mathbf{u}})\right\}=\frac{1}{d_1d_2d_3}\det\left\{\mu\mathscr{D}-\mathbf{F}_{\mathbf{u}}(\tilde{\mathbf{u}})\right\},\tag{3.12}$$

we note, furthermore, that if $H(\mu_i)\neq 0$, then for each $1\leq j\leq\dim E(\mu_i)$, the number of negative eigenvalues of $D_{\mathbf{u}}\mathbf{\Phi}(\tilde{\mathbf{u}})$ on $\mathbf{X}_{ij}$ is odd if and only if $H(\mu_i)<0$. In conclusion, we have the following result. It also can be found in [6, 19, 23].

Proposition 3.6 *Suppose that, for all $i\geq 0$, the matrix $\mu_i\mathbf{I}-\mathscr{D}^{-1}\boldsymbol{F}_{\boldsymbol{u}}(\tilde{\mathbf{u}})$ is non-singular. Then*

$$\mathrm{index}\left(\mathbf{\Phi}(\cdot),\tilde{\mathbf{u}}\right)=(-1)^{\sigma},\quad \textit{where } \sigma=\sum_{i\geq 0,H(\mu_i)<0}\dim E(\mu_i).$$

According to this proposition, we should consider carefully the sign of $H(\mu_i)$ in order to calculate $\mathrm{index}\left(\mathbf{\Phi}(\cdot),\tilde{\mathbf{u}}\right)$. A direct calculation gives

$$\begin{aligned}\det\left\{\mu\mathscr{D}-\mathbf{F}_{\mathbf{u}}(\tilde{\mathbf{u}})\right\} &= A_3(\Lambda,d_2)\mu^3+A_2(\Lambda,d_2)\mu^2+A_1(\Lambda,d_2)\mu-\det\left\{\mathbf{F}_{\mathbf{u}}(\tilde{\mathbf{u}})\right\}\\ &\triangleq \mathcal{A}(\Lambda,d_2;\mu),\end{aligned}\tag{3.13}$$

with

$$A_3(\Lambda,d_2)=d_1d_2d_3,\quad A_2(\Lambda,d_2)=-(a_{33}d_1d_2+a_{11}d_2d_3),$$

$$A_1(\Lambda, d_2) = (a_{11}a_{33} - a_{13}a_{31})d_2 - a_{12}a_{21}d_3, \quad \det\{\mathbf{F_u}(\tilde{\mathbf{u}})\} = -a_{12}a_{21}a_{33},$$

where a_{ij} are as given in (2.3).

We consider the dependence of $\mathcal{A}$ on d_2. Let $\tilde{\mu}_1(d_2)$, $\tilde{\mu}_2(d_2)$, $\tilde{\mu}_3(d_2)$ be the three roots of $\mathcal{A}(\Lambda, d_2; \mu) = 0$ with $\mathrm{Re}\{\tilde{\mu}_1(d_2)\} \leq \mathrm{Re}\{\tilde{\mu}_2(d_2)\} \leq \mathrm{Re}\{\tilde{\mu}_3(d_2)\}$, then $\tilde{\mu}_1(d_2)\tilde{\mu}_2(d_2)\tilde{\mu}_3(d_2) = \det\{\mathbf{F_u}(\tilde{\mathbf{u}})\}$. Note that $\det\{\mathbf{F_u}(\tilde{\mathbf{u}})\} < 0$ and $A_3(\Lambda, d_2) > 0$. Thus, one of $\tilde{\mu}_1(d_2)$, $\tilde{\mu}_2(d_2)$, $\tilde{\mu}_3(d_2)$ is real and negative, and the product of the other two is positive.

Perform the following limits:

$$\begin{aligned}\lim_{d_2\to\infty} \frac{\mathcal{A}(\Lambda, d_2; \mu)}{d_2} &= \lim_{d_2\to\infty} \frac{A_3(\Lambda, d_2)\mu^3 + A_2(\Lambda, d_2)\mu^2 + A_1(\Lambda, d_2)\mu}{d_2} \\ &= \mu\left[d_1d_3\mu^2 - (a_{33}d_1 + a_{11}d_3)\mu + (a_{11}a_{33} - a_{13}a_{31})\right].\end{aligned}$$

Note that $\mathscr{A} \triangleq (a_{33}d_1 + a_{11}d_3)^2 - 4d_1d_3(a_{11}a_{33} - a_{13}a_{31}) > 0$ provided $a_{11}a_{33} - a_{13}a_{31} < 0$. Furthermore, if the parameters Λ, d_1 and d_3 satisfy $a_{11}d_3 + a_{33}d_1 = a_{11}d_3 - \delta d_1 > 0$, we have $a_{11} > \frac{\delta d_1}{d_3}$. So we can establish the following proposition.

Proposition 3.7 *Assume $a_{11} > \frac{\delta d_1}{d_3}$ and $a_{11}a_{33} - a_{13}a_{31} < 0$. Then there exists a positive constant D_2, such that when $d_2 \geq D_2$, the three roots $\tilde{\mu}_1(d_2)$, $\tilde{\mu}_2(d_2)$, $\tilde{\mu}_3(d_2)$ of $\mathcal{A}(\Lambda, d_2; \mu) = 0$ are all real and satisfy*

$$\begin{cases} \lim\limits_{d_2\to\infty} \tilde{\mu}_1(d_2) = \frac{a_{11}d_3 + a_{33}d_1 - \sqrt{\mathscr{A}}}{2d_1d_3} < 0, \\ \lim\limits_{d_2\to\infty} \tilde{\mu}_2(d_2) = 0, \\ \lim\limits_{d_2\to\infty} \tilde{\mu}_3(d_2) = \frac{a_{11}d_3 + a_{33}d_1 + \sqrt{\mathscr{A}}}{2d_1d_3} \triangleq \bar{\mu} > 0. \end{cases} \tag{3.14}$$

Moreover,

$$\begin{cases} -\infty < \tilde{\mu}_1(d_2) < 0 < \tilde{\mu}_2(d_2) < \tilde{\mu}_3(d_2), \\ \mathcal{A}(\Lambda, d_2; \mu) < 0, \quad \textit{when } \mu \in (-\infty, \tilde{\mu}_1(d_2)) \cup (\tilde{\mu}_2(d_2), \tilde{\mu}_3(d_2)), \\ \mathcal{A}(\Lambda, d_2; \mu) > 0, \quad \textit{when } \mu \in (\tilde{\mu}_1(d_2), \tilde{\mu}_2(d_2)) \cup (\tilde{\mu}_3(d_2), \infty). \end{cases} \tag{3.15}$$

Now we establish the global existence of non-constant positive solutions of (3.1) with respect to the diffusion coefficient d_2, as the other parameters are all fixed positive constants. Our result is as follows.

Theorem 3.8 *Assume that the parameters Λ, d_1 and d_3 are fixed, $\frac{\beta - A}{\beta} > \frac{1}{e-b}$, $a_{11} > \frac{\delta d_1}{d_3}$ and $a_{11}a_{33} - a_{13}a_{31} < 0$ hold. If $\bar{\mu} \in (\mu_n, \mu_{n+1})$ for some $n \geq 1$, and the sum $\sigma_n = \sum_{i=1}^n \dim E(\mu_i)$ is odd, then there exists a positive constant D_2 such that, if $d_2 \geq D_2$, problem (3.1) has at least one non-constant positive solution.*

Proof. By Proposition 3.7, there exists a positive constant D_2, if $d_2 \geq D_2$, (3.15) holds and

$$\tilde{\mu}_1(d_2) < 0 = \mu_0 < \tilde{\mu}_2(d_2) < \mu_1, \quad \tilde{\mu}_3(d_2) \in (\mu_n, \mu_{n+1}). \tag{3.16}$$

We will prove that for any $d_2 \geq D_2$, (3.1) has at least one non-constant positive solution. By way of contradiction, assume that the assertion is not true for some $d_2 = \tilde{d}_2 \geq D_2$. By using the homotopy argument, we can derive a contradiction in the sequel.

Fixing $d_2 = \tilde{d}_2$, taking $d_2^* = d_3^* = \frac{m(e-b-1)}{\mu_1(1+b)} + \frac{\delta}{\mu_1}$ in Theorem 3.5, we obtain a positive constant $D_1 = D_1(\Lambda, d_2^*, d_3^*)$. Fix $\hat{d}_2 \geq d_2^*$, $\hat{d}_3 \geq d_3^* + d_3$ and $\hat{d}_1 \geq D_1(\Lambda, d_2^*, d_3^*)$. For $t \in [0,1]$, define $\mathscr{D}(t) = \mathrm{diag}(d_1(t), d_2(t), d_3(t))$ with $d_i(t) = td_i + (1-t)\hat{d}_i$, $i = 1,2,3$, and consider the problem

$$\begin{cases} -\mathscr{D}(t)\Delta \mathbf{u} = \mathbf{F}(\mathbf{u}), & x \in \Omega, \\ \partial_\nu \mathbf{u} = \mathbf{0}, & x \in \partial\Omega. \end{cases} \tag{3.17}$$

Then $\mathbf{u}$ is a non-constant positive solution of (3.1) if and only if it is a positive solution of (3.17) for $t = 1$. It is obvious that $\tilde{\mathbf{u}}$ is the unique constant positive solution of (3.17) for any $t \in [0,1]$. We know that for any $t \in [0,1]$, $\mathbf{u}$ is a positive solution of (3.17) if and only if

$$\mathbf{\Phi}(t;\mathbf{u}) \triangleq \mathbf{u} - (\mathbf{I} - \Delta)^{-1}\left\{\mathscr{D}^{-1}(t)\mathbf{F}(\mathbf{u}) + \mathbf{u}\right\} = \mathbf{0} \ \text{ in } \mathbf{X}^+.$$

It is obvious that $\mathbf{\Phi}(1;\mathbf{u}) = \mathbf{\Phi}(\mathbf{u})$, Theorem 3.5 indicates that $\mathbf{\Phi}(0;\mathbf{u}) = \mathbf{0}$ has only the positive solution $\tilde{\mathbf{u}}$ in $\mathbf{X}^+$. By a direct computation, we have

$$D_{\mathbf{u}}\mathbf{\Phi}(t;\tilde{\mathbf{u}}) = \mathbf{I} - (\mathbf{I} - \Delta)^{-1}\left\{\mathscr{D}^{-1}(t)\mathbf{F}_{\mathbf{u}}(\tilde{\mathbf{u}}) + \mathbf{I}\right\}.$$

In particular,

$$\begin{aligned} D_{\mathbf{u}}\mathbf{\Phi}(0;\tilde{\mathbf{u}}) &= \mathbf{I} - (\mathbf{I} - \Delta)^{-1}\left\{\widehat{\mathscr{D}}^{-1}\mathbf{F}_{\mathbf{u}}(\tilde{\mathbf{u}}) + \mathbf{I}\right\}, \\ D_{\mathbf{u}}\mathbf{\Phi}(1;\tilde{\mathbf{u}}) &= \mathbf{I} - (\mathbf{I} - \Delta)^{-1}\left\{\mathscr{D}^{-1}\mathbf{F}_{\mathbf{u}}(\tilde{\mathbf{u}}) + \mathbf{I}\right\} = D_{\mathbf{u}}\mathbf{\Phi}(\tilde{\mathbf{u}}), \end{aligned}$$

where $\widehat{\mathscr{D}} = \mathrm{diag}(\hat{d}_1, \hat{d}_2, \hat{d}_3)$. From (3.12) and (3.13), we already know that

$$H(\mu) = \frac{1}{d_1 d_2 d_3}\mathcal{A}(\Lambda, d_2; \mu). \tag{3.18}$$

For $t = 1$, by (3.15), (3.16) and (3.18), we have

$$\begin{cases} H(\mu_0) = H(0) > 0, \\ H(\mu_i) < 0 & \text{when } 1 \leq i \leq n, \\ H(\mu_{i+1}) > 0 & \text{when } i \geq n+1. \end{cases}$$

Thus, 0 is not an eigenvalue of the matrix $\mu_i \mathbf{I} - \mathscr{D}^{-1}\mathbf{F}_{\mathbf{u}}(\tilde{\mathbf{u}})$ for all $i \geq 0$, and

$$\sum_{i\geq 0, H(\mu_i)<0} \mathrm{dim}E(\mu_i) = \sum_{i=1}^{n} \mathrm{dim}E(\mu_i) = \sigma_n$$

is odd. Thanks to Proposition 3.6, we have

$$\operatorname{index}\left(\mathbf{\Phi}(1;\cdot),\tilde{\mathbf{u}}\right)=(-1)^{\gamma}=(-1)^{\sigma_n}=-1. \tag{3.19}$$

Now, we prove that

$$\operatorname{index}\left(\mathbf{\Phi}(0;\cdot),\tilde{\mathbf{u}}\right)=(-1)^{0}=1. \tag{3.20}$$

Fix b_0 such that $b<b_0<e$ and $a_{11}(b_0)<0$. Define $b(s)=sb+(1-s)b_0$ for $s\in[0,1]$, and consider problem (3.1), where (d_1,d_2,d_3) and b are replaced by $(\hat{d}_1,\hat{d}_2,\hat{d}_3)$ and $b(s)$, respectively. Precisely, we label this problem as (3.1s), and denote the corresponding non-linear term $\mathbf{F}(\mathbf{u})$ by $\mathbf{F}(s;\mathbf{u})$. As $b\le b(s)\le e$ for any $s\in[0,1]$. Similar to the proof of Theorem 3.5 we have that $\tilde{\mathbf{u}}$ is only positive solution of (3.1s) for $s\in[0,1]$. Same as above, we define

$$\widetilde{\mathbf{\Phi}}(s;\mathbf{u})\triangleq\mathbf{u}-(\mathbf{I}-\Delta)^{-1}\left\{\widehat{\mathscr{D}}^{-1}\mathbf{F}(s;\mathbf{u})+\mathbf{u}\right\}=\mathbf{0}\ \text{ in } \mathbf{X}^{+}.$$

Then $\widetilde{\mathbf{\Phi}}(1;\cdot)=\mathbf{\Phi}(0;\cdot)$, and $\tilde{\mathbf{u}}$ is the only positive solution of $\widetilde{\mathbf{\Phi}}(s;\mathbf{u})=0$ for all $s\in[0,1]$. The homotopy invariance of the topological degree asserts that

$$\operatorname{index}(\widetilde{\mathbf{\Phi}}(1;\cdot),\tilde{\mathbf{u}})=\operatorname{index}(\widetilde{\mathbf{\Phi}}(0;\cdot),\tilde{\mathbf{u}}). \tag{3.21}$$

Since $b(0)=b_0$ and b_0 satisfies $a_{11}(b_0)<0$, then $\det(\mu_i\widehat{\mathscr{D}}-\mathbf{F}_{\mathbf{u}}(0;\tilde{\mathbf{u}}))>0$ for all $i\ge 1$. Consequently, by Proposition 3.6, $\operatorname{index}(\widetilde{\mathbf{\Phi}}(0;\cdot),\tilde{\mathbf{u}})=(-1)^0=1$ because, in this case, the corresponding $\gamma=0$. Applying $\widetilde{\mathbf{\Phi}}(1;\cdot)=\widetilde{\mathbf{\Phi}}(0;\cdot)$ and (3.21) we see that (3.20) holds.

On the other hand, by Theorem 3.3 and 3.4, there exists a positive constant M such that, for all $t\in[0,1]$, the positive solutions of (3.17) satisfy $M^{-1}<u_1,u_2,u_3<M$. Therefore, $\mathbf{\Phi}(t;\mathbf{u})\neq\mathbf{0}$ on $\partial\mathscr{B}(M)$ for all $0\le t\le 1$. By the homotopy invariance of the topological degree, we can obtain

$$\deg\left(\mathbf{\Phi}(1;\cdot),\mathbf{0},\mathscr{B}(M)\right)=\deg\left(\mathbf{\Phi}(0;\cdot),\mathbf{0},\mathscr{B}(M)\right). \tag{3.22}$$

By our assumption, both equations $\mathbf{\Phi}(1;\mathbf{u})=\mathbf{0}$ and $\mathbf{\Phi}(0;\mathbf{u})=\mathbf{0}$ have only the positive solution $\tilde{\mathbf{u}}$ in $\mathscr{B}(M)$, and hence, by (3.19) and (3.20),

$$\begin{cases}\deg\left(\mathbf{\Phi}(0;\cdot),\mathbf{0},\mathscr{B}(M)\right)=\operatorname{index}\left(\mathbf{\Phi}(0;\cdot),\tilde{\mathbf{u}}\right)=1,\\ \deg\left(\mathbf{\Phi}(1;\cdot),\mathbf{0},\mathscr{B}(M)\right)=\operatorname{index}\left(\mathbf{\Phi}(1;\cdot),\tilde{\mathbf{u}}\right)=-1,\end{cases} \tag{3.23}$$

which contradicts (3.22). The proof is complete. □

Remark 3.9 *In present paper, we consider only the dependence of $\mathcal{A}$ on d_2 in* (3.13) *in order to calculate* $\operatorname{index}\left(\mathbf{\Phi}(\cdot),\tilde{\mathbf{u}}\right)$. *In fact, assume that the parameters Λ, d_1 and d_2 are fixed, one can establish the global existence of non-constant positive solutions of* (3.1) *with respect to the diffusion coefficient d_3, which is similarly as Theorem 3.8.*

Acknowledgment

The authors would like to thank the anonymous referees and the editors for their suggestions which greatly improved the presentation of this paper.

References

[1] M.A. Aziz-Alaoui, Study of a Leslie-Gower type tritrophic population model, *Chaos Solitons & Fractals*, **8** (2002) 1275-1293.

[2] R.S. Cantrell, C. Cosner, S. Ruan, Intraspecific interference and consumer-resource dynamics, *Discrete Contin. Dyn. Syst.*, **4** (2004) 527-546.

[3] W. Chen, M. Wang, Qualitative analysis of predator-prey models with Beddington-DeAngelis functional response and diffusion, *Math. Comput. Modelling*, **42** (2005) 31-44.

[4] D. Gilbarg, N.S.Trudinger, Elliptic Partial Differential Equations of Second Order, *Springer-Verlag*, 2001.

[5] D. Henry, Geometric Theory of Semilinear Parabolic Equations, Lecture Notes in Mathematics, vol. 840, *Springer-Verlag, Berlin, New York*, 1981.

[6] L.J. Hei, Y. Yu, Non-constant positive steady state of one resource and two consumers model with diffusion, *J. Math. Anal. Appl.*, **339** (2008) 566-581.

[7] S.B. Hsu, T.W. Huang, Global stability for a class of predator-predator-prey system, *SIAM J. Appl. Math.*, **55** (1995) 763-783.

[8] H.I. Freedman, G.S.K. Wolkowicz, PredatorCprey systems with group defense: The paradox of enrichment revisited, *Bull. Math. Biol.*, **57** (1986) 493-508.

[9] W. Ko, K. Ryu, Non-constant positive steady-state of a diffusive predator-prey system in homogeneous environment, *J. Math. Anal. Appl.*, **327** (2007) 539-549.

[10] W. Ko, K. Ryu, Qualitative analysis of a predator-prey model with Holling type II functinal response incorporating a prey refuge, *J. Differential Equations*, **231** (2006) 534-550.

[11] P.H. Leslie, Some further notes on the use of matrices in population mathematics, *Biometrica*, **35** (1948) 213-245.

[12] P.H. Leslie, J.C. Gower, A properties of a stochastic model for the predator-prey type of interaction between two species, *Biometrica*, **47** (1960) 219-234.

[13] L.S. Lin, W.M. Ni, I. Takagi, Large amplitude stationary solutions to a chemotaxis systems, *J. Differential Equations*, **72** (1988) 1-27.

[14] Y. Li, M. Wang, Stationary pattern of a diffusive prey-predator model with trophic intersections of three levels, *Nonlinear Anal. Real World Appl.*, **14** (2013) 1806-1816.

[15] Y. Lou, W.M. Ni, Diffusion, self-diffusion and cross-diffusion, *J. Differential Equations*, **131** (1996) 79-131.

[16] A.F. Nindjin, M.A. Aziz-Alaoui, Persistence and global stability in a delayed Leslie-Gower type three species food chain, *J. Math. Anal. Appl.*, **340** (2008) 340-357.

[17] R. Peng, M. Wang, Positive steady states of the Holling-Tanner prey-predator model with diffusion, *Proc. Roy. Soc. Edinburgh*, **135A** (2005) 149-164.

[18] R. Peng, M. Wang, Global stability of the equilibrium of a diffusive Holling-Tanner prey-predator model, *Appl. Math. Lett.*, **20** (2007) 664-670.

[19] P.Y.H. Pang, M. Wang, Non-constant positive steady states of a predator-prey system with non-monotonic functional response and diffusion, *Proc. Lond. Math. Soc.*, **88** (2004) 135-157.

[20] P.Y.H. Pang, M. Wang, Strategy and stationary pattern in a three-species predator-prey, *J. Differential Equations*, **200** (2004) 245-273.

[21] G. Seo, M. Kot, A comparison of two predator-prey models with Holling's type I functional response, *Mathematical Biosciences*, **212** (2008) 161-179.

[22] H.B. Shi, W.T. Li, G. Lin, Positive steady states of a diffusive predator-prey system with modified Holling-Tanner functional response, *Nonlinear Anal. Real Word Appl*, **11** (2010) 3711-3721.

[23] M.X. Wang, Stationary patterns for a prey-predator model with prey-dependent and ratio-dependent functional responses and diffusion, *Phys. D*, **196** (2004) 172-192.

[24] Y.X. Wang, W.T. Li, H.B. Shi, Stationary Patterns of a Ratio-Dependent Predator-Prey System with Cross-Diffusion, *Math. Model. Anal.*, **16** (2011) 461-474.

[25] J.D. Wollkind, J.B. Collings, J.A. Logan, Metastability in a temperature-dependent model system for predator-prey mite outbreak interactions on fruit trees, *Bull. Math. Biol.*, **50** (1988) 379-409.

[26] J.D. Wollkind, J.A. Logan, Temperature-dependent predator-prey mite ecosystem on apple tree foliage, *J. Math. Biol.*, **6** (1978) 265-284.