On a Subclass of Strongly Starlike Functions
Associated with Exponential Function

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Abstract. Let \( S^*_{e} \) denote the class of analytic functions \( f \) in the open unit disk normalized by \( f(0) = f'(0) - 1 = 0 \) and satisfying the condition \( zf'(z)/f(z) < e^z \) for \( |z| < 1 \). The structural formula, inclusion relations, coefficient estimates, growth and distortion results, subordination theorems and various radii constants for functions in the class \( S^*_{e} \) are obtained. In addition, the sharp \( S^*_{e} \)-radii for functions belonging to several interesting classes are also determined.

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1. Introduction

Let \( A_n \) denote the class of analytic functions in the open unit disk \( D := \{ z \in \mathbb{C} : |z| < 1 \} \) of the form \( f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \), and let \( \mathcal{A} := A_1 \). Let \( \mathcal{S} \) be the subclass of \( \mathcal{A} \) consisting of univalent functions. Using subordination, Ma and Minda [22] gave a unified representation of various geometric subclasses of \( \mathcal{S} \) which are characterized by the quantities \( zf'(z)/f(z) \) or \( 1 + zf''(z)/f'(z) \) lying in a domain in the right half-plane. They considered the class \( \Phi \) of analytic univalent functions \( \varphi \) with positive real part mapping \( D \) onto domains symmetric with respect to the real axis and starlike with respect to \( \varphi(0) = 1 \) such that \( \varphi'(0) > 0 \). For \( \varphi \in \Phi \), they introduced the following classes that include several well-known classes as special cases:

\[
\mathcal{S}^*(\varphi) = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} < \varphi(z) \right\} \quad \text{and} \quad \mathcal{H}(\varphi) = \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} < \varphi(z) \right\}.
\]

For \(-1 \leq B < A \leq 1\), \( \mathcal{S}^*[A, B] := \mathcal{S}^*((1+Az)/(1+Bz)) \) is the familiar class consisting of Janowski [16] starlike functions. The corresponding class of convex functions is denoted by \( \mathcal{H}[A, B] \). The special case \( A = 1 - 2\alpha, B = -1 \) with \( 0 \leq \alpha < 1 \) yield the subclasses \( \mathcal{S}^*(\alpha) \) and \( \mathcal{H}(\alpha) \) \( (0 \leq \alpha < 1) \) of \( \mathcal{S} \) consisting of starlike functions of order \( \alpha \) and convex functions of order \( \alpha \), respectively, introduced by Robertson [32]. The classes \( \mathcal{S}^* := \mathcal{S}^*(0) \) and \( \mathcal{H} := \mathcal{H}(0) \) are the classes of starlike and convex functions respectively. For \( 0 < \gamma \leq 1 \),

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\[ \mathcal{S}(\gamma) := \mathcal{S}^s((1+z)/(1-z))^{\gamma} \text{ and } \mathcal{K}(\gamma) := \mathcal{K}(((1+z)/(1-z))^{\gamma}) \] are the classes of strongly starlike and strongly convex functions of order \( \gamma \). If
\[ \varphi(z) = 1 + \frac{2}{\pi^2} \left( \log \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right)^2 \]
then \( \mathcal{U} \subseteq \mathcal{V} := \mathcal{K}(\varphi) \) is the class of uniformly convex functions introduced by Goodman [14]. The corresponding class \( \mathcal{S}_p := \mathcal{S}^s(\varphi) \) of parabolic starlike functions, was studied by Rønning [34]. Similarly, \( \mathcal{S}_L^s := \mathcal{S}^s(\sqrt{1+z}) \) is the subclass of \( \mathcal{S}^s \) introduced by Sokół and Stankiewicz [45], consisting of functions \( f \in \mathcal{A} \) such that \( zf'(z)/f(z) \) lies in the domain bounded by the right-half of the lemniscate of Bernoulli given by \(|w^2 - 1| < 1\). More results regarding these classes can be found in [2, 4, 7, 10, 13, 21, 28, 29, 31, 39–44]. Recently, the authors [26] discussed the properties of the class
\[ \mathcal{S}_{RL}^s = \mathcal{S}^s \left( \sqrt{2 - (\sqrt{2} - 1)} \sqrt{\frac{1 - z}{1 + 2(\sqrt{2} - 1)z}} \right). \]
Precisely, \( f \in \mathcal{S}_{RL}^s \) provided \( zf'(z)/f(z) \) lies in the interior of the left-half of the shifted lemniscate of Bernoulli given by \(|w^2 - 1| < 1\). In the similar fashion, many new interesting subclasses of starlike and convex functions can be defined by altering the subordinate function \( \varphi \in \Phi \). This paper aims to investigate the properties of the classes \( \mathcal{S}^s(e^z) \) and \( \mathcal{K}(e^z) \).

The exponential function \( \varphi_0(z) = e^z \) has positive real part in \( \mathbb{D} \), \( \varphi_0(\mathbb{D}) = \{ w \in \mathbb{C} : |\log w| < 1 \} \) (see Figure 1) is symmetric with respect to the real axis and starlike with respect to 1, and \( \varphi_0'(0) > 0 \). Hence \( \varphi_0 \in \Phi \) so that the classes \( \mathcal{S}^s(\varphi_0) \) and \( \mathcal{K}(\varphi_0) \) are well-defined. Set
\[ \mathcal{S}^s_e := \mathcal{S}^s(e^z) \text{ and } \mathcal{K}_e := \mathcal{K}(e^z). \]
In view of the Alexander relation between the classes \( \mathcal{S}^s_e \) and \( \mathcal{K}_e \): \( f \in \mathcal{K}_e \) if and only if \( zf'/f(0) \in \mathcal{S}^s_e \), the properties of functions in the class \( \mathcal{K}_e \) can be obtained from the corresponding results for \( \mathcal{S}^s_e \). Therefore, it is enough to focus our attention to the class \( \mathcal{S}^s_e \). For a function \( f \in \mathcal{A} \), we have the equivalences:
\[ f \in \mathcal{S}^s_e \iff \frac{zf'(z)}{f(z)} < e^z \quad (z \in \mathbb{D}) \iff \left| \log \frac{zf'(z)}{f(z)} \right| < 1 \quad (z \in \mathbb{D}). \]
This immediately yields the following structural formula for functions in the class \( \mathcal{S}^s_e \).

**Theorem 1.1.** A function \( f \) belongs to the class \( \mathcal{S}^s_e \) if and only if there exists an analytic function \( q \), \( q < e^z \) such that
\[ f(z) = z \exp \left( \int_0^z \frac{q(t)-1}{t} \, dt \right). \]
Define the functions \( h_n \) \((n = 2, 3, \ldots)\) by \( h(0) = h'(0) = 0 = 1 \) and
\[ \frac{zh_n(z)}{h_n(z)} = e^{z^{n-1}} \quad (z \in \mathbb{D}). \]
Then \( h_n \in \mathcal{S}^s_e \) \((n = 2, 3, \ldots)\). In terms of the Taylor series expansion, the functions \( h_n \) takes the form
\[ h_n(z) = z + \frac{1}{n-1}z^{n-1} + \cdots. \]
In particular, the function
\begin{equation}
(1.2) \quad h(z) := h_2(z) = z \exp \left( \int_0^z \frac{e^t - 1}{t} \, dt \right) = z + z^2 + \frac{3}{4} z^3 + \frac{17}{36} z^4 + \cdots
\end{equation}
plays the role of extremal function for many extremal problems over the class \( S_e^* \).

In Section 2, we investigate the geometric properties of functions in class \( S_e^* \). In particular, the relations between \( S_e^* \) and other classes geometrically defined are considered. The sharp radii of starlikeness of order \( \alpha (0 \leq \alpha < 1) \), parabolic starlikeness (and some of others) of \( f \in S_e^* \) are determined in the last section of the paper. The sharp \( S_e^* \)-radii for certain well-known classes of functions are also obtained.

2. Properties of functions in the class \( S_e^* \)

In this section, we will determine the inclusion relations, coefficient estimates, growth and distortion results and convolution properties of functions in the class \( S_e^* \). The following two lemmas will be needed in our investigation.

**Lemma 2.1.** For \( r \in (0, 1) \), the function \( \varphi_0(z) = e^z \) satisfies
\[
\min_{|z|=r} \Re \varphi_0(z) = \varphi_0(-r) = \min_{|z|=r} |\varphi_0(z)| \quad \text{and} \quad \max_{|z|=r} \Re \varphi_0(z) = \varphi_0(r) = \max_{|z|=r} |\varphi_0(z)|.
\]

**Proof.** For \( \theta \in [0, 2\pi) \), the function \( \psi_0(\theta) = \Re \varphi_0(re^{i\theta}) = e^{r \cos \theta} \cos(r \sin \theta) \) attains its minimum at \( \theta = \pi \) and maximum at \( \theta = 0 \). Consequently,
\[
\min_{|z|=r} \Re \varphi_0(z) = e^{-r} = \varphi_0(-r) \quad \text{and} \quad \max_{|z|=r} \Re \varphi_0(z) = e^r = \varphi_0(r).
\]
The other equality follows by observing that the real-valued function \( |e^z| = e^{\Re z} \) is strictly increasing in the interval \([r-r, r] \).

**Lemma 2.2.** For \( 1/e < a < e \), let \( r_a \) be given by
\[
r_a = \begin{cases} 
   a - e^{-1}, & e^{-1} < a \leq (e + e^{-1}) / 2; \\
   e - a, & (e + e^{-1}) / 2 \leq a < e.
\end{cases}
\]
Then
\[
\{ w : |w - a| < r_a \} \subset \{ w : |\log w| < 1 \}.
\]

**Proof.** Let \( \varphi_0(z) = e^z \). Then any point on the boundary of \( \varphi_0(D) \) is of the form \( \varphi_0(e^{i\theta}) = e^{\cos \theta} \cos(\sin \theta) + ie^{\cos \theta} \sin(\sin \theta) \) for \( \theta \in [-\pi, \pi] \). Note that the curve \( w = \varphi_0(e^{i\theta}) \) is symmetric with respect to the real axis, so it is sufficient to consider the interval \( 0 \leq \theta \leq \pi \). The square of the distance from the point \((a, 0)\) to the points on the curve \( w = \varphi_0(e^{i\theta}) \) is given by
\[
z(\theta) = (a - e^{\cos \theta} \cos(\sin \theta))^2 + e^{2\cos \theta} \sin^2(\sin \theta) = e^{2\cos \theta} - 2ae^{\cos \theta} \cos(\sin \theta) + a^2.
\]

Let us first assume that \( 1/e < a \leq e/2 \). Then \( z(\theta) \) is a decreasing function of \( \theta \in [0, \pi] \). Consequently, we have
\[
r_a = \min_{\theta \in [0, \pi]} \sqrt{z(\theta)} = \sqrt{z(\pi)} = a - \frac{1}{e}.
\]
Next, assume that \( e/2 < a < e \). A calculation shows that
\[
z'(\theta) = -2e^{\cos \theta} (e^{\cos \theta} \sin \theta - a \sin(\theta + \sin \theta))
\]
and hence $z'(0) = z'\left(\pi\right) = \frac{1}{e}$, where $\theta_0 \in (0, \pi)$ is the real root of the equation $e^{\cos \theta} \sin \theta = a \sin(\theta + \sin \theta)$. Observe that $\theta_{a_1} < \theta_{a_2}$ for $a_1 < a_2$. Moreover, the function $z(\theta)$ is increasing for $\theta \in [0, \theta_a]$ and decreasing for $\theta \in [\theta_a, \pi]$. Also,

$$z(\pi) - z(0) = 2 \left( e - \frac{1}{e} \right) \left( a - \frac{1}{2} \left( e + \frac{1}{e} \right) \right).$$

These observations lead to two cases:

Case 1: $e/2 < a \leq (e + e^{-1})/2$. In this case $\min\{z(0), z(\theta_0), z(\pi)\} = z(\pi)$. Thus $z(\theta)$ attains its minimum value at $\theta = \pi$ and $r_a = \min \sqrt{z(\theta)} = a - 1/e$.

Case 2: $(e + e^{-1})/2 \leq a < e$. It is easy to see that $\min\{z(0), z(\theta_0), z(\pi)\} = z(0)$ and hence $r_a = \min \sqrt{z(\theta)} = e - a$ in this case. This completes the proof of the lemma.

**Remark 2.1.** Following the notation and method of the proof of Lemma 2.2, it is easy to deduce that

$$\{w : |\log w| < 1\} \subset \{w : |w - a| < R_a\},$$

where $R_a$ is given by

$$R_a = \begin{cases} e - a, & e^{-1} < a \leq e/2; \\ z(\theta_0), & e/2 < a < e. \end{cases}$$

### 2.1. Inclusion Relations

Recall that starlike functions of order $\alpha$ ($0 \leq \alpha < 1$) and strongly starlike functions of order $\gamma$ ($0 < \gamma \leq 1$) are characterized by the conditions $\Re(zf'(z)/f(z)) > \alpha$ and $|\arg(zf'(z)/f(z))| \leq \gamma \pi/2$ respectively. Kanas and Wisniowska [17] introduced the class $\mathcal{K}_*^\gamma$ of $k$-starlike $(k \geq 0)$ functions $f \in \mathcal{A}$ defined by the condition

$$\Re \frac{zf'(z)}{f(z)} > k - \frac{z|f'(z)|}{f(z)} - 1 \quad (z \in \mathbb{D}),$$

which provides a continuous passage from starlike functions $(k = 0)$ to parabolic starlike functions $(k = 1)$. Another interesting class is $\mathcal{M}(\beta)$, $\beta > 1$, defined by

$$\mathcal{M}(\beta) = \left\{ f \in \mathcal{A} : \Re \frac{zf'(z)}{f(z)} < \beta, z \in \mathbb{D} \right\}.$$

This class was investigated by Urielgaddi et al. [47]. The following theorem investigates the inclusion relations between the classes $\mathcal{S}_*^\alpha$ $(0 \leq \alpha < 1)$, $\mathcal{S}_*^\gamma$ $(0 < \gamma \leq 1)$, $\mathcal{M}(\beta)$ $(\beta > 1)$, $k - \mathcal{S}_*^\gamma (k \geq 0)$ and $\mathcal{S}_*^\alpha$.

**Theorem 2.1.** The class $\mathcal{S}_*^\alpha$ satisfies the following relationships:

(i) $\mathcal{S}_*^\alpha \subset \mathcal{S}_*^\alpha(0 \leq \alpha \leq 1/e)$;

(ii) $\mathcal{S}_*^\alpha \subset \mathcal{M}(\beta)$ for $\beta \geq e$;

(iii) $\mathcal{S}_*^\alpha \subset \mathcal{S}_*^\gamma$ for $2/e \leq \gamma \leq 1$;

(iv) $k - \mathcal{S}_*^\gamma \subset \mathcal{S}_*^\alpha$ for $k \geq e/(e - 1)$.

The constants $1/e$, $2/e$ and $e/(e - 1)$ in parts (i), (ii), (iii) and (iv) respectively are best possible.

**Proof.** Let $f \in \mathcal{S}_*^\alpha$. Then $zf'(z)/f(z) < e^z$. By Lemma 2.1, it is easy to deduce that

$$\frac{1}{e} = \min_{|z|=1} \Re e^z < \Re \frac{zf'(z)}{f(z)} < \max_{|z|=1} \Re e^z = e \quad (z \in \mathbb{D}).$$
Thus \( f \in \mathcal{S}^* (1/e) \cap \mathcal{M}(e) \). Also,
\[
\left| \arg \frac{z f'(z)}{f(z)} \right| \leq \max_{|z|=1} \arg e^z = 1 \quad (z \in \mathbb{D})
\]
which shows that \( f \in \mathcal{S}^* (2/\pi) \).

For (iv), let \( f \in k - \mathcal{S}^* \) and consider the conic domain \( \Gamma_k = \{ w \in \mathbb{C} : \Re w > k|w - 1| \} \).
For \( k > 1 \), the curve \( \partial \Gamma_k \) is the ellipse \( \gamma_k : x^2 = k^2(x - 1)^2 + k^2y^2 \) which may be rewritten as
\[
\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} = 1,
\]
where \( x_0 = k^2/(k^2 - 1) \), \( y_0 = 0 \), \( a = k/(k^2 - 1) \) and \( b = 1/\sqrt{k^2-1} \). For the ellipse \( \gamma_k \) to lie inside \( |\log w| \leq 1 \), it is necessary that \( x_0 + a \leq e \), which is equivalent to the condition \( k \geq e/(e-1) \).
Figure 1 depicts that the ellipse \( \gamma_{e/(e-1)} \) lies completely inside \( |\log w| \leq 1 \).

\[ \begin{align*}
A &= 1/e \\
B &= e \\
\arg C &= -\arg D = 1 \\
k &= e/(e-1)
\end{align*} \]

\[ \begin{array}{c}
 \text{Figure 1. Image of the unit circle under the mapping } e^z. \\
\end{array} \]

\[ \begin{array}{c}
\text{For (iv), let } f \in k - \mathcal{S}^* \text{ and consider the conic domain } \Gamma_k = \{ w \in \mathbb{C} : \Re w > k|w - 1| \}. \\
\text{For } k > 1, \text{ the curve } \partial \Gamma_k \text{ is the ellipse } \gamma_k : x^2 = k^2(x - 1)^2 + k^2y^2 \text{ which may be rewritten as} \\
\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} = 1, \\
\text{where } x_0 = k^2/(k^2 - 1), y_0 = 0, a = k/(k^2 - 1) \text{ and } b = 1/\sqrt{k^2-1}. \text{ For the ellipse } \gamma_k \text{ to} \\
\text{lie inside } |\log w| \leq 1, \text{ it is necessary that } x_0 + a \leq e, \text{ which is equivalent to the condition} \\
k \geq e/(e-1). \text{ Figure 1 depicts that the ellipse } \gamma_{e/(e-1)} \text{ lies completely inside } |\log w| \leq 1. \\
\text{Also, since } \Gamma_{k_1} \subset \Gamma_{k_2} \text{ for } k_1 \geq k_2, \text{ it follows that } k - \mathcal{S}^* \subset \mathcal{S}^*_e \text{ for } k \geq e/(e-1). \]

\textbf{Remark 2.2.} In [10], Aouf, Dziok and Sokół investigated the properties of functions in the class \( \mathcal{S}^*(q_c) \), where \( q_c(z) = \sqrt{1 + cz}, c \in (0, 1] \). In particular, \( \mathcal{S}^*(q_1) = \mathcal{S}^*_L \). The function \( q_c \) maps \( \mathbb{D} \) onto the domain
\[ \mathcal{O}_c = \{ w \in \mathbb{C} : \Re w > 0, |w^2 - 1| < c \} \]
and its boundary \( \partial \mathcal{O}_c \) is the right loop of the Cassinian Ovals
\[ (x^2 + y^2)^2 - 2(x^2 - y^2) = c^2 - 1. \]

Using the similar analysis carried out in Theorem 2.1, it can be shown that \( \mathcal{S}^*(q_c) \subset \mathcal{S}^*_c \) for \( c \leq 1 - (1/e^2) \approx 0.864665 \), and this bound is best possible.

For \( -1 \leq B < A \leq 1 \), let \( \mathcal{P}[A, B] \) be the class of analytic functions \( p \) of the form \( p(z) = 1 + c_1z + c_2z^2 + \cdots \) satisfying \( p(z) \prec (1 + Az)/(1 + Bz) \) for all \( z \in \mathbb{D} \). We write \( \mathcal{P}[1 - 2\alpha, -1] = \mathcal{P}(\alpha) \) \( (0 \leq \alpha < 1) \) and \( \mathcal{P}(0) = \mathcal{P} \). The following lemma will be needed to determine the conditions on parameters \( A \) and \( B \) so that \( \mathcal{S}^*[A, B] \) is a subclass of \( \mathcal{S}^*_e \).

\textbf{Lemma 2.3.} ([31, Lemma 2.1, p. 267], [38]) If \( p \in \mathcal{P}[A, B] \), then
\[ \left| p(z) - \frac{1 - ABr^2}{1 - B^2r^2} \right| \leq \frac{(A - B)r}{1 - B^2r^2} \quad (|z| = r < 1). \]
Moreover, if \( p \in \mathcal{D}(\alpha) \), then
\[
|p(z) - \frac{1 + (1 - 2\alpha)r^2}{1 - r^2}| \leq \frac{2(1 - \alpha)r}{1 - r^2},
\]
and
\[
\left| \frac{zp'(z)}{p(z)} \right| \leq \frac{2r(1 - \alpha)}{(1 - r)(1 + (1 - 2\alpha)r)} \quad (|z| = r < 1).
\]

**Theorem 2.2.** Let \(-1 < B < A \leq 1\) and either

(i) \(2(1 - B^2) < 2e(1 - AB) \leq (e^2 + 1)(1 - B^2)\) and \((1 - B) \leq (1 - A)e\); or

(ii) \((1 + e^2)(1 - B^2) \leq 2(1 - AB)e < 2(1 - B^2)e^2\) and \((1 + A) \leq e(1 + B)\),

then \( \mathcal{S}^*[A, B] \subset \mathcal{S}^* \).

**Proof.** Let \( f \in \mathcal{S}^*[A, B] \). Then \( zf'(z)/f(z) \in \mathcal{S}[A, B] \) so that Lemma 2.3 gives
\[
(2.1) \quad \left| \frac{zf'(z)}{f(z)} - \frac{1 - AB}{1 - B^2} \right| < \frac{A - B}{1 - B^2}.
\]

Let \( a := (1 - AB)/(1 - B^2) \) and suppose that the two conditions in (i) hold. By multiplying both sides of the inequality \((1 - B) \leq (1 - A)e\) by the positive constant \(1 + B\) and rewriting, we obtain \((A - B)e \leq (1 - AB)e - (1 - B^2)\). A division by \(e(1 - B^2)\) shows that the condition \((1 - B) \leq (1 - A)e\) is equivalent to \((A - B)/(1 - B^2) \leq a - 1/e\). Similarly, the condition \(2(1 - B^2) < 2e(1 - AB) \leq (e^2 + 1)(1 - B^2)\) is equivalent to \(1/e < a \leq (e + e^{-1})/2\). From (2.1), it follows that the values of \( w = zf'(z)/f(z) \) lies in the disk \(|w - a| < r_a\), where \(r_a := a - 1/e\) and \(1/e < a \leq (e + e^{-1})/2\). Hence \( f \in \mathcal{S}^* \) by Lemma 2.2. A similar argument shows that \( f \in \mathcal{S}^* \) if condition (ii) is satisfied and therefore its details are omitted.

### 2.2. Coefficient Estimates

The estimation of coefficient bounds is one of the classical problems in univalent function theory. The famous Bieberbach conjecture for the class \( \mathcal{S} \) which stood as a challenge for several years, was finally settled by de Branges [12] in 1984. There are still many open problems concerning determination of sharp coefficient bounds for various subclasses of \( \mathcal{S} \) such as \( \mathcal{S}_C \), \( \mathcal{S}_R^e \) and \( \mathcal{S}_{RIL}^e \) (see [7, 26, 43, 48]).

The correspondence between the classes \( \mathcal{S}^* \) and \( \mathcal{K}_e \) and [22, Theorem 3, p. 164] yield the sharp upper bound for the Fekete-Szegő functional \(|a_3 - \mu a_2^3|\) in the class \( \mathcal{S}^* \) for all real \( \mu \). If \( f(z) = z + a_2z^2 + a_3z^3 + \cdots \in \mathcal{S}^* \), then
\[
|a_3 - \mu a_2^3| \leq \begin{cases} (3 - 4\mu)/4, & \mu \leq 1/4; \\ 1/2, & 1/4 \leq \mu \leq 5/4; \\ -(3 - 4\mu)/4, & \mu \geq 5/4. \end{cases}
\]

Suppose that the functions \( h_\lambda \) are given by (1.1). For \( 0 \leq \lambda \leq 1 \), define \( f_\lambda \) and \( g_\lambda \) by \( f_\lambda(0) = f'_\lambda(0) - 1 = g_\lambda(0) = g'_\lambda(0) - 1 = 0 \),
\[
\frac{zf'_\lambda(z)}{f_\lambda(z)} = \exp \left( \frac{z(z + \lambda)}{1 + \lambda z} \right) \quad \text{and} \quad \frac{zg'_\lambda(z)}{g_\lambda(z)} = \exp \left( -\frac{z(z + \lambda)}{1 + \lambda z} \right),
\]
respectively. If \( \mu < 1/4 \) or \( \mu > 5/4 \), then equality holds if and only if \( f \) is \( h_2 \) or one of its rotation. If \( 1/4 < \mu < 5/4 \), then equality holds if and only if \( f \) is equal to \( h_3 \) or one of its rotation. If \( \mu = 1/4 \), equality holds if and only if \( f \) is equal to \( f_\lambda \) or one of its rotation. When \( \mu = 5/4 \), inequality becomes equality if and only if \( f \) equals \( g_\lambda \) or one of its rotation.
These observations together with [8, Theorem 1, p. 38] yield the sharp upper bound on the absolute value of second, third and fourth coefficient of functions in the class \( \mathcal{S}_e^* \).

**Theorem 2.4.** If \( f(z) = z + a_2z^2 + a_3z^3 + a_4z^4 + \cdots \in \mathcal{S}_e^* \), then \(|a_2| \leq 1, |a_3| \leq 3/4 \) and \(|a_4| \leq 17/36 \). These bounds are sharp for the function \( h \) given by (1.2).

Since the function \( h \) given by (1.2) maximizes \(|a_n|\) for \( n = 2, 3, 4 \) in the class \( \mathcal{S}_e^* \), it is natural to suspect that \( h \) maximizes \(|a_n|\) for each \( n \). But we are not able to prove it for \( n > 4 \). However, we may obtain bounds on \(|a_n|\) \((n = 5, 6, \ldots)\), although they are not sharp. By Theorem 2.1(i), \( \mathcal{S}_e^* \subset \mathcal{S}(1/e) \) and hence

\[
|a_n| \leq \frac{1}{(n-1)!} \prod_{k=2}^{n} \left( k - \frac{2}{e^2} \right) \quad (n = 2, 3, \ldots)
\]

for a function \( f \in \mathcal{S}_e^* \) (see [37]). These bounds can be further improved by making use of the result by Rogosinski [33]: if \( h(z) = 1 + \sum_{k=1}^{\infty} c_k z^k \) is subordinate to \( H(z) = 1 + \sum_{k=1}^{\infty} C_k z^k \) in \( \mathbb{D} \), where \( H \) is univalent in \( \mathbb{D} \) and \( H(\mathbb{D}) \) is convex, then \(|c_n| \leq |C_1|\) for \( n = 1, 2, \ldots \).

**Theorem 2.4.** If \( f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in \mathcal{S}_e^* \), then \(|a_n| \leq 1\) for \( n \geq 2 \).

**Proof.** Note that

\[
1 + \sum_{k=2}^{\infty} c_k z^k = q(z) := \frac{zf'(z)}{f(z)}, \quad \varphi_0(z) = e^z = 1 + \frac{z^2}{2!} + \cdots.
\]

Since \( \varphi_0 \) maps \( \mathbb{D} \) univalently onto a convex domain, by Rogosinski’s result, \(|c_n| \leq 1\) for each \( n \). Writing \( zf'(z) = q(z)f(z) \) and comparing the coefficient of \( z^n \) on both sides, we obtain

\[
(n-1)a_n = \sum_{k=1}^{n-1} c_n c_{n-k} a_k.
\]

Therefore, \(|a_2| = |c_1| \leq 1\). Assume that \(|a_k| \leq 1\) for \( k = 3, 4, \ldots, n-1 \). Then it is easy to see that

\[
(n-1)|a_n| = \sum_{k=1}^{n-1} |c_n - a_k| \leq \sum_{k=1}^{n-1} |a_k| \leq n - 1.
\]

The result now follows by induction.

If \( f \in \mathcal{S}_e^* \cap \mathcal{A}_3 \), then the result of Theorem 2.4 can be further strengthened as seen by the following theorem.

**Theorem 2.5.** If \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}_e^* \), then

\[
\sum_{n=2}^{\infty} \left( n^2 - 2 \right)|a_n|^2 \leq e^2 - 1.
\]

**Proof.** Since \( f \in \mathcal{S}_e^* \), therefore \( zf'(z)/f(z) = e^{w(z)} \) where \( w \) is an analytic function in \( \mathbb{D} \) with \( w(0) = 0 \) and \(|w(z)| < 1\) for all \( z \in \mathbb{D} \). Using the identity \( f^2(z) = e^{-2w(z)} \left( zf'(z) \right)^2 \), we have

\[
2\pi \sum_{n=1}^{\infty} |a_n|^2 r^{2n} = \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta = \int_0^{2\pi} |e^{-2w(re^{i\theta})}| |re^{i\theta} f'(re^{i\theta})|^2 d\theta
\]

\[
\geq e^{-2} \int_0^{2\pi} |re^{i\theta} f'(re^{i\theta})|^2 d\theta = 2\pi e^{-2} \sum_{n=1}^{\infty} n^2 |a_n|^2 r^{2n}
\]
where $0 < r < 1$ and $a_1 = 1$. Thus
\[ \sum_{n=1}^{\infty} (n^2 - e^2)|a_n|^2 r^{2n} \leq 0. \]
On letting $r \to 1^-$, we obtain the required result. 

**Corollary 2.1.** Let $f(z) = z + \sum_{k=1}^{\infty} a_k z^k \in \mathcal{K}_e$. Then
\[ |a_n| \leq \sqrt{\frac{e^2 - 1}{n^2 - e^2}} < 1 \quad \text{for } n = 4, 5, \ldots. \]

**Remark 2.3.** Since $\sum_{n=1}^{\infty} \frac{1}{(n^2)^2} < \infty$, therefore the function $\varphi_0(z) = e^z$ belongs to $\mathcal{H}^2$, the Hardy class of analytic functions in $\mathbb{D}$. Hence for a function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{K}_e$, the sharp order of growth $|a_n| = O(1/n)$ by [22, Corollary 4, p. 166].

By making use of Lemma 2.2, we next determine necessary and sufficient conditions for some special members to be in the class $\mathcal{K}_e$.

**Theorem 2.6.** (i) A function $f(z) = z + a_n z^n \quad (n = 2, 3, \ldots)$ belongs to the class $\mathcal{K}_e$ if and only if $|a_n| \leq (e - 1)/(ne - 1)$.
(ii) The function $z/(1 - Az)^2$ is in $\mathcal{K}_e$ if and only if $|A| \leq (e - 1)/(e + 1)$.

**Proof.** (i) Since $\mathcal{K}_e \subset \mathcal{K}_0$, $|a_n| \leq 1/n$. To find the sharp estimate, observe that $w = zf'(z)/f(z) = (1 + na_ne^{n-1})/(1 + a_n e^{n-1})$ maps $\mathbb{D}$ onto the disk
\[ w < \frac{1 - n|a_n|^2}{1 - |a_n|^2} < \frac{(n - 1)|a_n|}{1 - |a_n|^2}. \]
Since $(1 - n|a_n|^2)/(1 - |a_n|^2) \leq 1$, therefore, in view of Lemma 2.2, the disk (2.2) lies inside $|\log w| \leq 1$ if and only if
\[ \frac{(n - 1)|a_n|}{1 - |a_n|^2} \leq \frac{1 - n|a_n|^2}{1 - |a_n|^2} - \frac{1}{e}. \]
This yields $|a_n| \leq (e - 1)/(ne - 1)$.

(ii) Clearly, the Koebe function $k(z) = z/(1 - z)^2 \notin \mathcal{K}_e$. Let $g(z) = z/(1 - Az)^2 \quad (A \neq 1)$. Then the bilinear transformation $w = zg'(z)/g(z) = (1 + Az)/(1 - Az)$ maps $\mathbb{D}$ onto the disk
\[ w < \frac{1 + |A|^2}{1 - |A|^2} \leq \frac{2|A|}{1 - |A|^2}. \]
with diametric end points $x_L = (1 - |A|)/(1 + |A|)$ and $x_R = (1 + |A|)/(1 - |A|)$. If $g \in \mathcal{K}_e$, then the disk (2.3) lies inside $|\log w| \leq 1$. Consequently, it is necessary that $x_R \leq e$ which gives $|A| \leq (e - 1)/(e + 1)$. Conversely, if $|A| \leq (e - 1)/(e + 1)$, then $a := (1 + |A|^2)/(1 - |A|^2) \leq (e + e^{-1})/2$ and $2|A|/(1 - |A|^2) \leq a - 1/e$. By again applying Lemma 2.2, we conclude that the disk (2.3) lies inside $|\log w| \leq 1$ and hence $g \in \mathcal{K}_e$.

### 2.3. Subordination Results and their consequences

If $f \in \mathcal{K}_e$, then $f(z)/z < h(z)/z$ by [22, Theorem 1’, p. 161], where $h$ is given by (1.2). Since the function $e^z$ is convex univalent, this result can also be obtained as a special case of [35, Theorem 1, p. 275]. Using this subordination relation, or by directly applying the results of [22], we obtain the following result.
Theorem 2.7. Let \( f \in \mathcal{S}_e^* \) and \( h \) be given by (1.2). Then, for \( |z| = r \), we have the following:

(i) (Growth Theorem)
\[
-h(r) \leq |f(z)| \leq h(r).
\]
In particular, \( f(\mathbb{D}) \supset \{ w : |w| < -h(-1) \approx 0.450859 \} \).

(ii) (Rotation Theorem)
\[
\left| \arg \left( \frac{f(z)}{z} \right) \right| \leq \max_{|z|=r} \left| \frac{h(z)}{z} \right|.
\]

(iii) (Distortion Theorem)
\[
h'(r) \leq |f'(z)| \leq h'(r).
\]
Equality holds for some \( z \neq 0 \) if and only if \( f \) is a rotation of \( h \).

It is worth to remark that the proof of Theorem 2.7(iii) makes use of Lemma 2.1. In [34], Rønning proved that \( |f(z)| \leq K|z| \) \( (z \in \mathbb{D}) \) for functions \( f \in \mathcal{S}_p \), where \( K = \exp(14\zeta(3)/\pi^2) \approx 5.502 \) (\( \zeta(t) \) is the Reimann Zeta function). The next corollary proves the corresponding result for \( \mathcal{S}_e^* \).

Corollary 2.2. If \( f \in \mathcal{S}_e^* \), then
\[
|f(z)| \leq |z| \exp \left( \sum_{n=1}^{\infty} \frac{1}{n(n!)} \right) = |z|K,
\]
for all \( z \in \mathbb{D} \), where \( K \approx 3.73558 \). The function \( h \) given by (1.2) shows that this bound cannot be further improved.

Proof. By Theorem 2.7(i), for \( |z| = r \), we have
\[
\log \left| \frac{f(z)}{z} \right| \leq \int_0^r \frac{e^t - 1}{t} \, dt \leq \int_0^1 \frac{e^t - 1}{t} \, dt = \sum_{n=1}^{\infty} \frac{1}{n(n!)}.\]
The series on the right side of the above inequality is convergent and hence we obtain the desired result.

In terms of subordination, Tuneski [46] gave an interesting criteria for analytic functions to be in the class \( \mathcal{S}_e^*[A,B] \) \( (-1 \leq B < A \leq 1) \). In 2007, Sokół [41] generalized this result using Jack lemma [15] and obtained a sufficient condition for functions \( f \in \mathcal{A} \) to be in a more general class of functions. As an application, observe that the function \( \varphi_0(z) = e^z \) is univalent and non-vanishing in \( \mathbb{D} \) with \( \varphi_0(0) = 1 \) and such that
\[
\text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) > 2 \text{Re} \left( \frac{z\varphi''_0(z)}{\varphi'_0(z)} \right) \quad (z \in \mathbb{D}).
\]
Therefore, a function \( f \in \mathcal{S}_e^* \) satisfying
\[
\frac{1+zf''(z)}{zf'(z)/f(z)} - 1 < ze^{-z} \quad (z \in \mathbb{D})
\]
belongs to the class \( \mathcal{S}_e^* \), by [41, Corollary 1, p. 239].

Let \( p \) be an analytic function in \( \mathbb{D} \) with \( p(0) = 1 \). Recently, Ali et al. [3] and Kumar et al. [18] determined conditions on \( \beta \) so that \( p(z) \prec \sqrt{1+z} \) when \( 1 + \beta z p'(z)/p(z) \) is subordinated to \( \sqrt{1+z} \) or \( (1+Az)/(1+Bz) \). This, in turn, provide sufficient conditions for analytic functions \( f \in \mathcal{S}_e^* \) to belong to the class \( \mathcal{S}_L^* \). Motivated by their work, we prove the corresponding subordination results involving the exponential function.
Theorem 2.8. Let \( p \) be an analytic function in \( \mathbb{D} \) with \( p(0) = 1 \). If either of the following three conditions is satisfied:

(a) \( 1 + \beta z p'(z) / p(z) < e^z \) \( (\beta \geq e - 1) \),
(b) \( 1 + \beta z p'(z) / p(z) < (1 + Az) / (1 + Bz) \) \( ( -1 < B < A \leq 1, |\beta| \geq (A - B) / (1 - |B|)) \),
(c) \( 1 + \beta z p'(z) / p(z) < \sqrt{1 + z} \) \( (\beta \geq 1) \),

then \( p(z) \prec e^z \).

Proof. Let \( \varphi_0 \) be the convex univalent function defined by \( \varphi_0(z) = e^z \). Then it is clear that \( \beta \varphi_0(z) \) is starlike. The main crux of the proof relies on the observation that if the subordination

\[
1 + \beta \frac{z p'(z)}{p(z)} < 1 + \beta \frac{z \varphi_0'(z)}{\varphi_0(z)} = 1 + \beta z := \psi(z)
\]
is satisfied, then \( p(z) \prec \varphi_0(z) \) by [27, Corollary 3.4h.1, p. 135].

(a) It suffices to show that \( \varphi_0(z) \prec \psi(z) \). Since \( \psi(\mathbb{D}) = \{ w \in \mathbb{C} : |w - 1| < \beta \} \), it follows that \( \varphi_0(\mathbb{D}) \subset \psi(\mathbb{D}) \) if \( \beta \geq e - 1 \) by Remark 2.1. Hence \( \varphi_0(z) \prec \psi(z) \) and consequently \( p(z) \prec e^z \).

(b) Set \( \phi(z) = (1 + Az) / (1 + Bz) \). Then \( \phi^{-1}(w) = (w - 1) / (A - Bw) \). Since the subordination \( \phi(z) \prec \psi(z) \) is equivalent to \( z \prec \phi^{-1}(\psi(z)) \), we only need to show that \( |\phi^{-1}(\psi(e^t))| \geq 1 \) for \( -\pi \leq t \leq \pi \). For \( t \in [-\pi, \pi] \), we have

\[
|\phi^{-1}(\psi(e^t))| = \left| \frac{\beta e^t}{(A - B) - \beta Be^t} \right| \geq \frac{|\beta|}{A - B + |\beta B|} \geq 1
\]
provided \( |\beta| \geq (A - B) / (1 - |B|) \). Thus \( \phi(z) \prec \psi(z) \) and hence \( p(z) \prec e^z \).

(c) Let \( \chi(z) = \sqrt{1 + z} \). Since \( \chi(\mathbb{D}) \subset \psi(\mathbb{D}) \) if \( \beta \geq 1 \) by [4, Lemma 2.2, p. 6559], it follows that \( \chi(z) \prec \psi(z) \) and so \( p(z) \prec e^z \). \qed

For \( f \in \mathcal{A} \), the function \( p(z) = z f''(z) / f(z) \) is analytic in \( \mathbb{D} \) with \( p(0) = 1 \). As a result, Theorem 2.8 immediately yields the following corollary.

Corollary 2.3. Let \( f \in \mathcal{A} \) and set

\[
\Psi_\beta(z) = 1 + \beta \left( 1 + \frac{z f''(z)}{f'(z)} - \frac{z f'(z)}{f(z)} \right).
\]

If either of the following three conditions is satisfied:

(a) \( \Psi_\beta(z) \prec e^z \) \( (\beta \geq e - 1) \);
(b) \( \Psi_\beta(z) \prec (1 + Az) / (1 + Bz) \) \( (|\beta| \geq (A - B) / (1 - |B|)) \); or
(c) \( \Psi_\beta(z) \prec \sqrt{1 + z} \) \( (\beta \geq 1) \),

then \( f \in \mathcal{S}_e^* \).

These results can be extended to functions with fixed second coefficient by using the results of [6].

2.4. Convolution Properties

For analytic functions \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) and \( F(z) = z + \sum_{n=2}^{\infty} A_n z^n \), their convolution (or Hadamard product) is defined as \( (f * F)(z) = z + \sum_{n=2}^{\infty} a_n A_n z^n \). The classes of starlike, convex and close-to-convex functions are closed under convolution with convex functions (see [36]). A similar result holds for \( \mathcal{S}_e^* \). By [22, Theorem 5, p. 167], we have

Theorem 2.9. If \( f \in \mathcal{S}_e^* \) and \( g \in \mathcal{K} \), then \( f * g \in \mathcal{S}_e^* \).
Let $\Gamma_i : \mathcal{A} \to \mathcal{A}$ $(i = 1, 2, 3)$ be the operators defined by
\[ \Gamma_1[f](z) = zf'(z), \quad \Gamma_2[f](z) = \frac{1}{2}(f(z) + zf'(z)) \quad \text{and} \]
\[ \Gamma_3[f](z) = \frac{k+1}{e^k} \int_0^\infty \xi^{k-1} f(\xi) d\xi \quad (\text{Re}k > 0). \]
These operators were introduced by Alexander [1], Livingston [20] and Bernardi [11] respectively. Each of these operators can be written as a convolution operator: $\Gamma_i = f \ast g_i$ ($i = 1, 2, 3$), where
\[ g_1(z) = \frac{z}{(1-z)^2}, \quad g_2(z) = \frac{z - z^2/2}{(1-z)^2}, \quad \text{and} \quad g_3(z) = \sum_{n=1}^\infty \frac{k+1}{k+n} z^n. \]
The function $g_1$ is convex in $|z| < 2 - \sqrt{3}$, $g_2$ is convex in $|z| < 1/2$ while $g_3$ is convex in $\mathbb{D}$. Hence Theorem 2.9 gives

**Corollary 2.4.** Let $f \in \mathcal{S}_{e}^\ast$. Then $\Gamma_1[f] \in \mathcal{S}_{e}^\ast$ in $|z| < r_i$ $(i = 1, 2, 3)$ where $r_1 = 2 - \sqrt{3}$, $r_2 = 1/2$ and $r_3 = 1$.

The convolution of two starlike functions need not be univalent in $\mathbb{D}$. Let $f, g \in \mathcal{S}^\ast$ and $h_\rho(z) = (f \ast g)(\rho z)/\rho$. Ling and Ding [19, Theorem 1, p. 404] proved that $h_\rho \in \mathcal{S}^\ast$ for $0 \leq \rho \leq 2 - \sqrt{3}$. Ali et al. [9] determined conditions on $\rho$ so that $h_\rho$ belongs to the classes $\mathcal{S}_\rho, \mathcal{S}^\ast(\alpha)$ $(0 \leq \alpha < 1), \mathcal{S}_\rho^\ast(\gamma)$ $(0 < \gamma \leq 1)$ and $\mathcal{S}_{L\rho}^\ast$. The next theorem investigates the corresponding result for $\mathcal{S}_{e}^\ast$.

**Theorem 2.10.** If $f, g \in \mathcal{S}^\ast$, then $f \ast g \in \mathcal{S}_{e}^\ast$ in $|z| < \rho_0$, where
\begin{equation}
\rho_0 = \frac{e - 1}{2e + \sqrt{1 + 3e^2}} \approx 0.167641.
\end{equation}
The number $\rho_0$ is best possible.

**Proof.** Consider the function $H : \mathbb{D} \to \mathbb{C}$ defined by
\[ H(z) = \frac{z(1+z)}{(1-z)^3} = z + \sum_{n=2}^\infty n^2 z^n. \]
The function $H$ is analytic in $\mathbb{D}$ and satisfies
\[ \left| \frac{zH'(z)}{H(z)} - \frac{1 + r^2}{1 - r^2} \right| \leq \frac{4r}{1 - r^2}, \quad |z| = r < 1. \]
Let $0 \leq r \leq \rho_0$, where $\rho_0$ is given by (2.4). Then it is a simple exercise to show that if $a := (1+r^2)/(1-r^2)$ then $1/e < a \leq (e + e^{-1})/2$ and $4r/(1-r^2) \leq a - 1/e$. By Lemma 2.2, it follows that $H \in \mathcal{S}_{e}^\ast$ in $|z| < \rho_0$. Also, at the point $z = -\rho_0$, we have
\[ |\log \left( \frac{zH'(z)}{H(z)} \right)| = |\log \left( \frac{1 - 4\rho_0 + \rho_0^2}{1 - \rho_0^2} \right)| = |\log(1/e)| = 1. \]
This shows that the number $\rho_0$ is sharp.

Now, let $f, g \in \mathcal{S}^\ast$. Then the functions $F$ and $G$ defined by $zF'(z) = f(z)$ and $zG'(z) = g(z)$ belong to $\mathcal{K}$. Since the convolution of two convex functions is convex, $F \ast G \in \mathcal{K}$. Since $H(\rho_0 z)/\rho_0 \in \mathcal{S}_{e}^\ast$, therefore $(F \ast G \ast H)(\rho_0 z)/\rho_0 \in \mathcal{S}_{e}^\ast$ by Theorem 2.9. But $f \ast g = F \ast G \ast H$. Hence $f \ast g \in \mathcal{S}_{e}^\ast$ in $|z| < \rho_0$. 

\[ \square \]
3. Radius Problems

Let $\mathcal{F}$ and $\mathcal{G}$ be subsets of $\mathcal{A}$. Then the $G$-radius in $\mathcal{F}$, denoted by $R_g(\mathcal{F})$ is the largest $R$ such that for every $f \in \mathcal{F}$, $r^{-1}f(rz) \in \mathcal{G}$ for each $r \leq R$. In particular, if $\mathcal{F} \subset \mathcal{G}$ then $R_g(\mathcal{F}) = 1$. In this section, we compute $R_g(\mathcal{I}_e^*)$ and $R_g(\mathcal{I}_e^*)$ for various subclasses $\mathcal{F}$ and $\mathcal{G}$ of $\mathcal{A}$.

The first theorem of this section determines sharp $I_e^*(\alpha) (0 \leq \alpha < 1)$, $I_e^*(\gamma) (0 < \gamma \leq 1)$, $M(\beta) (\beta > 1)$ and $k - I_e^*(k \geq 0)$ radii in the class $I_e^*$. By Theorem 2.1, it is known that $R_{\mathcal{I}_e^*} = R_{\mathcal{F}_e^*} = R_{\mathcal{G}_e^*} = R_{\mathcal{M}(\beta)} = R_{\mathcal{M}(\beta)}(\mathcal{I}_e^*) = 1$ for $0 \leq \alpha \leq 1/e, 2/\pi \leq \gamma \leq 1$ and $\beta \geq e$.

**Theorem 3.1.** Let $f \in I_e^*$. Then we have the following.

- (i) If $1/e \leq \alpha < 1$, then $f$ is starlike of order $\alpha$ in $|z| < (-\log \alpha)$;
- (ii) If $1 < \beta \leq e$, then $f \in M(\beta)$ in $|z| < \log \beta$;
- (iii) If $0 < \gamma < 2/\pi$, then $f$ is strongly starlike of order $\gamma$ in $|z| < \gamma \pi / 2$;
- (iv) If $k > 0$, then $f$ is $k$-starlike in $|z| < \log((1+k)/k)$. In particular, $f$ is parabolic starlike in $|z| < \log 2$.

The results are all sharp.

**Proof.** Since $f \in I_e^*$, $zf'(z)/f(z) < e^r$ and hence Lemma 2.1 gives

$$e^{-r} \leq \Re \frac{zf'(z)}{f(z)} \leq e^r, |z| = r < 1$$

which verifies the validity of parts (i) and (ii). The function $h$ given by (1.2) shows that the constants $-\log \alpha$ and $\log \beta$ are best possible.

Also, we can write $zf'(z)/f(z) = e^{w(z)}$ where $w$ is an analytic function in $D$ with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in D$. By Schwarz lemma, $|w(z)| \leq |z|$ for all $z \in D$. If $|z| < \gamma \pi / 2$, then

$$|\arg e^{w(z)}| = |\Im w(z)| \leq |w(z)| \leq |z| < \gamma \pi / 2.$$  

Thus $f \in I_e^*(\gamma)$ in $|z| < \gamma \pi / 2$. At the point $z_0 = i \gamma \pi / 2$, the function $h$ given by (1.2) gives

$$|\arg \frac{z_0 h'(z_0)}{h(z_0)}| = |\arg e^{\gamma \pi / 2} = |\Im z_0| = \frac{\gamma \pi}{2}.$$  

This proves (iii).

For the proof of (iv), note that $f$ is $k$-starlike in $|z| < r$ whenever $\Re e^{w(z)} > k|e^{w(z)} - 1|$. Since $\Re e^{w(z)} > e^{-r}$ and $|e^{w(z)} - 1| < 1 - e^{-r}$, we conclude that the condition $e^{-r} > k(1 - e^{-r})$ is sufficient for the inequality $\Re e^{w(z)} > k|e^{w(z)} - 1|$ to hold. Hence solving $e^{-r} > k(1 - e^{-r})$, we obtain $r < \log((1+k)/k)$. For the function $h$ given by (1.2) and $z_0 = -\log((1+k)/k)$, we have

$$\Re \frac{z_0 h'(z_0)}{h(z_0)} = \Re e^{\gamma \pi / 2} = \frac{k}{1+k} = k|1 - e^{\gamma \pi / 2}| = k \left| 1 - \frac{z_0 h'(z_0)}{h(z_0)} \right|.$$  

This completes the proof of the theorem.

**Remark 3.1.** Let $f \in I_e^*$. Then $zf'(z)/f(z) = e^{w(z)}$ where $w$ is Schwarz function. Differentiation gives $1 + zf''(z)/f'(z) = e^{w(z)} + zw'(z)$ so that

$$\Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) \geq \Re e^{w(z)} - |zw'(z)|.$$
By using the identity \( |w'(z)| \leq (1 - |w(z)|^2)/(1 - |z|^2) \), we deduce that
\[
\text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) \geq e^{-|z|} - \frac{|z|}{1 - |z|^2}.
\]

The function \( g(r) = e^{-r} - r/(1 - r^2) \) is decreasing in \([0, 1]\) and \( g(0) = 1 \). Hence \( \text{Re}(1 + zf''(z)/f'(z)) > \alpha \) in \(|z| < r(\alpha)\) where \( r(\alpha) \) is the real root of the equation \( g(r) = \alpha \) in \((0, 1)\). In particular, \( r(0) \approx 0.478172 \).

Now, we will determine the \( \mathcal{S}_e^* \)-radii for several interesting subclasses of analytic functions. We begin with the fundamental class \( \mathcal{S} \) of normalized univalent functions.

**Theorem 3.2.** The \( \mathcal{S}_e^* \)-radius for the class \( \mathcal{S} \) is given by \( \mathcal{R}_{\mathcal{S}_e^*}(\mathcal{S}) = (e - 1)/(e + 1) \approx 0.462117. \) This radius is sharp.

**Proof.** A function \( f \in \mathcal{S} \) satisfies the sharp inequality (see [30, Theorem 6.5, p. 168]):
\[
\left| \log \frac{zf'(z)}{f(z)} \right| \leq \log \frac{1 + |z|}{1 - |z|} \quad (z \in \mathbb{D}).
\]
If \(|z| < (e - 1)/(e + 1)\), then \(|\log(zf'(z)/f(z))| < 1\). Thus \( f \in \mathcal{S}_e^* \) in \(|z| < (e - 1)/(e + 1)\). To show that the bound \((e - 1)/(e + 1)\) cannot be increased, consider the Koebe function \( k(z) = z/(1 - z)^2 \). At the point \( z_0 = (e - 1)/(e + 1) \), a computation shows that
\[
\left| \log \frac{k(z_0)}{z_0k(z_0)} \right| = \left| \log \frac{1 + z_0}{1 - z_0} \right| = 1.
\]
This proves that \((e - 1)/(e + 1)\) is the \( \mathcal{S}_e^* \)-radius for the class \( \mathcal{S} \) and that the result is sharp.

Since \( \mathcal{S}_e^* \subset \mathcal{S} \) and Koebe function \( k(z) = z/(1 - z)^2 \) is starlike, Theorem 3.2 shows that \( \mathcal{R}_{\mathcal{S}_e^*}(\mathcal{S}_e^*) = (e - 1)/(e + 1) \). We next determine the \( \mathcal{S}_e^* \)-radius for the class \( \mathcal{S}_e^*[A,B] \) \((-1 \leq B < A \leq 1\)) with the cases \( B \geq 0 \) and \( B < 0 \).

**Theorem 3.3.** Let \( 0 \leq B < A \leq 1 \). Then the \( \mathcal{S}_e^* \)-radius for the class \( \mathcal{S}_e^*[A,B] \) is given by
\[
\mathcal{R}_{\mathcal{S}_e^*}(\mathcal{S}_e^*[A,B]) = \min \left\{ 1, \frac{e - 1}{Ae - B} \right\}.
\]
The result is sharp. In particular, if \( 1 - B \leq (1 - A)e \), then \( \mathcal{S}_e^*[A,B] \subset \mathcal{S}_e^* \).

**Proof.** Let \( f \in \mathcal{S}_e^*[A,B] \). Then \( zf'(z)/f(z) \in \mathcal{P}[A,B] \) so that Lemma 2.3 gives
\[
\left| \frac{zf'(z)}{f(z)} \right| \geq \frac{1 - ABr^2}{1 - B^2r^2} \leq \frac{(A - B)r}{1 - B^2r^2}, \quad |z| = r < 1.
\]
Since \( B \geq 0 \), \( a := (1 - ABr^2)/(1 - B^2r^2) \leq 1 \). Using Lemma 2.2, the function \( f \) satisfies \(|\log(zf'(z)/f(z))| \leq 1\) provided
\[
\frac{(A - B)r}{1 - B^2r^2} \leq \frac{1 - ABr^2}{1 - B^2r^2} < \frac{1}{e}.
\]
This yields \( r \leq (e - 1)/(Ae - B) \). The result is sharp for the function
\[
f(z) = \begin{cases} 
  z(1 + Bz)^{A/B}, & B \neq 0; \\
  ze^{A/2}, & B = 0.
\end{cases}
\]
The function \( f \in \mathcal{S}^*[A,B] \) and at the point \( z_0 = (1 - e)/(Ae - B) \), we have
\[
\left| \log \frac{zf'(z_0)}{f(z_0)} \right| = \left| \log \frac{1 + Az_0}{1 + Bz_0} \right| = |\log(1/e)| = 1.
\]

**Theorem 3.4.** Let \(-1 \leq B < A \leq 1\), with \( B < 0 \). Let
\[
R_1 = \min \left\{ 1, \sqrt{\frac{1 + e^2}{(1 + e^2)B^2 + 2e(1 - AB)}} \right\}, \quad R_2 = \min \left\{ 1, \frac{e - 1}{Ae - B} \right\}
\]
and
\[
R_3 = \min \left\{ 1, \frac{2(e - 1)}{(A - B) + \sqrt{(A - B)^2 + 4B(eB - A)(e - 1)}} \right\}.
\]

Then the \( \mathcal{S}_e^* \)-radius for the class \( \mathcal{S}^*[A,B] \) is given by
\[
\mathcal{R}_{\mathcal{S}_e^*}(\mathcal{S}^*[A,B]) = \left\{ \begin{array}{ll}
R_2, & R_2 \leq R_1; \\
R_3, & R_2 > R_1.
\end{array} \right.
\]

**Proof.** Let \( f \in \mathcal{S}^*[A,B] \). Then Lemma 2.3 shows that the quantity \( w = zf'(z)/f(z) \) lies in the disk \( |w - a| \leq R \), where \( a := (1 - ABr^2)/(1 - B^2r^2) > 1 \) and \( R := (A - B)r/(1 - B^2r^2) \).

Also, observe that the numbers \( R_1 \), \( R_2 \) and \( R_3 \) are determined so that \( r \leq R_1 \) if and only if \( a \leq (e + e^{-1})/2 \), \( r \leq R_2 \) if and only if \( R \leq a - 1/e \) and \( r \leq R_3 \) if and only if \( R \leq e - a \).

Suppose that \( R_2 \leq R_1 \). Since \( r \leq R_1 \) is equivalent to \( a \leq (e + e^{-1})/2 \), \( 0 \leq r \leq R_2 \), it follows that \( a \leq (e + e^{-1})/2 \). From Lemma 2.2 the \( \mathcal{S}_e^* \)-radius satisfies the inequality \( R \leq a - 1/e \). This shows that \( f \in \mathcal{S}_e^* \) in \( |z| \leq R_2 \).

Next, assume that \( R_2 > R_1 \). In this case, since \( r \geq R_1 \) if and only if \( a \geq (e + e^{-1})/2 \), for \( r = R_2 \), we have \( a \geq (e + e^{-1})/2 \). Lemma 2.2 shows that \( f \in \mathcal{S}_e^* \) in \( |z| \leq r \) if \( R \leq e - a \), or equivalently if \( r \leq R_3 \).

For the function \( f \) given by (3.1), \( \{ zf'(z)/f(z) : |z| < r \} = \{ w : |w - a| < R \} \), where \( a \) and \( R \) are as defined above. This shows that the result is sharp.

**Corollary 3.1.** \( \mathcal{R}_{\mathcal{S}_e^*}(\mathcal{K}) = (e - 1)/e \approx 0.632121 \). The result is sharp.

**Proof.** By the well-known Marx Strohhäcker theorem [27, Theorem 2.6(a), p. 57], \( \mathcal{K} \subset \mathcal{S}^*(1/2) \) and \( \mathcal{S}^*[0, -1] = \mathcal{S}^*(1/2) \), therefore by Theorem 3.4, the \( \mathcal{S}_e^* \)-radius for the class \( \mathcal{K} \) is at least \( (e - 1)/e \). The half-plane mapping \( l(z) = z/(1 - z) \in \mathcal{K} \) satisfies \( zl'(z)/l(z) = 1/(1 - z) \). In particular, at the point \( z_0 = (e - 1)/e \), we have
\[
\left| \log \frac{z_0l'(z_0)}{l(z_0)} \right| = |\log e| = 1.
\]
This establishes the sharpness of the result.

Let \( \mathcal{W} \) be the class of functions \( f \in \mathcal{A} \) satisfying \( f(z)/z \in \mathcal{P} \). MacGregor [23], Ali et al. [4] and Mendiratta [26] determined \( \mathcal{S}^*, \mathcal{S}_L^* \) and \( \mathcal{S}_{RL}^* \) radii respectively for the class \( \mathcal{W} \). The following result determines the sharp \( \mathcal{S}_e^* \)-radius for the class \( \mathcal{W} \).

**Theorem 3.5.** The \( \mathcal{S}_e^* \)-radius for the class \( \mathcal{W} \) is given by
\[
\mathcal{R}_{\mathcal{S}_e^*}(\mathcal{W}) = \frac{e - 1}{e + \sqrt{e^2 + (e - 1)^2}} \approx 0.28956.
\]
This bound is best possible.
Thus Lemma 2.3, it is easy to deduce that
\[ |z f'(z) - 1| \leq \frac{2r}{1 - r^2}. \]
In view of Lemma 2.2, the disk \(|w - 1| \leq 2r/(1 - r^2)|\) lies inside \(|\log w| \leq 1\) if
\[ \frac{2r}{1 - r^2} \leq \frac{1}{e}. \]
This yields \(r \leq R := (e - 1)/(e + \sqrt{e^2 + (e - 1)^2})\). Hence \( \mathcal{R}_{e^*}(\mathcal{W}) \geq R\). The function \(f(z) = z(1 + z)/(1 - z) \in \mathcal{W}\) and at the point \(z_0 = -R\), we obtain
\[ \begin{align*}
|\log z_0 f'(z_0)| &= |\log \left( \frac{1 - 2R - R^2}{1 - R^2} \right)| = |\log(1/e)| = 1.
\end{align*} \]
Thus \( \mathcal{R}_{e^*}(\mathcal{W}) \leq R\).}

Motivated by [5, 24, 25], we close this section by determining the \( \mathcal{S}_{e^*}\)-radii for several classes of functions \(f \in \mathcal{A}\) characterized by its ratio with a certain function \(g\). Let \(\mathcal{F}_1\) be the class of functions \(f \in \mathcal{A}\) satisfying \(f/g \in \mathcal{P}\) for some \(g \in \mathcal{W}\).

**Theorem 3.6.** The \( \mathcal{S}_{e^*}\)-radius for the class \(\mathcal{F}_1\) is
\[ \mathcal{R}_{e^*}(\mathcal{F}_1) = \frac{e - 1}{2e + \sqrt{4e^2 + (e - 1)^2}} \approx 0.154269. \]
The result is sharp.

**Proof.** Let \(f \in \mathcal{F}_1\) and define \(p, q : \mathbb{D} \to \mathbb{C}\) by \(p(z) = g(z)/z\) and \(q(z) = f(z)/g(z)\). Then \(p, q \in \mathcal{P}\) and using Lemma 2.3, it follows that
\[ \begin{align*}
\left| \frac{z f'(z) - 1}{f(z)} \right| &\leq \left| \frac{z p'(z)}{p(z)} \right| + \left| \frac{z q'(z)}{q(z)} \right| \leq \frac{4r}{1 - r^2} \quad (|z| = r)
\end{align*} \]
In view of Lemma 2.2, \(f \in \mathcal{S}_{e^*}\) provided \(4r/(1 - r^2) \leq 1 - 1/e\), which gives \(r \leq R := (e - 1)/(2e + \sqrt{4e^2 + (e - 1)^2})\). To show that \(R\) is the sharp \( \mathcal{S}_{e^*}\)-radius for \(\mathcal{F}_1\), consider the function \(f_0(z) = z(1 + z)^2/(1 - z)^2\) with \(g_0(z) = z(1 + z)/(1 - z)\). Clearly \(f_0 \in \mathcal{F}\) and at the point \(z_0 = -R\), a routine calculation shows that
\[ \begin{align*}
|\log z_0 f_0'(z_0)| &= |\log \left( \frac{1 - 4R - R^2}{1 - R^2} \right)| = |\log(1/e)| = 1.
\end{align*} \]
Let \(\mathcal{F}_2\) be the class of functions \(f \in \mathcal{A}\) satisfying the inequality
\[ \left| \frac{f(z)}{g(z)} - 1 \right| < 1 \quad (z \in \mathbb{D}) \]
for some \(g \in \mathcal{W}\).

**Theorem 3.7.** The \( \mathcal{S}_{e^*}\)-radius for the class \(\mathcal{F}_2\) is given by
\[ \mathcal{R}_{e^*}(\mathcal{F}_2) = \frac{2(e - 1)}{3e + \sqrt{9e^2 + 4(2e - 1)(e - 1)}} \approx 0.190884. \]
This bound is best possible.
Proof. Let \( f \in \mathcal{F}_2 \) and define functions \( p, q : \mathbb{D} \to \mathbb{C} \) by \( p(z) = g(z)/z \) and \( q(z) = g(z)/f(z) \). Since the inequality (3.2) is equivalent to \( \text{Re} g(z)/f(z) > 1/2 \), therefore \( p \in \mathcal{P} \) and \( q \in \mathcal{P}(1/2) \). Applying Lemma 2.3 to the identity

\[
\frac{zf'(z)}{f(z)} = 1 + \frac{zp'(z)}{p(z)} - \frac{zd'(z)}{q(z)}
\]

we obtain

\[
\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \frac{r(3 + r)}{1 - r^2}.
\]

By Lemma 2.2, the function \( f \) satisfies \( |\log(zf'(z)/f(z))| \leq 1 \) if \( r(3 + r)/(1 - r^2) \leq 1 - 1/e \). This yields \( r \leq R := 2(e - 1)/(3e + \sqrt{9e^2 + 4(2e - 1)(e - 1)}) \). For sharpness, note that the function \( f_0(z) = z(1 + z)^2/(1 - z) \) belongs to class \( \mathcal{F}_2 \) with \( g_0(z) = z(1 + z)/(1 - z) \). At the point \( z_0 = -R \), we have

\[
\left| \log \frac{zf_0'(z_0)}{f_0(z_0)} \right| = \left| \log \frac{1 - 3R - 2R^2}{1 - R^2} \right| = |\log(1/e)| = 1.
\]

Hence the result is sharp. \( \square \)

The other related radius problems carried out in [5] can also be performed on the similar lines as that of Theorems 3.6 and 3.7 for the class \( \mathcal{I}^* \).

Remark 3.2. Let \( \mathcal{C} \mathcal{I}^* \) be the class of close-to-star functions defined by

\[
\mathcal{C} \mathcal{I}^* = \left\{ f \in \mathcal{A} : \frac{f}{g} \in \mathcal{P} \text{ and } g \in \mathcal{I}^* \right\}.
\]

Then \( \mathcal{R}_{\mathcal{I}^*} (\mathcal{C} \mathcal{I}^*) = \frac{(e - 1)/(2e + \sqrt{1 + 3e^2})}{e} \approx 0.167641 \). To see this, let \( f \in \mathcal{C} \mathcal{I}^* \) and \( g \in \mathcal{I}^* \) such that \( p(z) = f(z)/g(z) \) belongs to \( \mathcal{P} \). Then \( zg'(z)/g(z) \in \mathcal{P} \) so that

\[
\left| \frac{zp'(z)}{p(z)} \right| \leq \frac{2r}{1 - r^2} \quad \text{and} \quad \left| \frac{zg'(z)}{g(z)} \frac{1 + r^2}{1 - r^2} \right| \leq \frac{2r}{1 - r^2},
\]

by applying Lemma 2.3. Using these estimates in the identity

\[
\frac{zf'(z)}{f(z)} = \frac{zg'(z)}{g(z)} + \frac{zp'(z)}{p(z)}
\]

it is easy to see that

\[
\left| \frac{zf'(z)}{f(z)} - \frac{1 + r^2}{1 - r^2} \right| \leq \frac{4r}{1 - r^2}.
\]

The rest of the proof is similar to Theorem 2.10 and so its details are omitted.

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References


On a subclass of strongly starlike functions


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