

APPROXIMATING THE FINITE HILBERT TRANSFORM VIA SOME COMPANIONS OF OSTROWSKI'S INEQUALITIES

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ABSTRACT. The finite Hilbert transform is a helpful tool in fields like aerodynamics, the theory of elasticity and other areas of the engineering sciences. In this paper, by using some companions of Ostrowski's inequalities for absolutely continuous functions due to Dragomir, we give some new inequalities and approximations for the finite Hilbert transform, which may have the better error bounds than the known results.

1. INTRODUCTION

Cauchy principal value integrals (finite Hilbert transform) of the form

$$(1.1) \quad (Tf)(a, b; t) = \frac{1}{\pi} PV \int_a^b \frac{f(\tau)}{\tau - t} d\tau := \frac{1}{\pi} \lim_{\varepsilon \rightarrow 0^+} \left[\int_a^{t-\varepsilon} \frac{f(\tau)}{\tau - t} d\tau + \int_{t+\varepsilon}^b \frac{f(\tau)}{\tau - t} d\tau \right].$$

play an important role in many fields like aerodynamics, the theory of elasticity and other areas of the engineering sciences. When the interval (a, b) coincides with the interval $(-\infty, \infty)$, this integral is called the Hilbert transform of $f(\tau)$, which appears frequently in tomographic image reconstruction [33], signal processing, disordered alloys, magnetic properties of disordered alloys, and phase determination in bright-field electron microscopy [27]. For different approaches in approximating the finite Hilbert transform (1.1) including: noninterpolatory, interpolatory, Gaussian, Chebychevian and spline methods, see for example the papers [6, 7, 8, 10, 12, 13, 14, 15, 18] and the references therein.

In [10], Dragomir proved the following Ostrowski type inequalities for absolutely continuous functions.

Theorem 1.1. *Let $u : [a, b] \rightarrow \mathbb{R}$ be an absolutely continuous function on $[a, b]$. Then we have*

$$(1.2) \quad \left| u(x)(b-a) - \int_a^b u(t) dt \right| \leq \begin{cases} \left[\frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2} \right)^2 \right] \|u'\|_{[a,b],\infty}, & u' \in L^\infty[a, b], \\ \frac{1}{(q+1)^{\frac{1}{q}}} \left[(x-a)^{1+\frac{1}{q}} + (b-x)^{1+\frac{1}{q}} \right] \|u'\|_{[a,b],p}, & u' \in L^p[a, b], p > 1, \frac{1}{p} + \frac{1}{q} = 1, \\ \left[\frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right] \|u'\|_{[a,b],1}, & u' \in L^1[a, b], \end{cases}$$

where $\|\cdot\|_{[a,b],r}$ ($r \in [1, \infty)$) are the usual Lebesgue norms, i.e., for $c < d$,

$$\|h\|_{[c,d],\infty} := \operatorname{ess\,sup}_{t \in [c,d]} |h(t)| \quad \text{and} \quad \|h\|_{[c,d],r} := \left(\int_c^d |h(t)|^r dt \right)^{\frac{1}{r}}, \quad r \geq 1.$$

Then, Dragomir pointed out a new method in approximating the finite Hilbert transform and proved the following estimates for the finite Hilbert transform via the above Ostrowski type inequalities for absolutely continuous function.

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Theorem 1.2. Let $f : [a, b] \rightarrow \mathbb{R}$ be a function such that its derivative $f' : [a, b] \rightarrow \mathbb{R}$ is absolutely continuous on $[a, b]$. Then we have the inequality:

$$(1.3) \quad \left| (Tf)(a, b; t) - \frac{f(t)}{\pi} \ln \left(\frac{b-t}{t-a} \right) - \frac{b-a}{\pi} [f; \lambda t + (1-\lambda)b, \lambda t + (1-\lambda)a] \right| \leq \frac{1}{\pi} \begin{cases} \left[\frac{1}{4} + \left(\lambda - \frac{1}{2} \right)^2 \right] \left[\frac{1}{4}(b-a)^2 + \left(t - \frac{a+b}{2} \right)^2 \right] \|f''\|_{[a,b],\infty}, & f'' \in L^\infty[a, b], \\ \frac{q}{(q+1)^{\frac{1}{q}}} \left[\lambda^{1+\frac{1}{q}} + (1-\lambda)^{1+\frac{1}{q}} \right] \left[(t-a)^{1+\frac{1}{q}} + (b-t)^{1+\frac{1}{q}} \right] \|f''\|_{[a,b],p}, & f'' \in L^p[a, b], p > 1, \frac{1}{p} + \frac{1}{q} = 1, \\ \left[\frac{1}{2} + \left| \lambda - \frac{1}{2} \right| \right] \left[\frac{1}{2}(b-a) + \left| t - \frac{a+b}{2} \right| \right] \|f''\|_{[a,b],1}, & f'' \in L^1[a, b] \end{cases}$$

for any $t \in [a, b]$ and $\lambda \in [0, 1)$, where $[f; \alpha, \beta]$ is the divided difference, i.e.,

$$[f; \alpha, \beta] := \frac{f(\alpha) - f(\beta)}{\alpha - \beta}.$$

Moreover, Dragomir obtained the following theorem in approximating the finite Hilbert transform of a differentiable function whose derivative is absolutely continuous.

Theorem 1.3. Let $f : [a, b] \rightarrow \mathbb{R}$ be a differentiable function such that its derivative f' is absolutely continuous on $[a, b]$. If $\lambda = (\lambda_i)_{i=0, n-1}$, $\lambda_i \in [0, 1)$ ($i = 0, n-1$) and

$$(1.4) \quad S_n(f; \lambda, t) = \frac{b-a}{\pi n} \sum_{i=0}^{n-1} \left[f; t + (i+1-\lambda_i) \frac{b-t}{n}, t - (i+1-\lambda_i) \frac{a-t}{n} \right],$$

then we have

$$(1.5) \quad (Tf)(a; b, t) = \frac{f(t)}{\pi} \ln \left(\frac{b-t}{t-a} \right) + S_n(f; \lambda, t) + R_n(f; \lambda, t),$$

and the remainder $R_n(f; \lambda, t)$ satisfies the estimate:

$$(1.6) \quad |R_n(f; \lambda, t)| \leq \frac{1}{\pi} \begin{cases} \frac{1}{n} \left[\frac{1}{4} + \frac{1}{n} \sum_{i=0}^{n-1} \left(\lambda_i - \frac{1}{2} \right)^2 \right] \left[\frac{1}{4}(b-a)^2 + \left(t - \frac{a+b}{2} \right)^2 \right] \|f''\|_{[a,b],\infty}, & f'' \in L^\infty[a, b], \\ \frac{1}{n} \frac{q}{(q+1)^{1+\frac{1}{q}}} \left[\frac{1}{n} \sum_{i=0}^{n-1} \left(\lambda_i^{1+\frac{1}{q}} + (1-\lambda_i)^{1+\frac{1}{q}} \right)^q \right]^{\frac{1}{q}} \left[(t-a)^{1+\frac{1}{q}} + (b-t)^{1+\frac{1}{q}} \right] \|f''\|_{[a,b],p}, & f'' \in L^p[a, b], p > 1, \frac{1}{p} + \frac{1}{q} = 1, \\ \frac{1}{n} \left[\frac{1}{2} + \max \left| \lambda_i - \frac{1}{2} \right| \right] \left[\frac{1}{2}(b-a) + \left| t - \frac{a+b}{2} \right| \right] \|f''\|_{[a,b],1}, & f'' \in L^1[a, b]; \end{cases}$$

$$(1.7) \quad \leq \frac{1}{\pi} \begin{cases} \frac{1}{4n} (b-a)^2 \|f''\|_{[a,b],\infty}, & f'' \in L^\infty[a, b], \\ \frac{q}{(q+1)^{1+\frac{1}{q}}} (b-a)^{1+\frac{1}{q}} \|f''\|_{[a,b],p}, & f'' \in L^p[a, b], p > 1, \frac{1}{p} + \frac{1}{q} = 1, \\ \frac{1}{n} (b-a) \|f''\|_{[a,b],1}, & f'' \in L^1[a, b]. \end{cases}$$

Recently, some companions of Ostrowski's inequality were established in [1], [2], [3], [11], [25] and [28] (see also [4, 5, 16, 17, 19, 20, 21, 22, 23, 24, 26, 29, 30, 32, 31] for other related Ostrowski type inequalities).

In this paper, we shall provide some new inequalities and approximations for the finite Hilbert transform by the use of some companions of Ostrowski's inequalities for absolutely continuous functions

in various L^p spaces due to Dragomir in [11] (see Lemma 2.1 below). For our purpose, we will use the idea of Dragomir in [10]. In Section 2, we will estimate the finite Hilbert transform on the interval $[a, b]$. Our result can give a smaller error bound than the similar result in [10] (see Remark 1 below). In Section 3, we will prove a quadrature formula for equidistant division of $[a, b]$ in approximating the finite Hilbert transform.

2. SOME INEQUALITIES IN ESTIMATING THE FINITE HILBERT TRANSFORM

For the sake of completeness, we state and prove the following lemma which provides some companions of Ostrowski's inequalities for absolutely continuous functions in various L^p spaces (see [9, 11]).

Lemma 2.1. *Let $u : [a, b] \rightarrow \mathbb{R}$ be an absolutely continuous function on $[a, b]$. Then, for all $x \in [a, \frac{a+b}{2}]$, we have*

$$(2.1) \quad \left| \frac{u(x) + u(a+b-x)}{2}(b-a) - \int_a^b u(t)dt \right| \leq \begin{cases} \left[\frac{1}{8}(b-a)^2 + 2 \left(x - \frac{3a+b}{4} \right)^2 \right] \|u'\|_{[a,b],\infty}, & u' \in L^\infty[a, b], \\ \frac{1}{(q+1)^{\frac{1}{q}}} \left[2(x-a)^{1+\frac{1}{q}} + 2^{\frac{1}{q}} \left(\frac{a+b}{2} - x \right)^{1+\frac{1}{q}} \right] \|u'\|_{[a,b],p}, & u' \in L^p[a, b], p > 1, \frac{1}{p} + \frac{1}{q} = 1, \\ \left[\frac{1}{4}(b-a) + \left| x - \frac{3a+b}{4} \right| \right] \|u'\|_{[a,b],1}, & u' \in L^1[a, b], \end{cases}$$

where $\|\cdot\|_{[a,b],r}$ ($r \in [1, \infty]$) are the usual Lebesgue norms.

Proof. Using the integration by parts formula, we have

$$(2.2) \quad \begin{aligned} & \frac{u(x) + u(a+b-x)}{2}(b-a) - \int_a^b u(t)dt \\ &= \int_a^x (t-a)u'(t)dt + \int_x^{a+b-x} \left(t - \frac{a+b}{2} \right) u'(t)dt + \int_{a+b-x}^b (t-b)u'(t)dt \end{aligned}$$

for any $x \in [a, b]$. Taking the modulus, we have

$$\begin{aligned} & \left| \frac{u(x) + u(a+b-x)}{2}(b-a) - \int_a^b u(t)dt \right| \\ & \leq \int_a^x (t-a)|u'(t)|dt + \int_x^{a+b-x} \left| t - \frac{a+b}{2} \right| |u'(t)|dt + \int_{a+b-x}^b (b-t)|u'(t)|dt := M(x). \end{aligned}$$

Now, it is obvious that

$$\begin{aligned} M(x) & \leq \|u'\|_{[a,x],\infty} \int_a^x (t-a)dt + \|u'\|_{[x,a+b-x],\infty} \int_x^{a+b-x} \left| t - \frac{a+b}{2} \right| dt + \|u'\|_{[a+b-x,b],\infty} \int_{a+b-x}^b (b-t)dt \\ & = \|u'\|_{[a,x],\infty} \frac{(x-a)^2}{2} + \|u'\|_{[x,a+b-x],\infty} \left(\frac{a+b}{2} - x \right)^2 dt + \|u'\|_{[a+b-x,b],\infty} \frac{(x-a)^2}{2} \\ & \leq \|u'\|_{[a,b],\infty} \left[\frac{(x-a)^2}{2} + \left(\frac{a+b}{2} - x \right)^2 + \frac{(x-a)^2}{2} \right] \\ & = \|u'\|_{[a,b],\infty} \left[\frac{1}{8}(b-a)^2 + 2 \left(x - \frac{3a+b}{4} \right)^2 \right], \end{aligned}$$

proving the first part of (2.1).

Using Hölder's inequality, we may write:

$$M(x) \leq \|u'\|_{[a,x],p} \left(\int_a^x (t-a)^q dt \right)^{\frac{1}{q}} + \|u'\|_{[x,a+b-x],p} \left(\int_x^{a+b-x} \left| t - \frac{a+b}{2} \right|^q dt \right)^{\frac{1}{q}}$$

$$\begin{aligned}
& + \|u'\|_{[a+b-x,b],p} \left(\int_{a+b-x}^b (b-t)^q dt \right)^{\frac{1}{q}} \\
& = \|u'\|_{[a,x],p} \frac{(x-a)^{1+\frac{1}{q}}}{(q+1)^{\frac{1}{q}}} + \|u'\|_{[x,a+b-x],p} \left[2^{\frac{1}{q}} \frac{\left(\frac{a+b}{2}-x\right)^{1+\frac{1}{q}}}{(q+1)^{\frac{1}{q}}} \right] + \|u'\|_{[a+b-x,b],p} \frac{(x-a)^{1+\frac{1}{q}}}{(q+1)^{\frac{1}{q}}} \\
& \leq \|u'\|_{[a,b],p} \frac{1}{(q+1)^{\frac{1}{q}}} \left[2(x-a)^{1+\frac{1}{q}} + 2^{\frac{1}{q}} \left(\frac{a+b}{2}-x\right)^{1+\frac{1}{q}} \right],
\end{aligned}$$

proving the second part of (2.1).

Finally, we observe that

$$\begin{aligned}
M(x) & \leq (x-a) \|u'\|_{[a,x],1} + \left(\frac{a+b}{2}-x\right) \|u'\|_{[x,a+b-x],1} + (x-a) \|u'\|_{[a+b-x,b],1} \\
& \leq \max \left\{ x-a, \frac{a+b}{2}-x \right\} \|u'\|_{[a,b],1} \\
& = \left[\frac{1}{4}(b-a) + \left| x - \frac{3a+b}{4} \right| \right] \|u'\|_{[a,b],1},
\end{aligned}$$

and the lemma is proved. \square

The best inequalities we can get from (2.1) are embodied in the following corollary.

Corollary 2.1. *With the assumption in Lemma 2.1, we have*

$$\begin{aligned}
(2.3) \quad & \left| \frac{u\left(\frac{3a+b}{4}\right) + u\left(\frac{a+3b}{4}\right)}{2} (b-a) - \int_a^b u(t) dt \right| \\
& \leq \begin{cases} \frac{1}{8}(b-a)^2 \|u'\|_{[a,b],\infty}, & u' \in L^\infty[a,b], \\ \frac{1}{4(q+1)^{\frac{1}{q}}} (b-a)^{1+\frac{1}{q}} \|u'\|_{[a,b],p}, & u' \in L^p[a,b], p > 1, \frac{1}{p} + \frac{1}{q} = 1, \\ \frac{1}{4}(b-a) \|u'\|_{[a,b],1}, & u' \in L^1[a,b]. \end{cases}
\end{aligned}$$

Using the above inequalities (2.1), we may point out the following result in estimating the finite Hilbert transform.

Theorem 2.1. *Let $f : [a,b] \rightarrow \mathbb{R}$ be a function such that its derivative $f' : [a,b] \rightarrow \mathbb{R}$ is absolutely continuous on $[a,b]$. Then we have the inequality:*

$$\begin{aligned}
(2.4) \quad & \left| (Tf)(a,b;t) - \frac{f(t)}{\pi} \ln \left(\frac{b-t}{t-a} \right) \right. \\
& \quad \left. - \frac{b-a}{\pi} \frac{[f; \lambda t + (1-\lambda)b, \lambda t + (1-\lambda)a] + [f; \lambda b + (1-\lambda)t, \lambda a + (1-\lambda)t]}{2} \right| \\
& \leq \frac{1}{\pi} \begin{cases} \left[\frac{1}{8} + 2 \left(\lambda - \frac{3}{4} \right)^2 \right] \left[\frac{1}{4}(b-a)^2 + \left(t - \frac{a+b}{2} \right)^2 \right] \|f''\|_{[a,b],\infty}, & f'' \in L^\infty[a,b], \\ \frac{q}{(q+1)^{\frac{1}{q}}} \left[2(1-\lambda)^{1+\frac{1}{q}} + 2^{\frac{1}{q}} \left(\frac{1}{2} - \lambda \right)^{1+\frac{1}{q}} \right] \left[(t-a)^{1+\frac{1}{q}} + (b-t)^{1+\frac{1}{q}} \right] \|f''\|_{[a,b],p}, & f'' \in L^p[a,b], p > 1, \frac{1}{p} + \frac{1}{q} = 1, \\ \left[\frac{1}{4} + \left| \lambda - \frac{3}{4} \right| \right] \left[\frac{1}{2}(b-a) + \left| t - \frac{a+b}{2} \right| \right] \|f''\|_{[a,b],1}, & f'' \in L^1[a,b] \end{cases}
\end{aligned}$$

for any $t \in [a,b]$ and $\lambda \in [\frac{1}{2}, 1)$, where $[f; \alpha, \beta]$ is the divided difference.

Proof. Since f' is bounded on $[a, b]$, it follows that f is Lipschitzian on $[a, b]$ and thus the finite Hilbert transform exists everywhere in (a, b) .

As in [10], for the function $f_0 : (a, b) \rightarrow \mathbb{R}$, $f_0(t) = 1$, $t \in (a, b)$, we have

$$(Tf)(a, b; t) = \frac{1}{\pi} \ln \left(\frac{b-t}{t-a} \right), \quad t \in (a, b),$$

then obviously

$$(2.5) \quad (Tf)(a, b; t) - \frac{f(t)}{\pi} \ln \left(\frac{b-t}{t-a} \right) = \frac{1}{\pi} PV \int_a^b \frac{f(\tau) - f(t)}{\tau - t} d\tau.$$

Now, if we choose in (2.1), $u = f'$, $x = \lambda c + (1 - \lambda)d$, $\lambda \in [\frac{1}{2}, 1]$, $c, d \in [a, b]$, then we get

$$\begin{aligned} & \left| f(d) - f(c) - \frac{f'(\lambda c + (1 - \lambda)d) + f'(\lambda d + (1 - \lambda)c)}{2} (d - c) \right| \\ & \leq \begin{cases} (d - c)^2 \left[\frac{1}{8} + 2 \left(\lambda - \frac{3}{4} \right)^2 \right] \|f''\|_{[c, d], \infty}, & f'' \in L^\infty[a, b], \\ \frac{|d - c|^{1 + \frac{1}{q}}}{(q + 1)^{\frac{1}{q}}} \left[2(1 - \lambda)^{1 + \frac{1}{q}} + 2^{\frac{1}{q}} \left(\frac{1}{2} - \lambda \right)^{1 + \frac{1}{q}} \right] \|f''\|_{[c, d], p}, & f'' \in L^p[a, b], p > 1, \frac{1}{p} + \frac{1}{q} = 1, \\ |d - c| \left[\frac{1}{4} + \left| \lambda - \frac{3}{4} \right| \right] \|f''\|_{[c, d], 1}, & f'' \in L^1[a, b], \end{cases} \end{aligned}$$

which is equivalent to

$$(2.6) \quad \begin{aligned} & \left| \frac{f(d) - f(c)}{d - c} - \frac{f'(\lambda c + (1 - \lambda)d) + f'(\lambda d + (1 - \lambda)c)}{2} \right| \\ & \leq \begin{cases} |d - c| \left[\frac{1}{8} + 2 \left(\lambda - \frac{3}{4} \right)^2 \right] \|f''\|_{[c, d], \infty}, & f'' \in L^\infty[a, b], \\ \frac{|d - c|^{\frac{1}{q}}}{(q + 1)^{\frac{1}{q}}} \left[2(1 - \lambda)^{1 + \frac{1}{q}} + 2^{\frac{1}{q}} \left(\frac{1}{2} - \lambda \right)^{1 + \frac{1}{q}} \right] \|f''\|_{[c, d], p}, & f'' \in L^p[a, b], p > 1, \frac{1}{p} + \frac{1}{q} = 1, \\ \left[\frac{1}{4} + \left| \lambda - \frac{3}{4} \right| \right] \|f''\|_{[c, d], 1}, & f'' \in L^1[a, b]. \end{cases} \end{aligned}$$

Using (2.6), we may write

$$\begin{aligned} & \left| \frac{1}{\pi} PV \int_a^b \frac{f(\tau) - f(t)}{\tau - t} d\tau - \frac{1}{2\pi} PV \int_a^b [f'(\lambda t + (1 - \lambda)\tau) + f'(\lambda\tau + (1 - \lambda)t)] d\tau \right| \\ & \leq \begin{cases} \frac{1}{\pi} \left[\frac{1}{8} + 2 \left(\lambda - \frac{3}{4} \right)^2 \right] PV \int_a^b |t - \tau| \|f''\|_{[t, \tau], \infty} d\tau, & f'' \in L^\infty[a, b], \\ \frac{1}{\pi} \frac{1}{(q + 1)^{\frac{1}{q}}} \left[2(1 - \lambda)^{1 + \frac{1}{q}} + 2^{\frac{1}{q}} \left(\frac{1}{2} - \lambda \right)^{1 + \frac{1}{q}} \right] PV \int_a^b |t - \tau|^{\frac{1}{q}} \|f''\|_{[t, \tau], p} d\tau, & f'' \in L^p[a, b], p > 1, \frac{1}{p} + \frac{1}{q} = 1, \\ \frac{1}{\pi} \left[\frac{1}{4} + \left| \lambda - \frac{3}{4} \right| \right] PV \int_a^b \|f''\|_{[t, \tau], 1} d\tau, & f'' \in L^1[a, b]; \end{cases} \end{aligned}$$

$$\begin{aligned}
(2.7) \quad & \leq \begin{cases} \left[\frac{1}{\pi} \left[\frac{1}{8} + 2 \left(\lambda - \frac{3}{4} \right)^2 \right] \|f''\|_{[a,b],\infty} PV \int_a^b |t-\tau| d\tau, & f'' \in L^\infty[a,b], \\ \frac{1}{\pi} \frac{1}{(q+1)^{\frac{1}{q}}} \left[2(1-\lambda)^{1+\frac{1}{q}} + 2^{\frac{1}{q}} \left(\frac{1}{2} - \lambda \right)^{1+\frac{1}{q}} \right] \|f''\|_{[a,b],p} PV \int_a^b |t-\tau|^{\frac{1}{q}} d\tau, \\ & f'' \in L^p[a,b], p > 1, \frac{1}{p} + \frac{1}{q} = 1, \\ \frac{1}{\pi} \left[\frac{1}{4} + \left| \lambda - \frac{3}{4} \right| \right] PV \left[\int_a^t \|f''\|_{[\tau,t],1} d\tau + \int_t^b \|f''\|_{[t,\tau],1} d\tau \right], & f'' \in L^1[a,b]; \\ \left[\frac{1}{\pi} \left[\frac{1}{8} + 2 \left(\lambda - \frac{3}{4} \right)^2 \right] \|f''\|_{[a,b],\infty} \left[\frac{1}{4}(b-a)^2 + \left(t - \frac{a+b}{2} \right)^2 \right], & f'' \in L^\infty[a,b], \\ \frac{1}{\pi} \frac{1}{(q+1)^{\frac{1}{q}}} \left[2(1-\lambda)^{1+\frac{1}{q}} + 2^{\frac{1}{q}} \left(\frac{1}{2} - \lambda \right)^{1+\frac{1}{q}} \right] \|f''\|_{[a,b],p} \\ \quad \times \frac{q}{q+1} \left[(t-a)^{1+\frac{1}{q}} + (b-t)^{1+\frac{1}{q}} \right], & f'' \in L^p[a,b], p > 1, \frac{1}{p} + \frac{1}{q} = 1, \\ \frac{1}{\pi} \left[\frac{1}{4} + \left| \lambda - \frac{3}{4} \right| \right] \left[\frac{1}{2}(b-a) + \left| t - \frac{a+b}{2} \right| \right] \|f''\|_{[a,b],1}, & f'' \in L^1[a,b]. \end{cases}
\end{aligned}$$

Since (note that $\lambda \neq 1$) (see [10])

$$\begin{aligned}
& \frac{1}{2\pi} PV \int_a^b [f'(\lambda t + (1-\lambda)\tau) + f'(\lambda\tau + (1-\lambda)t)] d\tau \\
&= \frac{1}{2\pi} \lim_{\varepsilon \rightarrow 0^+} \left[\int_a^{t-\varepsilon} + \int_{t+\varepsilon}^b \right] [f'(\lambda t + (1-\lambda)\tau) + f'(\lambda\tau + (1-\lambda)t)] d\tau \\
&= \frac{1}{2\pi} \left[\frac{-f(\lambda t + (1-\lambda)a) + f(\lambda t + (1-\lambda)b)}{1-\lambda} + \frac{-f(\lambda a + (1-\lambda)t) + f(\lambda b + (1-\lambda)t)}{\lambda} \right] \\
&= \frac{b-a}{\pi} \frac{[f; \lambda t + (1-\lambda)b, \lambda t + (1-\lambda)a] + [f; \lambda b + (1-\lambda)t, \lambda a + (1-\lambda)t]}{2}.
\end{aligned}$$

Using (2.5) and (2.7), we deduce the desired result (2.4). \square

The best inequality one may obtain from (2.4) is embodied in the following corollary.

Corollary 2.2. *With the assumption of Theorem 2.1, for $\lambda = \frac{3}{4}$, one has the inequality*

$$\begin{aligned}
(2.8) \quad & \left| (Tf)(a,b;t) - \frac{f(t)}{\pi} \ln \left(\frac{b-t}{t-a} \right) - \frac{b-a}{\pi} \frac{[f; \frac{3t+b}{4}, \frac{3t+a}{4}] + [f; \frac{3b+t}{4}, \frac{3a+t}{4}]}{2} \right| \\
& \leq \frac{1}{\pi} \begin{cases} \left[\frac{1}{8} \left[\frac{1}{4}(b-a)^2 + \left(t - \frac{a+b}{2} \right)^2 \right] \|f''\|_{[a,b],\infty}, & f'' \in L^\infty[a,b], \\ \frac{q}{4^{\frac{2}{q}}(q+1)^{1+\frac{1}{q}}} \left[(t-a)^{1+\frac{1}{q}} + (b-t)^{1+\frac{1}{q}} \right] \|f''\|_{[a,b],p}, & f'' \in L^p[a,b], p > 1, \frac{1}{p} + \frac{1}{q} = 1, \\ \frac{1}{4} \left[\frac{1}{2}(b-a) + \left| t - \frac{a+b}{2} \right| \right] \|f''\|_{[a,b],1}, & f'' \in L^1[a,b] \end{cases}
\end{aligned}$$

for any $t \in [a, b]$.

Remark 1. *As stated in [10], the best estimator one may obtain from (1.3) is, for $\lambda = \frac{1}{2}$,*

$$\left| (Tf)(a,b;t) - \frac{f(t)}{\pi} \ln \left(\frac{b-t}{t-a} \right) - \frac{b-a}{\pi} \left[f; \frac{t+b}{2}, \frac{t+a}{2} \right] \right|$$

$$(2.9) \quad \leq \frac{1}{\pi} \begin{cases} \frac{1}{4} \left[\frac{1}{4} (b-a)^2 + \left(t - \frac{a+b}{2} \right)^2 \right] \|f''\|_{[a,b],\infty}, & f'' \in L^\infty[a,b], \\ \frac{q}{2^{\frac{p}{q}}(q+1)^{1+\frac{1}{q}}} \left[(t-a)^{1+\frac{1}{q}} + (b-t)^{1+\frac{1}{q}} \right] \|f''\|_{[a,b],p}, & f'' \in L^p[a,b], p > 1, \frac{1}{p} + \frac{1}{q} = 1, \\ \frac{1}{2} \left[\frac{1}{2} (b-a) + \left| t - \frac{a+b}{2} \right| \right] \|f''\|_{[a,b],1}, & f'' \in L^1[a,b]. \end{cases}$$

We note that inequality (2.8) we obtained here gives a new estimate of the finite Hilbert transform and a smaller error bound than that of (2.9).

3. A QUADRATURE FORMULA FOR EQUIDISTANT DIVISIONS

The following lemma is of interest in itself.

Lemma 3.1. *Let $u : [a, b] \rightarrow \mathbb{R}$ be an absolutely continuous function on $[a, b]$. Then for all $n \geq 1$, $\lambda_i \in [\frac{1}{2}, 1)$ ($i = 0, \dots, n-1$) and $t, \tau \in [a, b]$ with $t \neq \tau$, we have the inequality*

$$(3.1) \quad \begin{cases} \left| \frac{1}{\tau-t} \int_t^\tau u(s) ds - \frac{1}{n} \sum_{i=0}^{n-1} \left[\frac{u(t + (i+1-\lambda_i)\frac{\tau-t}{n}) + u(t + (i+\lambda_i)\frac{\tau-t}{n})}{2} \right] \right| \\ \left\{ \begin{array}{ll} \frac{|t-\tau|}{n} \left[\frac{1}{8} + \frac{2}{n} \sum_{i=0}^{n-1} \left(\lambda_i - \frac{3}{4} \right)^2 \right] \|u'\|_{[t,\tau],\infty}, & u' \in L^\infty[a,b], \\ \frac{|t-\tau|^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}} n} \left[\frac{1}{n} \sum_{i=0}^{n-1} \left(2(1-\lambda_i)^{1+\frac{1}{q}} + 2^{\frac{1}{q}} \left(\lambda_i - \frac{1}{2} \right)^{1+\frac{1}{q}} \right)^q \right]^{\frac{1}{q}} \|u'\|_{[t,\tau],p}, & u' \in L^p[a,b], \\ \frac{1}{n} \left[\frac{1}{4} + \max \left| \lambda_i - \frac{3}{4} \right| \right] \|u'\|_{[t,\tau],1}, & u' \in L^1[a,b], \end{array} \right. \end{cases}$$

where $\|\cdot\|_{[t,\tau],r}$ ($r \in [1, \infty)$) are the usual Lebesgue norms, $p > 1, \frac{1}{p} + \frac{1}{q} = 1$.

Proof. Consider the equidistant division of $[t, \tau]$ (if $t < \tau$) or $[\tau, t]$ (if $\tau < t$) given by

$$(3.2) \quad E_n : x_i = t + i \cdot \frac{\tau-t}{n}, \quad i = \overline{0, n}.$$

Then the points $\xi_i = \lambda_i[t + i \cdot \frac{\tau-t}{n}] + (1-\lambda_i)[t + (i+1) \cdot \frac{\tau-t}{n}]$ ($\lambda_i \in [\frac{1}{2}, 1)$, $i = \overline{0, n-1}$) are between x_i and x_{i+1} . We observe that we may write for simplicity $\xi_i = t + (i+1-\lambda_i)\frac{\tau-t}{n}$ ($i = \overline{0, n-1}$). We also have

$$\begin{aligned} x_i + x_{i+1} - \xi_i &= t + (i+\lambda_i)\frac{\tau-t}{n}, \\ \xi_i - \frac{3x_i + x_{i+1}}{4} &= \frac{\tau-t}{n} \left(\frac{3}{4} - \lambda_i \right), \\ \xi_i - x_i &= (1-\lambda_i)\frac{\tau-t}{n} \end{aligned}$$

and

$$\frac{x_i + x_{i+1}}{2} - \xi_i = \left(\lambda_i - \frac{1}{2} \right) \frac{\tau-t}{n}$$

for any $i = \overline{0, n-1}$.

If we apply the inequality (2.1) on the intervals $[x_i, x_{i+1}]$ and the intermediate points ξ_i ($i = \overline{0, n-1}$), then we may write that

$$\left| \frac{\tau-t}{n} \left[\frac{u(t + (i+1-\lambda_i)\frac{\tau-t}{n}) + u(t + (i+\lambda_i)\frac{\tau-t}{n})}{2} \right] - \int_{x_i}^{x_{i+1}} u(s) ds \right|$$

$$(3.3) \quad \leq \begin{cases} \left[\frac{1}{8} \frac{(t-\tau)^2}{n^2} + 2 \frac{(t-\tau)^2}{n^2} \left(\frac{3}{4} - \lambda_i \right)^2 \right] \|u'\|_{[x_i, x_{i+1}], \infty}, & u' \in L^\infty[a, b], \\ \frac{1}{(q+1)^{\frac{1}{q}}} \left[\frac{|t-\tau|^{1+\frac{1}{q}}}{n^{1+\frac{1}{q}}} 2(1-\lambda_i)^{1+\frac{1}{q}} + 2^{\frac{1}{q}} \frac{|t-\tau|^{1+\frac{1}{q}}}{n^{1+\frac{1}{q}}} \left(\lambda_i - \frac{1}{2} \right)^{1+\frac{1}{q}} \right] \|u'\|_{[x_i, x_{i+1}], p}, & \\ & u' \in L^p[a, b], p > 1, \frac{1}{p} + \frac{1}{q} = 1, \\ \left[\frac{1}{4} \frac{|\tau-t|}{n} + \frac{|\tau-t|}{n} \left| \frac{3}{4} - \lambda_i \right| \right] \|u'\|_{[x_i, x_{i+1}], 1}, & u' \in L^1[a, b]. \end{cases}$$

Summing (3.3), we get

$$(3.4) \quad \leq \begin{cases} \left| \int_t^\tau u(s) ds - \frac{\tau-t}{n} \sum_{i=0}^{n-1} \left[\frac{u(t+(i+1-\lambda_i)\frac{\tau-t}{n}) + u(t+(i+\lambda_i)\frac{\tau-t}{n})}{2} \right] \right| \\ \left(\frac{(t-\tau)^2}{n^2} \sum_{i=0}^{n-1} \left[\frac{1}{8} + 2 \left(\lambda_i - \frac{3}{4} \right)^2 \right] \|u'\|_{[x_i, x_{i+1}], \infty}, \right. & u' \in L^\infty[a, b], \\ \left. \frac{|t-\tau|^{1+\frac{1}{q}}}{(q+1)^{\frac{1}{q}} n^{1+\frac{1}{q}}} \sum_{i=0}^{n-1} \left[2(1-\lambda_i)^{1+\frac{1}{q}} + 2^{\frac{1}{q}} \left(\lambda_i - \frac{1}{2} \right)^{1+\frac{1}{q}} \right] \|u'\|_{[x_i, x_{i+1}], p}, \right. & \\ & u' \in L^p[a, b], p > 1, \frac{1}{p} + \frac{1}{q} = 1, \\ \left. \frac{|\tau-t|}{n} \sum_{i=0}^{n-1} \left[\frac{1}{4} + \left| \lambda_i - \frac{3}{4} \right| \right] \|u'\|_{[x_i, x_{i+1}], 1}, \right. & u' \in L^1[a, b]. \end{cases}$$

However,

$$(3.5) \quad \sum_{i=0}^{n-1} \left[\frac{1}{8} + 2 \left(\lambda_i - \frac{3}{4} \right)^2 \right] \|u'\|_{[x_i, x_{i+1}], \infty} \leq \|u'\|_{[t, \tau], \infty} \left[\frac{1}{8} n + 2 \sum_{i=0}^{n-1} \left(\lambda_i - \frac{3}{4} \right)^2 \right],$$

$$\sum_{i=0}^{n-1} \left[2(1-\lambda_i)^{1+\frac{1}{q}} + 2^{\frac{1}{q}} \left(\lambda_i - \frac{1}{2} \right)^{1+\frac{1}{q}} \right] \|u'\|_{[x_i, x_{i+1}], p}$$

$$\leq \left(\sum_{i=0}^{n-1} \left[2(1-\lambda_i)^{1+\frac{1}{q}} + 2^{\frac{1}{q}} \left(\lambda_i - \frac{1}{2} \right)^{1+\frac{1}{q}} \right]^q \right)^{\frac{1}{q}} \left(\sum_{i=0}^{n-1} \|u'\|_{[x_i, x_{i+1}], p}^p \right)^{\frac{1}{p}}$$

$$(3.6) \quad = \|u'\|_{[t, \tau], p} \left(\sum_{i=0}^{n-1} \left[2(1-\lambda_i)^{1+\frac{1}{q}} + 2^{\frac{1}{q}} \left(\lambda_i - \frac{1}{2} \right)^{1+\frac{1}{q}} \right]^q \right)^{\frac{1}{q}}$$

and

$$(3.7) \quad \sum_{i=0}^{n-1} \left[\frac{1}{4} + \left| \lambda_i - \frac{3}{4} \right| \right] \|u'\|_{[x_i, x_{i+1}], 1} \leq \left[\frac{1}{4} + \max \left| \lambda_i - \frac{3}{4} \right| \right] \sum_{i=0}^{n-1} \|u'\|_{[x_i, x_{i+1}], 1}$$

$$= \left[\frac{1}{4} + \max \left| \lambda_i - \frac{3}{4} \right| \right] \|u'\|_{[t, \tau], 1}.$$

Now, using (3.4)-(3.7), we deduce the desired result (3.1). \square

We may now state the following theorem in approximating the finite Hilbert transform of a differentiable function whose derivative is absolutely continuous.

Theorem 3.1. *Let $f : [a, b] \rightarrow \mathbb{R}$ be a differentiable function such that its derivative f' is absolutely continuous on $[a, b]$. If $\lambda = (\lambda_i)_{i=0, n-1}$, $\lambda_i \in [\frac{1}{2}, 1)$ ($i = 0, n-1$),*

$$(3.8) \quad S_n(f; \lambda, t) = \frac{b-a}{\pi n} \sum_{i=0}^{n-1} \left[f; t + (i+1-\lambda_i) \frac{b-t}{n}, t + (i+1-\lambda_i) \frac{a-t}{n} \right]$$

and

$$(3.9) \quad T_n(f; \lambda, t) = \frac{b-a}{\pi n} \sum_{i=0}^{n-1} \left[f; t + (i + \lambda_i) \frac{b-t}{n}, t + (i + \lambda_i) \frac{a-t}{n} \right],$$

then we have

$$(3.10) \quad (Tf)(a, b; t) = \frac{f(t)}{\pi} \ln \left(\frac{b-t}{t-a} \right) + \frac{S_n(f; \lambda, t) + T_n(f; \lambda, t)}{2} + R_n(f; \lambda, t),$$

and the reminder $R_n(f; \lambda, t)$ satisfies the estimate:

$$(3.11) \quad |R_n(f; \lambda, t)| \leq \frac{1}{\pi} \begin{cases} \frac{1}{n} \left[\frac{1}{8} + \frac{2}{n} \sum_{i=0}^{n-1} \left(\lambda_i - \frac{3}{4} \right)^2 \right] \left[\frac{1}{4} (b-a)^2 + \left(t - \frac{a+b}{2} \right)^2 \right] \|f''\|_{[a,b],\infty}, & f'' \in L^\infty[a, b], \\ \frac{1}{n} \frac{q}{(q+1)^{1+\frac{1}{q}}} \left[\frac{1}{n} \sum_{i=0}^{n-1} \left(2(1-\lambda_i)^{1+\frac{1}{q}} + 2^{\frac{1}{q}} \left(\lambda_i - \frac{1}{2} \right)^{1+\frac{1}{q}} \right)^q \right]^{\frac{1}{q}} \\ \quad \times \left[(t-a)^{1+\frac{1}{q}} + (b-t)^{1+\frac{1}{q}} \right] \|f''\|_{[a,b],p}, & f'' \in L^p[a, b], p > 1, \frac{1}{p} + \frac{1}{q} = 1, \\ \frac{1}{n} \left[\frac{1}{4} + \max \left| \lambda_i - \frac{3}{4} \right| \right] \left[\frac{1}{2} (b-a) + \left| t - \frac{a+b}{2} \right| \right] \|f''\|_{[a,b],1}, & f'' \in L^1[a, b]. \end{cases}$$

Proof. Applying Lemma 3.1 for the function f' , we may write that

$$(3.12) \quad \left| \frac{f(\tau) - f(t)}{\tau - t} - \frac{1}{n} \sum_{i=0}^{n-1} \frac{f'(t + (i+1 - \lambda_i) \frac{\tau-t}{n}) + f'(t + (i + \lambda_i) \frac{\tau-t}{n})}{2} \right| \leq \begin{cases} \frac{|t-\tau|}{n} \left[\frac{1}{8} + \frac{2}{n} \sum_{i=0}^{n-1} \left(\lambda_i - \frac{3}{4} \right)^2 \right] \|f''\|_{[t,\tau],\infty}, & f'' \in L^\infty[a, b], \\ \frac{|t-\tau|^{\frac{1}{q}}}{n(q+1)^{\frac{1}{q}}} \left[\frac{1}{n} \sum_{i=0}^{n-1} \left(2(1-\lambda_i)^{1+\frac{1}{q}} + 2^{\frac{1}{q}} \left(\lambda_i - \frac{1}{2} \right)^{1+\frac{1}{q}} \right)^q \right]^{\frac{1}{q}} \|f''\|_{[t,\tau],p}, & f'' \in L^p[a, b], \\ & p > 1, \frac{1}{p} + \frac{1}{q} = 1, \\ \frac{1}{n} \left[\frac{1}{4} + \max \left| \lambda_i - \frac{3}{4} \right| \right] \|f''\|_{[t,\tau],1}, & f'' \in L^1[a, b] \end{cases}$$

for any $t, \tau \in [a, b], t \neq \tau$.

Taking the PV, we may write

$$(3.13) \quad \left| \frac{1}{\pi} PV \int_a^b \frac{f(\tau) - f(t)}{\tau - t} d\tau - \frac{1}{\pi n} \sum_{i=0}^{n-1} PV \int_a^b \frac{f'(t + (i+1 - \lambda_i) \frac{\tau-t}{n}) + f'(t + (i + \lambda_i) \frac{\tau-t}{n})}{2} d\tau \right| \leq \frac{1}{\pi} \begin{cases} \frac{1}{n} \left[\frac{1}{8} + \frac{2}{n} \sum_{i=0}^{n-1} \left(\lambda_i - \frac{3}{4} \right)^2 \right] PV \int_a^b |t - \tau| \|f''\|_{[t,\tau],\infty} d\tau, & f'' \in L^\infty[a, b], \\ \frac{1}{n(q+1)^{\frac{1}{q}}} \left[\frac{1}{n} \sum_{i=0}^{n-1} \left(2(1-\lambda_i)^{1+\frac{1}{q}} + 2^{\frac{1}{q}} \left(\lambda_i - \frac{1}{2} \right)^{1+\frac{1}{q}} \right)^q \right]^{\frac{1}{q}} \\ \quad \times PV \int_a^b |t - \tau|^{\frac{1}{q}} \|f''\|_{[t,\tau],p} d\tau, & f'' \in L^p[a, b], p > 1, \frac{1}{p} + \frac{1}{q} = 1, \\ \frac{1}{n} \left[\frac{1}{4} + \max \left| \lambda_i - \frac{3}{4} \right| \right] PV \int_a^b \|f''\|_{[t,\tau],1} d\tau, & f'' \in L^1[a, b]. \end{cases}$$

However (see [10]),

$$PV \int_a^b |t - \tau| \|f''\|_{[t,\tau],\infty} d\tau \leq \|f''\|_{[a,b],\infty} PV \int_a^b |t - \tau| d\tau = \|f''\|_{[a,b],\infty} \left[\frac{(t-a)^2 + (b-t)^2}{2} \right],$$

$$\begin{aligned} PV \int_a^b |t - \tau|^{\frac{1}{q}} \|f''\|_{[t,\tau],p} d\tau &\leq \|f''\|_{[a,b],p} PV \int_a^b |t - \tau|^{\frac{1}{q}} d\tau \\ &= \|f''\|_{[a,b],p} \left[\frac{(t+a)^{\frac{1}{q}+1} + (b-t)^{\frac{1}{q}+1}}{\frac{1}{q}+1} \right] \\ &= \frac{q}{q+1} \left[(t-a)^{\frac{1}{q}+1} + (b-t)^{\frac{1}{q}+1} \right] \|f''\|_{[a,b],p}, \end{aligned}$$

$$\begin{aligned} PV \int_a^b \|f''\|_{[t,\tau],1} d\tau &= PV \left[\int_a^t \|f''\|_{[\tau,t],1} d\tau + \int_a^t \|f''\|_{[t,\tau],1} d\tau \right] \\ &\leq \max\{t-a, b-t\} \|f''\|_{[a,b],1}, \\ &= \left[\frac{1}{2}(b-a) + \left| t - \frac{a+b}{2} \right| \right] \|f''\|_{[a,b],1}, \end{aligned}$$

and using the inequality (3.13) we obtain the desired estimate (3.11). \square

Remark 2. For $n = 1$, we recapture the inequality (2.4).

For $\lambda_i = \frac{3}{4}$ ($i = \overline{0, n-1}$), the following particular case, which may be easily numerically implemented, holds.

Corollary 3.1. Let f be as in Theorem 3.1. Define

$$(3.14) \quad S_{M,n}(f;t) = \frac{b-a}{\pi n} \sum_{i=0}^{n-1} \left[f; t + \left(i + \frac{1}{4} \right) \frac{b-t}{n}, t + \left(i + \frac{1}{4} \right) \frac{a-t}{n} \right]$$

and

$$(3.15) \quad T_{M,n}(f;t) = \frac{b-a}{\pi n} \sum_{i=0}^{n-1} \left[f; t + \left(i + \frac{3}{4} \right) \frac{b-t}{n}, t + \left(i + \frac{3}{4} \right) \frac{a-t}{n} \right],$$

and the remainder $R_{M,n}(f;t)$ satisfies the estimate

$$(3.16) \quad (Tf)(a,b;t) = \frac{f(t)}{\pi} \ln \left(\frac{b-t}{t-a} \right) + \frac{S_{M,n}(f;t) + T_{M,n}(f;t)}{2} + R_{M,n}(f;t).$$

Then we have the representation:

$$(3.17) \quad |R_{M,n}(f;t)| \leq \frac{1}{\pi} \begin{cases} \frac{1}{8n} \left[\frac{1}{4}(b-a)^2 + \left(t - \frac{a+b}{2} \right)^2 \right] \|f''\|_{[a,b],\infty}, & f'' \in L^\infty[a,b], \\ \frac{1}{n} \frac{(2+2^{\frac{1}{q}})q}{4^{1+\frac{1}{q}}(q+1)^{1+\frac{1}{q}}} \left[(t-a)^{1+\frac{1}{q}} + (b-t)^{1+\frac{1}{q}} \right] \|f''\|_{[a,b],p}, & f'' \in L^p[a,b], \\ \frac{1}{4n} \left[\frac{1}{2}(b-a) + \left| t - \frac{a+b}{2} \right| \right] \|f''\|_{[a,b],1}, & f'' \in L^1[a,b] \end{cases}$$

for any $t \in [a, b]$, where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

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