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On a Subclass of Strongly Starlike Functions Associated with Exponential Function

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Abstract. Let $\mathcal{S}_e^*$ denote the class of analytic functions f in the open unit disk normalized by $f(0)=f'(0)-1=0$ and satisfying the condition $zf'(z)/f(z)\prec e^z$ for $|z|<1$. The structural formula, inclusion relations, coefficient estimates, growth and distortion results, subordination theorems and various radii constants for functions in the class $\mathcal{S}_e^*$ are obtained. In addition, the sharp $\mathcal{S}_e^*$-radii for functions belonging to several interesting classes are also determined.

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1. Introduction

Let $\mathcal{A}_n$ denote the class of analytic functions in the open unit disk $\mathbb{D}:=\{z\in\mathbb{C}:|z|<1\}$ of the form $f(z)=z+\sum_{k=n+1}^{\infty}a_kz^k$, and let $\mathcal{A}:=\mathcal{A}_1$. Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of univalent functions. Using subordination, Ma and Minda [22] gave a unified representation of various geometric subclasses of $\mathcal{S}$ which are characterized by the quantities $zf'(z)/f(z)$ or $1+zf''(z)/f'(z)$ lying in a domain in the right half-plane. They considered the class Φ of analytic univalent functions φ with positive real part mapping $\mathbb{D}$ onto domains symmetric with respect to the real axis and starlike with respect to $\varphi(0)=1$ such that $\varphi'(0)>0$. For $\varphi\in\Phi$, they introduced the following classes that include several well-known classes as special cases:

$$\mathcal{S}^*(\varphi)=\left\{f\in\mathcal{A}:\frac{zf'(z)}{f(z)}\prec\varphi(z)\right\}\quad\text{and}\quad\mathcal{K}(\varphi)=\left\{f\in\mathcal{A}:1+\frac{zf''(z)}{f'(z)}\prec\varphi(z)\right\}.$$

For $-1\le B<A\le 1$, $\mathcal{S}^*[A,B]:=\mathcal{S}^*((1+Az)/(1+Bz))$ is the familiar class consisting of Janowski [16] starlike functions. The corresponding class of convex functions is denoted by $\mathcal{K}[A,B]$. The special case $A=1-2\alpha$, $B=-1$ with $0\le\alpha<1$ yield the subclasses $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ $(0\le\alpha<1)$ of $\mathcal{S}$ consisting of starlike functions of order α and convex functions of order α, respectively, introduced by Robertson [32]. The classes $\mathcal{S}^*:=\mathcal{S}^*(0)$ and $\mathcal{K}:=\mathcal{K}(0)$ are the classes of starlike and convex functions respectively. For $0<\gamma\le 1$,

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$\mathcal{SS}^*(\gamma) := \mathcal{S}^*(((1+z)/(1-z))^\gamma)$ and $\mathcal{SK}(\gamma) := \mathcal{K}(((1+z)/(1-z))^\gamma)$ are the classes of strongly starlike and strongly convex functions of order γ. If

$$\varphi(z) = 1 + \frac{2}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2$$

then $\mathcal{UCV} := \mathcal{K}(\varphi)$ is the class of uniformly convex functions introduced by Goodman [14]. The corresponding class $\mathcal{S}_P := \mathcal{S}^*(\varphi)$ of parabolic starlike functions, was studied by Rønning [34]. Similarly, $\mathcal{S}_L^* := \mathcal{S}^*(\sqrt{1+z})$ is the subclass of $\mathcal{S}^*$ introduced by Sokół and Stankiewicz [45], consisting of functions $f \in \mathcal{A}$ such that $zf'(z)/f(z)$ lies in the domain bounded by the right-half of the lemniscate of Bernoulli given by $|w^2-1| < 1$. More results regarding these classes can be found in [2, 4, 7, 10, 13, 21, 28, 29, 31, 39–44]. Recently, the authors [26] discussed the properties of the class

$$\mathcal{S}_{RL}^* = \mathcal{S}^*\left(\sqrt{2} - (\sqrt{2}-1)\sqrt{\frac{1-z}{1+2(\sqrt{2}-1)z}}\right).$$

Precisely, $f \in \mathcal{S}_{RL}^*$ provided $zf'(z)/f(z)$ lies in the interior of the left-half of the shifted lemniscate of Bernoulli given by $|(w-\sqrt{2})^2 - 1| < 1$. In the similar fashion, many new interesting subclasses of starlike and convex functions can be defined by altering the subordinate function $\varphi \in \Phi$. This paper aims to investigate the properties of the classes $\mathcal{S}^*(e^z)$ and $\mathcal{K}(e^z)$.

The exponential function $\varphi_0(z) = e^z$ has positive real part in $\mathbb{D}$, $\varphi_0(\mathbb{D}) = \{w \in \mathbb{C} : |\log w| < 1\}$ (see Figure 1) is symmetric with respect to the real axis and starlike with respect to 1, and $\varphi_0'(0) > 0$. Hence $\varphi_0 \in \Phi$ so that the classes $\mathcal{S}^*(\varphi_0)$ and $\mathcal{K}(\varphi_0)$ are well-defined. Set

$$\mathcal{S}_e^* := \mathcal{S}^*(e^z) \quad \text{and} \quad \mathcal{K}_e := \mathcal{K}(e^z).$$

In view of the Alexander relation between the classes $\mathcal{S}_e^*$ and $\mathcal{K}_e$: $f \in \mathcal{K}_e$ if and only if $zf' \in \mathcal{S}_e^*$, the properties of functions in the class $\mathcal{K}_e$ can be obtained from the corresponding results for $\mathcal{S}_e^*$. Therefore, it is enough to focus our attention to the class $\mathcal{S}_e^*$. For a function $f \in \mathcal{A}$, we have the equivalences:

$$f \in \mathcal{S}_e^* \quad \Leftrightarrow \quad \frac{zf'(z)}{f(z)} \prec e^z \quad (z \in \mathbb{D}) \quad \Leftrightarrow \quad \left|\log\frac{zf'(z)}{f(z)}\right| < 1 \quad (z \in \mathbb{D}).$$

This immediately yields the following structural formula for functions in the class $\mathcal{S}_e^*$.

Theorem 1.1. *A function f belongs to the class $\mathcal{S}_e^*$ if and only if there exists an analytic function q, $q \prec e^z$ such that*

$$f(z) = z\exp\left(\int_0^z \frac{q(t)-1}{t}\,dt\right).$$

Define the functions h_n $(n = 2, 3, \ldots)$ by $h(0) = h'(0) - 1 = 0$ and

$$\frac{zh_n'(z)}{h_n(z)} = e^{z^{n-1}} \quad (z \in \mathbb{D}). \tag{1.1}$$

Then $h_n \in \mathcal{S}_e^*$ $(n = 2, 3, \ldots)$. In terms of the Taylor series expansion, the functions h_n takes the form

$$h_n(z) = z + \frac{1}{n-1}z^{n-1} + \cdots.$$

In particular, the function

$$h(z) := h_2(z) = z\exp\left(\int_0^z \frac{e^t-1}{t}\,dt\right) = z + z^2 + \frac{3}{4}z^3 + \frac{17}{36}z^4 + \cdots \tag{1.2}$$

plays the role of extremal function for many extremal problems over the class $\mathcal{S}_e^*$.

In Section 2, we investigate the geometric properties of functions in class $\mathcal{S}_e^*$. In particular, the relations between $\mathcal{S}_e^*$ and other classes geometrically defined are considered. The sharp radii of starlikeness of order α $(0 \le \alpha < 1)$, parabolic starlikeness (and some of others) of $f \in \mathcal{S}_e^*$ are determined in the last section of the paper. The sharp $\mathcal{S}_e^*$-radii for certain well-known classes of functions are also obtained.

2. Properties of functions in the class $\mathcal{S}_e^*$

In this section, we will determine the inclusion relations, coefficient estimates, growth and distortion results and convolution properties of functions in the class $\mathcal{S}_e^*$. The following two lemmas will be needed in our investigation.

Lemma 2.1. *For $r \in (0,1)$, the function $\varphi_0(z) = e^z$ satisfies*

$$\min_{|z|=r} \operatorname{Re}\varphi_0(z) = \varphi_0(-r) = \min_{|z|=r} |\varphi_0(z)| \quad \text{and} \quad \max_{|z|=r} \operatorname{Re}\varphi_0(z) = \varphi_0(r) = \max_{|z|=r} |\varphi_0(z)|.$$

Proof. For $\theta \in [0, 2\pi)$, the function $\psi_0(\theta) = \operatorname{Re}\varphi_0(re^{i\theta}) = e^{r\cos\theta}\cos(r\sin\theta)$ attains its minimum at $\theta = \pi$ and maximum at $\theta = 0$. Consequently,

$$\min_{|z|=r} \operatorname{Re}\varphi_0(z) = e^{-r} = \varphi_0(-r) \quad \text{and} \quad \max_{|z|=r} \operatorname{Re}\varphi_0(z) = e^r = \varphi_0(r).$$

The other equality follows by observing that the real-valued function $|e^z| = e^{\operatorname{Re} z}$ is strictly increasing in the interval $[-r, r]$. ∎

Lemma 2.2. *For $1/e < a < e$, let r_a be given by*

$$r_a = \begin{cases} a - e^{-1}, & e^{-1} < a \le (e+e^{-1})/2; \\ e - a, & (e+e^{-1})/2 \le a < e. \end{cases}$$

Then

$$\{w : |w-a| < r_a\} \subset \{w : |\log w| < 1\}.$$

Proof. Let $\varphi_0(z) = e^z$. Then any point on the boundary of $\varphi_0(\mathbb{D})$ is of the form $\varphi_0(e^{i\theta}) = e^{\cos\theta}\cos(\sin\theta) + ie^{\cos\theta}\sin(\sin\theta)$ for $\theta \in [-\pi, \pi]$. Note that the curve $w = \varphi_0(e^{i\theta})$ is symmetric with respect to the real axis, so it is sufficient to consider the interval $0 \le \theta \le \pi$. The square of the distance from the point $(a, 0)$ to the points on the curve $w = \varphi_0(e^{i\theta})$ is given by

$$z(\theta) = (a - e^{\cos\theta}\cos(\sin\theta))^2 + e^{2\cos\theta}\sin^2(\sin\theta) = e^{2\cos\theta} - 2ae^{\cos\theta}\cos(\sin\theta) + a^2.$$

Let us first assume that $1/e < a \le e/2$. Then $z(\theta)$ is a decreasing function of $\theta \in [0, \pi]$. Consequently, we have

$$r_a = \min_{\theta\in[0,\pi]} \sqrt{z(\theta)} = \sqrt{z(\pi)} = a - \frac{1}{e}.$$

Next, assume that $e/2 < a < e$. A calculation shows that

$$z'(\theta) = -2e^{\cos\theta}(e^{\cos\theta}\sin\theta - a\sin(\theta + \sin\theta))$$

and hence $z'(0) = z'(\pi) = z'(\theta_a) = 0$, where $\theta_a \in (0,\pi)$ is the real root of the equation $e^{\cos\theta}\sin\theta = a\sin(\theta+\sin\theta)$. Observe that $\theta_{a_1} < \theta_{a_2}$ for $a_1 < a_2$. Moreover, the function $z(\theta)$ is increasing for $\theta \in [0,\theta_a]$ and decreasing for $\theta \in [\theta_a,\pi]$. Also,

$$z(\pi) - z(0) = 2\left(e - \frac{1}{e}\right)\left(a - \frac{1}{2}\left(e + \frac{1}{e}\right)\right).$$

These observations lead to two cases:

Case 1: $e/2 < a \leq (e+e^{-1})/2$. In this case $\min\{z(0), z(\theta_a), z(\pi)\} = z(\pi)$. Thus $z(\theta)$ attains its minimum value at $\theta = \pi$ and $r_a = \min\sqrt{z(\theta)} = a - 1/e$.

Case 2: $(e+e^{-1})/2 \leq a < e$. It is easy to see that $\min\{z(0), z(\theta_a), z(\pi)\} = z(0)$ and hence $r_a = \min\sqrt{z(\theta)} = e - a$ in this case. This completes the proof of the lemma. ∎

Remark 2.1. Following the notation and method of the proof of Lemma 2.2, it is easy to deduce that

$$\{w : |\log w| < 1\} \subset \{w : |w - a| < R_a\},$$

where R_a is given by

$$R_a = \begin{cases} e - a, & e^{-1} < a \leq e/2; \\ z(\theta_a), & e/2 < a < e. \end{cases}$$

2.1. Inclusion Relations

Recall that starlike functions of order α $(0 \leq \alpha < 1)$ and strongly starlike functions of order γ $(0 < \gamma \leq 1)$ are characterized by the conditions $\operatorname{Re}(zf'(z)/f(z)) > \alpha$ and $|\arg(zf'(z)/f(z))| \leq \gamma\pi/2$ respectively. Kanas and Wisniowska [17] introduced the class $k-\mathcal{S}^*$ of k-starlike $(k \geq 0)$ functions $f \in \mathcal{A}$ defined by the condition

$$\operatorname{Re}\frac{zf'(z)}{f(z)} > k\left|\frac{zf'(z)}{f(z)} - 1\right| \quad (z \in \mathbb{D}),$$

which provides a continuous passage from starlike functions $(k=0)$ to parabolic starlike functions $(k=1)$. Another interesting class is $\mathcal{M}(\beta)$, $\beta > 1$, defined by

$$\mathcal{M}(\beta) = \left\{ f \in \mathcal{A} : \operatorname{Re}\frac{zf'(z)}{f(z)} < \beta, z \in \mathbb{D} \right\}.$$

This class was investigated by Uralegaddi *et al.* [47]. The following theorem investigates the inclusion relations between the classes $\mathcal{S}^*(\alpha)$ $(0 \leq \alpha < 1)$, $\mathcal{SS}^*(\gamma)$ $(0 < \gamma \leq 1)$, $\mathcal{M}(\beta)$ $(\beta > 1)$, $k-\mathcal{S}^*$ $(k \geq 0)$ and $\mathcal{S}^*_e$.

Theorem 2.1. *The class $\mathcal{S}^*_e$ satisfies the following relationships:*

(i) $\mathcal{S}^*_e \subset \mathcal{S}^*(\alpha) \subset \mathcal{S}^*$ *for* $0 \leq \alpha \leq 1/e$;
(ii) $\mathcal{S}^*_e \subset \mathcal{M}(\beta)$ *for* $\beta \geq e$;
(iii) $\mathcal{S}^*_e \subset \mathcal{SS}^*(\gamma) \subset \mathcal{S}^*$ *for* $2/\pi \leq \gamma \leq 1$;
(iv) $k-\mathcal{S}^* \subset \mathcal{S}^*_e$ *for* $k \geq e/(e-1)$.

The constants $1/e$, e, $2/\pi$ and $e/(e-1)$ in parts (i), (ii), (iii) and (iv) respectively are best possible.

Proof. Let $f \in \mathcal{S}^*_e$. Then $zf'(z)/f(z) \prec e^z$. By Lemma 2.1, it is easy to deduce that

$$\frac{1}{e} = \min_{|z|=1}\operatorname{Re} e^z < \operatorname{Re}\frac{zf'(z)}{f(z)} < \max_{|z|=1}\operatorname{Re} e^z = e \quad (z \in \mathbb{D}).$$

Thus $f \in \mathcal{S}^*(1/e) \cap \mathcal{M}(e)$. Also,

$$\left|\arg \frac{zf'(z)}{f(z)}\right| \leq \max_{|z|=1} \arg e^z = 1 \quad (z \in \mathbb{D})$$

which shows that $f \in \mathcal{SS}^*(2/\pi)$.

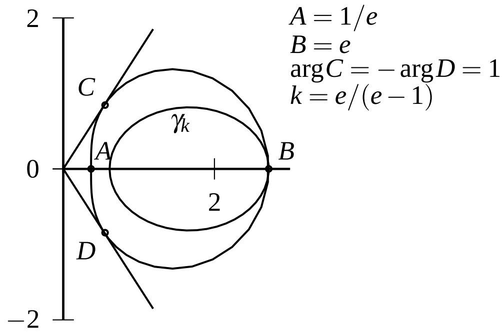

Figure 1. Image of the unit circle under the mapping e^z.

For (iv), let $f \in k-\mathcal{S}^*$ and consider the conic domain $\Gamma_k = \{w \in \mathbb{C} : \operatorname{Re} w > k|w-1|\}$. For $k > 1$, the curve $\partial\Gamma_k$ is the ellipse $\gamma_k : x^2 = k^2(x-1)^2 + k^2y^2$ which may be rewritten as

$$\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} = 1,$$

where $x_0 = k^2/(k^2-1)$, $y_0 = 0$, $a = k/(k^2-1)$ and $b = 1/\sqrt{k^2-1}$. For the ellipse γ_k to lie inside $|\log w| \leq 1$, it is necessary that $x_0 + a \leq e$, which is equivalent to the condition $k \geq e/(e-1)$. Figure 1 depicts that the ellipse $\gamma_{e/(e-1)}$ lies completely inside $|\log w| \leq 1$. Also, since $\Gamma_{k_1} \subset \Gamma_{k_2}$ for $k_1 \geq k_2$, it follows that $k-\mathcal{S}^* \subset \mathcal{S}^*_e$ for $k \geq e/(e-1)$. ∎

Remark 2.2. In [10], Aouf, Dziok and Sokół investigated the properties of functions in the class $\mathcal{S}^*(q_c)$, where $q_c(z) = \sqrt{1+cz}$, $c \in (0,1]$. In particular, $\mathcal{S}^*(q_1) = \mathcal{S}^*_L$. The function q_c maps $\mathbb{D}$ onto the domain

$$\mathcal{O}_c = \{w \in \mathbb{C} : \operatorname{Re} w > 0, |w^2-1| < c\}$$

and its boundary $\partial\mathcal{O}_c$ is the right loop of the Cassinian Ovals

$$(x^2+y^2)^2 - 2(x^2-y^2) = c^2 - 1.$$

Using the similar analysis carried out in Theorem 2.1, it can be shown that $\mathcal{S}^*(q_c) \subset \mathcal{S}^*_e$ for $c \leq 1-(1/e^2) \approx 0.864665$, and this bound is best possible.

For $-1 \leq B < A \leq 1$, let $\mathcal{P}[A,B]$ be the class of analytic functions p of the form $p(z) = 1 + c_1z + c_2z^2 + \cdots$ satisfying $p(z) \prec (1+Az)/(1+Bz)$ for all $z \in \mathbb{D}$. We write $\mathcal{P}[1-2\alpha,-1] = \mathcal{P}(\alpha)$ $(0 \leq \alpha < 1)$ and $\mathcal{P}(0) = \mathcal{P}$. The following lemma will be needed to determine the conditions on parameters A and B so that $\mathcal{S}^*[A,B]$ is a subclass of $\mathcal{S}^*_e$.

Lemma 2.3. *([31, Lemma 2.1, p. 267], [38]) If $p \in \mathcal{P}[A,B]$, then*

$$\left|p(z) - \frac{1-ABr^2}{1-B^2r^2}\right| \leq \frac{(A-B)r}{1-B^2r^2} \quad (|z| = r < 1).$$

Moreover, if $p \in \mathscr{P}(\alpha)$, then

$$\left| p(z) - \frac{1+(1-2\alpha)r^2}{1-r^2} \right| \leq \frac{2(1-\alpha)r}{1-r^2},$$

and

$$\left| \frac{zp'(z)}{p(z)} \right| \leq \frac{2r(1-\alpha)}{(1-r)(1+(1-2\alpha)r)} \quad (|z| = r < 1).$$

Theorem 2.2. *Let $-1 < B < A \leq 1$ and either*

(i) $2(1-B^2) < 2e(1-AB) \leq (e^2+1)(1-B^2)$ *and* $(1-B) \leq (1-A)e$*; or*

(ii) $(1+e^2)(1-B^2) \leq 2(1-AB)e < 2(1-B^2)e^2$ *and* $(1+A) \leq e(1+B)$,

then $\mathscr{S}^*[A,B] \subset \mathscr{S}^*_e$.

Proof. Let $f \in \mathscr{S}^*[A,B]$. Then $zf'(z)/f(z) \in \mathscr{P}[A,B]$ so that Lemma 2.3 gives

$$\left| \frac{zf'(z)}{f(z)} - \frac{1-AB}{1-B^2} \right| < \frac{A-B}{1-B^2}. \tag{2.1}$$

Let $a := (1-AB)/(1-B^2)$ and suppose that the two conditions in (i) hold. By multiplying both sides of the inequality $(1-B) \leq (1-A)e$ by the positive constant $1+B$ and rewriting, we obtain $(A-B)e \leq (1-AB)e - (1-B^2)$. A division by $e(1-B^2)$ shows that the condition $(1-B) \leq (1-A)e$ is equivalent to $(A-B)/(1-B^2) \leq a - 1/e$. Similarly, the condition $2(1-B^2) < 2e(1-AB) \leq (e^2+1)(1-B^2)$ is equivalent to $1/e < a \leq (e+e^{-1})/2$. From (2.1), it follows that the values of $w = zf'(z)/f(z)$ lies in the disk $|w-a| < r_a$, where $r_a := a - 1/e$ and $1/e < a \leq (e+e^{-1})/2$. Hence $f \in \mathscr{S}^*_e$ by Lemma 2.2. A similar argument shows that $f \in \mathscr{S}^*_e$ if condition (ii) is satisfied and therefore its details are omitted. ∎

2.2. Coefficient Estimates

The estimation of coefficient bounds is one of the classical problem in univalent univalent theory. The famous Bieberbach conjecture for the class $\mathscr{S}$ which stood as a challenge for several years, was finally settled by de Branges [12] in 1984. There are still many open problems concerning determination of sharp coefficient bounds for various subclasses of $\mathscr{S}$ such as $\mathscr{S}_P$, $\mathscr{S}^*_L$ and $\mathscr{S}^*_{RL}$ (see [7, 26, 43, 48]).

The correspondence between the classes $\mathscr{S}^*_e$ and $\mathscr{K}_e$ and [22, Theorem 3, p. 164] yield the sharp upper bound for the Fekete-Szegö functional $|a_3 - \mu a_2^2|$ in the class $\mathscr{S}^*_e$ for all real μ. If $f(z) = z + a_2z^2 + a_3z^3 + \cdots \in \mathscr{S}^*_e$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} (3-4\mu)/4, & \mu \leq 1/4; \\ 1/2, & 1/4 \leq \mu \leq 5/4; \\ -(3-4\mu)/4, & \mu \geq 5/4. \end{cases}$$

Suppose that the functions h_n are given by (1.1). For $0 \leq \lambda \leq 1$, define f_λ and g_λ by $f_\lambda(0) = f'_\lambda(0) - 1 = g_\lambda(0) = g'_\lambda(0) - 1 = 0$,

$$\frac{zf'_\lambda(z)}{f_\lambda(z)} = \exp\left(\frac{z(z+\lambda)}{1+\lambda z}\right) \quad \text{and} \quad \frac{zg'_\lambda(z)}{g_\lambda(z)} = \exp\left(-\frac{z(z+\lambda)}{1+\lambda z}\right),$$

respectively. If $\mu < 1/4$ or $\mu > 5/4$, then equality holds if and only if f is h_2 or one of its rotation. If $1/4 < \mu < 5/4$, then equality holds if and only if f is equal to h_3 or one of its rotation. If $\mu = 1/4$, equality holds if and only if f is equal to f_λ or one of its rotation. When $\mu = 5/4$, inequality becomes equality if and only if f equals g_λ or one of its rotation.

These observations together with [8, Theorem 1, p. 38] yield the sharp upper bound on the absolute value of second, third and fourth coefficient of functions in the class $\mathcal{S}_e^*$.

Theorem 2.3. *If* $f(z) = z + a_2z^2 + a_3z^3 + a_4z^4 + \cdots \in \mathcal{S}_e^*$, *then* $|a_2| \leq 1$, $|a_3| \leq 3/4$ *and* $|a_4| \leq 17/36$. *These bounds are sharp for the function h given by* (1.2).

Since the function h given by (1.2) maximizes $|a_n|$ for $n = 2,3,4$ in the class $\mathcal{S}_e^*$, it is natural to suspect that h maximizes $|a_n|$ for each n. But we are not able to prove it for $n > 4$. However, we may obtain bounds on $|a_n|$ $(n = 5,6,\ldots)$, although they are not sharp. By Theorem 2.1(i), $\mathcal{S}_e^* \subset \mathcal{S}^*(1/e)$ and hence

$$|a_n| \leq \frac{1}{(n-1)!}\prod_{k=2}^{n}\left(k - \frac{2}{e}\right) \quad (n = 2,3,\ldots)$$

for a function $f \in \mathcal{S}_e^*$ (see [37]). These bounds can be further improved by making use of the result by Rogosinski [33]: if $h(z) = 1+\sum_{k=1}^{\infty} c_kz^k$ is subordinate to $H(z) = 1+\sum_{k=1}^{\infty} C_kz^k$ in $\mathbb{D}$, where H is univalent in $\mathbb{D}$ and $H(\mathbb{D})$ is convex, then $|c_n| \leq |C_1|$ for $n = 1,2,\ldots$.

Theorem 2.4. *If* $f(z) = z+\sum_{k=2}^{\infty} a_kz^k \in \mathcal{S}_e^*$, *then* $|a_n| \leq 1$ *for* $n \geq 2$.

Proof. Note that

$$1+\sum_{k=1}^{\infty} c_kz^k = q(z) := \frac{zf'(z)}{f(z)} \prec \varphi_0(z) = e^z = 1+z+\frac{z^2}{2!}+\cdots.$$

Since φ_0 maps $\mathbb{D}$ univalently onto a convex domain, by Rogosinski's result, $|c_n| \leq 1$ for each n. Writing $zf'(z) = q(z)f(z)$ and comparing the coefficient of z^n on both sides, we obtain

$$(n-1)a_n = \sum_{k=1}^{n-1} c_{n-k}a_k.$$

Therefore, $|a_2| = |c_1| \leq 1$. Assume that $|a_k| \leq 1$ for $k = 3,4,\ldots,n-1$. Then it is easy to see that

$$(n-1)|a_n| = \sum_{k=1}^{n-1} |c_{n-k}||a_k| \leq \sum_{k=1}^{n-1} |a_k| \leq n-1.$$

The result now follows by induction. ∎

If $f \in \mathcal{S}_e^* \cap \mathcal{A}_3$, then the result of Theorem 2.4 can be further strengthened as seen by the following theorem.

Theorem 2.5. *If* $f(z) = z+\sum_{n=2}^{\infty} a_nz^n \in \mathcal{S}_e^*$, *then*

$$\sum_{n=2}^{\infty}(n^2-e^2)|a_n|^2 \leq e^2-1.$$

Proof. Since $f \in \mathcal{S}_e^*$, therefore $zf'(z)/f(z) = e^{w(z)}$ where w is an analytic function in $\mathbb{D}$ with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in \mathbb{D}$. Using the identity $f^2(z) = e^{-2w(z)}(zf'(z))^2$, we have

$$\begin{aligned}2\pi\sum_{n=1}^{\infty}|a_n|^2r^{2n} &= \int_0^{2\pi}|f(re^{i\theta})|^2\,d\theta = \int_0^{2\pi}|e^{-2w(re^{i\theta})}||re^{i\theta}f'(re^{i\theta})|^2\,d\theta\\ &\geq e^{-2}\int_0^{2\pi}|re^{i\theta}f'(re^{i\theta})|^2\,d\theta = 2\pi e^{-2}\sum_{n=1}^{\infty}n^2|a_n|^2r^{2n}\end{aligned}$$

where $0 < r < 1$ and $a_1 = 1$. Thus

$$\sum_{n=1}^{\infty}(n^2 - e^2)|a_n|^2 r^{2n} \leq 0.$$

On letting $r \to 1^-$, we obtain the required result. ∎

Corollary 2.1. *Let $f(z) = z + \sum_{k=4}^{\infty} a_k z^k \in \mathcal{S}_e^*$. Then*

$$|a_n| \leq \sqrt{\frac{e^2 - 1}{n^2 - e^2}} < 1 \quad \textit{for } n = 4, 5, \ldots.$$

Remark 2.3. Since $\sum_{n=1}^{\infty} \frac{1}{(n!)^2} < \infty$, therefore the function $\varphi_0(z) = e^z$ belongs to $\mathcal{H}^2$, the Hardy class of analytic functions in $\mathbb{D}$. Hence for a function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}_e^*$, the sharp order of growth $|a_n| = O(1/n)$ by [22, Corollary 4', p. 166].

By making use of Lemma 2.2, we next determine necessary and sufficient conditions for some special members to be in the class $\mathcal{S}_e^*$.

Theorem 2.6. *(i) A function $f(z) = z + a_n z^n$ $(n = 2, 3, \ldots)$ belongs to the class $\mathcal{S}_e^*$ if and only if $|a_n| \leq (e-1)/(ne-1)$.*
(ii) The function $z/(1-Az)^2$ is in $\mathcal{S}_e^*$ if and only if $|A| \leq (e-1)/(e+1)$.

Proof. (i) Since $\mathcal{S}_e^* \subset \mathcal{S}^*$, $|a_n| \leq 1/n$. To find the sharp estimate, observe that $w = zf'(z)/f(z) = (1 + na_n z^{n-1})/(1 + a_n z^{n-1})$ maps $\mathbb{D}$ onto the disk

$$\left| w - \frac{1 - n|a_n|^2}{1 - |a_n|^2} \right| < \frac{(n-1)|a_n|}{1 - |a_n|^2}. \tag{2.2}$$

Since $(1 - n|a_n|^2)/(1 - |a_n|^2) \leq 1$, therefore, in view of Lemma 2.2, the disk (2.2) lies inside $|\log w| \leq 1$ if and only if

$$\frac{(n-1)|a_n|}{1 - |a_n|^2} \leq \frac{1 - n|a_n|^2}{1 - |a_n|^2} - \frac{1}{e}.$$

This yields $|a_n| \leq (e-1)/(ne-1)$.

(ii) Clearly, the Koebe function $k(z) = z/(1-z)^2 \notin \mathcal{S}_e^*$. Let $g(z) = z/(1-Az)^2$ $(A \neq 1)$. Then the bilinear transformation $w = zg'(z)/g(z) = (1+Az)/(1-Az)$ maps $\mathbb{D}$ onto the disk

$$\left| w - \frac{1 + |A|^2}{1 - |A|^2} \right| < \frac{2|A|}{1 - |A|^2} \tag{2.3}$$

with diametric end points $x_L = (1 - |A|)/(1 + |A|)$ and $x_R = (1 + |A|)/(1 - |A|)$. If $g \in \mathcal{S}_e^*$, then the disk (2.3) lies inside $|\log w| \leq 1$. Consequently, it is necessary that $x_R \leq e$ which gives $|A| \leq (e-1)/(e+1)$. Conversely, if $|A| \leq (e-1)/(e+1)$, then $a := (1 + |A|^2)/(1 - |A|^2) \leq (e + e^{-1})/2$ and $2|A|/(1 - |A|^2) \leq a - 1/e$. By again applying Lemma 2.2, we conclude that the disk (2.3) lies inside $|\log w| \leq 1$ and hence $g \in \mathcal{S}_e^*$. ∎

2.3. Subordination Results and their consequences

If $f \in \mathcal{S}_e^*$, then $f(z)/z \prec h(z)/z$ by [22, Theorem 1', p. 161], where h is given by (1.2). Since the function e^z is convex univalent, this result can also be obtained as a special case of [35, Theorem 1, p. 275]. Using this subordination relation, or by directly applying the results of [22], we obtain the following result.

Theorem 2.7. *Let $f \in \mathscr{S}_e^*$ and h be given by* (1.2)*. Then, for $|z| = r$, we have the following:*

(i) *(Growth Theorem)*

$$-h(-r) \le |f(z)| \le h(r).$$

In particular, $f(\mathbb{D}) \supset \{w : |w| < -h(-1) \approx 0.450859\}$.

(ii) *(Rotation Theorem)*

$$\left|\arg\left(\frac{f(z)}{z}\right)\right| \le \max_{|z|=r} \arg\left(\frac{h(z)}{z}\right).$$

(iii) *(Distortion Theorem)*

$$h'(-r) \le |f'(z)| \le h'(r).$$

Equality holds for some $z \neq 0$ if and only if f is a rotation of h.

It is worth to remark that the proof of Theorem 2.7(iii) makes use of Lemma 2.1. In [34], Rønning proved that $|f(z)| \le K|z|$ $(z \in \mathbb{D})$ for functions $f \in \mathscr{S}_P$, where $K = \exp(14\zeta(3)/\pi^2) \approx 5.502$ ($\zeta(t)$ is the Reimann Zeta function). The next corollary proves the corresponding result for $\mathscr{S}_e^*$.

Corollary 2.2. *If $f \in \mathscr{S}_e^*$, then*

$$|f(z)| \le |z| \exp\left(\sum_{n=1}^{\infty} \frac{1}{n(n!)}\right) = |z|K,$$

for all $z \in \mathbb{D}$, where $K \approx 3.73558$. The function h given by (1.2) *shows that this bound cannot be further improved.*

Proof. By Theorem 2.7(i), for $|z| = r$, we have

$$\log\left|\frac{f(z)}{z}\right| \le \int_0^r \frac{e^t - 1}{t}\, dt \le \int_0^1 \frac{e^t - 1}{t}\, dt = \sum_{n=1}^{\infty} \frac{1}{n(n!)}.$$

The series on the right side of the above inequality is convergent and hence we obtain the desired result. ∎

In terms of subordination, Tuneski [46] gave an interesting criteria for analytic functions to be in the class $\mathscr{S}^*[A,B]$ $(-1 \le B < A \le 1)$. In 2007, Sokół [41] generalized this result using Jack lemma [15] and obtained a sufficient condition for functions $f \in \mathscr{A}$ to be in a more general class of functions. As an application, observe that the function $\varphi_0(z) = e^z$ is univalent and non-vanishing in $\mathbb{D}$ with $\varphi_0(0) = 1$ and such that

$$\operatorname{Re}\left(1 + \frac{z\varphi_0''(z)}{\varphi_0'(z)}\right) > 2 \operatorname{Re} \frac{z\varphi_0'(z)}{\varphi_0(z)} \quad (z \in \mathbb{D}).$$

Therefore, a function $f \in \mathscr{A}$ satisfying

$$\frac{1 + zf''(z)/f'(z)}{zf'(z)/f(z)} - 1 \prec ze^{-z} \quad (z \in \mathbb{D})$$

belongs to the class $\mathscr{S}_e^*$, by [41, Corollary 1, p. 239].

Let p be an analytic function in $\mathbb{D}$ with $p(0) = 1$. Recently, Ali *et al.* [3] and Kumar *et al.* [18] determined conditions on β so that $p(z) \prec \sqrt{1+z}$ when $1 + \beta zp'(z)/p(z)$ is subordinated to $\sqrt{1+z}$ or $(1+Az)/(1+Bz)$. This, in turn, provide sufficient conditions for analytic functions $f \in \mathscr{A}$ to belong to the class $\mathscr{S}_L^*$. Motivated by their work, we prove the corresponding subordination results involving the exponential function.

Theorem 2.8. *Let p be an analytic function in $\mathbb{D}$ with $p(0)=1$. If either of the following three conditions is satisfied:*

(a) $1+\beta zp'(z)/p(z) \prec e^z \quad (\beta \geq e-1)$,
(b) $1+\beta zp'(z)/p(z) \prec (1+Az)/(1+Bz) \quad (-1<B<A\leq 1, |\beta| \geq (A-B)/(1-|B|))$,
(c) $1+\beta zp'(z)/p(z) \prec \sqrt{1+z} \quad (\beta \geq 1)$,

then $p(z) \prec e^z$.

Proof. Let φ_0 be the convex univalent function defined by $\varphi_0(z)=e^z$. Then it is clear that $\beta z\varphi_0'(z)$ is starlike. The main crux of the proof relies on the observation that if the subordination

$$1+\beta\frac{zp'(z)}{p(z)} \prec 1+\beta\frac{z\varphi_0'(z)}{\varphi_0(z)} = 1+\beta z := \psi(z)$$

is satisfied, then $p(z) \prec \varphi_0(z)$ by [27, Corollary 3.4h.1, p. 135].

(a) It suffices to show that $\varphi_0(z) \prec \psi(z)$. Since $\psi(\mathbb{D}) = \{w \in \mathbb{C} : |w-1| < \beta\}$, it follows that $\varphi_0(\mathbb{D}) \subset \psi(\mathbb{D})$ if $\beta \geq e-1$ by Remark 2.1. Hence $\varphi_0(z) \prec \psi(z)$ and consequently $p(z) \prec e^z$.

(b) Set $\phi(z) = (1+Az)/(1+Bz)$. Then $\phi^{-1}(w) = (w-1)/(A-Bw)$. Since the subordination $\phi(z) \prec \psi(z)$ is equivalent to $z \prec \phi^{-1}(\psi(z))$, we only need to show that $|\phi^{-1}(\psi(e^{it}))| \geq 1$ for $-\pi \leq t \leq \pi$. Fot $t \in [-\pi,\pi]$, we have

$$|\phi^{-1}(\psi(e^{it}))| = \left|\frac{\beta e^{it}}{(A-B)-\beta Be^{it}}\right| \geq \frac{|\beta|}{A-B+|\beta B|} \geq 1$$

provided $|\beta| \geq (A-B)/(1-|B|)$. Thus $\phi(z) \prec \psi(z)$ and hence $p(z) \prec e^z$.

(c) Let $\chi(z) = \sqrt{1+z}$. Since $\chi(\mathbb{D}) \subset \psi(\mathbb{D})$ if $\beta \geq 1$ (by [4, Lemma 2.2, p. 6559]), it follows that $\chi(z) \prec \psi(z)$ and so $p(z) \prec e^z$. ∎

For $f \in \mathscr{A}$, the function $p(z) = zf'(z)/f(z)$ is analytic in $\mathbb{D}$ with $p(0)=1$. As a result, Theorem 2.8 immediately yields the following corollary.

Corollary 2.3. *Let $f \in \mathscr{A}$ and set*

$$\Psi_\beta(z) = 1+\beta\left(1+\frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right).$$

If either of the following three conditions is satisfied:

(a) $\Psi_\beta(z) \prec e^z$ $(\beta \geq e-1)$*;*
(b) $\Psi_\beta(z) \prec (1+Az)/(1+Bz)$ $(|\beta| \geq (A-B)/(1-|B|))$*; or*
(c) $\Psi_\beta(z) \prec \sqrt{1+z}$ $(\beta \geq 1)$,

then $f \in \mathscr{S}_e^*$.

These results can be extended to functions with fixed second coefficient by using the results of [6].

2.4. Convolution Properties

For analytic functions $f(z) = z+\sum_{n=2}^\infty a_n z^n$ and $F(z) = z+\sum_{n=2}^\infty A_n z^n$, their convolution (or Hadamard product) is defined as $(f*F)(z) = z+\sum_{n=2}^\infty a_n A_n z^n$. The classes of starlike, convex and close-to-convex functions are closed under convolution with convex functions (see [36]). A similar result holds for $\mathscr{S}_e^*$. By [22, Theorem 5, p. 167], we have

Theorem 2.9. *If $f \in \mathscr{S}_e^*$ and $g \in \mathscr{K}$, then $f*g \in \mathscr{S}_e^*$.*

Let $\Gamma_i : \mathscr{A} \to \mathscr{A}$ $(i = 1, 2, 3)$ be the operators defined by

$$\Gamma_1[f](z) = zf'(z), \quad \Gamma_2[f](z) = \frac{1}{2}(f(z) + zf'(z)) \quad \text{and}$$

$$\Gamma_3[f](z) = \frac{k+1}{z^k}\int_0^z \zeta^{k-1} f(\zeta)\,d\zeta \quad (\operatorname{Re} k > 0).$$

These operators were introduced by Alexander [1], Livingston [20] and Bernardi [11] respectively. Each of these operators can be written as a convolution operator: $\Gamma_i = f * g_i$ $(i = 1, 2, 3)$, where

$$g_1(z) = \frac{z}{(1-z)^2}, \quad g_2(z) = \frac{z - z^2/2}{(1-z)^2}, \quad \text{and} \quad g_3(z) = \sum_{n=1}^{\infty} \frac{k+1}{k+n} z^n.$$

The function g_1 is convex in $|z| < 2 - \sqrt{3}$, g_2 is convex in $|z| < 1/2$ while g_3 is convex in $\mathbb{D}$. Hence Theorem 2.9 gives

Corollary 2.4. *Let $f \in \mathscr{S}_e^*$. Then $\Gamma_i[f] \in \mathscr{S}_e^*$ in $|z| < r_i$ $(i = 1, 2, 3)$ where $r_1 = 2 - \sqrt{3}$, $r_2 = 1/2$ and $r_3 = 1$.*

The convolution of two starlike functions need not be univalent in $\mathbb{D}$. Let $f, g \in \mathscr{S}^*$ and $h_\rho(z) = (f * g)(\rho z)/\rho$. Ling and Ding [19, Theorem 1, p. 404] proved that $h_\rho \in \mathscr{S}^*$ for $0 \le \rho \le 2 - \sqrt{3}$. Ali *et al.* [9] determined conditions on ρ so that h_ρ belongs to the classes $\mathscr{S}_P$, $\mathscr{S}^*(\alpha)$ $(0 \le \alpha < 1)$, $\mathscr{S}\mathscr{S}^*(\gamma)$ $(0 < \gamma \le 1)$ and $\mathscr{S}_L^*$. The next theorem investigates the corresponding result for $\mathscr{S}_e^*$.

Theorem 2.10. *If $f, g \in \mathscr{S}^*$, then $f * g \in \mathscr{S}_e^*$ in $|z| < \rho_0$, where*

$$\rho_0 = \frac{e-1}{2e + \sqrt{1+3e^2}} \approx 0.167641. \tag{2.4}$$

The number ρ_0 is best possible.

Proof. Consider the function $H : \mathbb{D} \to \mathbb{C}$ defined by

$$H(z) = \frac{z(1+z)}{(1-z)^3} = z + \sum_{n=2}^{\infty} n^2 z^n.$$

The function H is analytic in $\mathbb{D}$ and satisfies

$$\left| \frac{zH'(z)}{H(z)} - \frac{1+r^2}{1-r^2} \right| \le \frac{4r}{1-r^2}, \quad |z| = r < 1.$$

Let $0 \le r \le \rho_0$, where ρ_0 is given by (2.4). Then it is a simple exercise to show that if $a := (1+r^2)/(1-r^2)$ then $1/e < a \le (e + e^{-1})/2$ and $4r/(1-r^2) \le a - 1/e$. By Lemma 2.2, it follows that $H \in \mathscr{S}_e^*$ in $|z| < \rho_0$. Also, at the point $z = -\rho_0$, we have

$$\left| \log \frac{zH'(z)}{H(z)} \right| = \left| \log \left(\frac{1 - 4\rho_0 + \rho_0^2}{1 - \rho_0^2} \right) \right| = |\log(1/e)| = 1.$$

This shows that the number ρ_0 is sharp.

Now, let $f, g \in \mathscr{S}^*$. Then the functions F and G defined by $zF'(z) = f(z)$ and $zG'(z) = g(z)$ belong to $\mathscr{K}$. Since the convolution of two convex functions is convex, $F * G \in \mathscr{K}$. Since $H(\rho_0 z)/\rho_0 \in \mathscr{S}_e^*$, therefore $(F * G * H)(\rho_0 z)/\rho_0 \in \mathscr{S}_e^*$ by Theorem 2.9. But $f * g = F * G * H$. Hence $f * g \in \mathscr{S}_e^*$ in $|z| < \rho_0$. ∎

3. Radius Problems

Let $\mathscr{F}$ and $\mathscr{G}$ be subsets of $\mathscr{A}$. Then the $\mathscr{G}$-radius in $\mathscr{F}$, denoted by $\mathscr{R}_{\mathscr{G}}(\mathscr{F})$ is the largest R such that for every $f \in \mathscr{F}$, $r^{-1}f(rz) \in \mathscr{G}$ for each $r \leq R$. In particular, if $\mathscr{F} \subset \mathscr{G}$ then $\mathscr{R}_{\mathscr{G}}(\mathscr{F}) = 1$. In this section, we compute $\mathscr{R}_{\mathscr{G}}(\mathscr{S}^*_e)$ and $\mathscr{R}_{\mathscr{S}^*_e}(\mathscr{F})$ for various subclasses $\mathscr{F}$ and $\mathscr{G}$ of $\mathscr{A}$.

The first theorem of this section determines sharp $\mathscr{S}^*(\alpha)$ $(0 \leq \alpha < 1)$, $\mathscr{S}\mathscr{S}^*(\gamma)$ $(0 < \gamma \leq 1)$, $\mathscr{M}(\beta)$ $(\beta > 1)$ and $k-\mathscr{S}^*$ $(k \geq 0)$ radii in the class $\mathscr{S}^*_e$. By Theorem 2.1, it is known that $\mathscr{R}_{\mathscr{S}^*(\alpha)}(\mathscr{S}^*_e) = \mathscr{R}_{\mathscr{S}\mathscr{S}^*(\gamma)}(\mathscr{S}^*_e) = \mathscr{R}_{\mathscr{M}(\beta)}(\mathscr{S}^*_e) = 1$ for $0 \leq \alpha \leq 1/e$, $2/\pi \leq \gamma \leq 1$ and $\beta \geq e$.

Theorem 3.1. *Let $f \in \mathscr{S}^*_e$. Then we have the following.*

(i) *If $1/e \leq \alpha < 1$, then f is starlike of order α in $|z| < (-\log\alpha)$;*
(ii) *If $1 < \beta \leq e$, then $f \in \mathscr{M}(\beta)$ in $|z| < \log\beta$;*
(iii) *If $0 < \gamma \leq 2/\pi$, then f is strongly starlike of order γ in $|z| < \gamma\pi/2$;*
(iv) *If $k > 0$, then f is k-starlike in $|z| < \log((1+k)/k)$. In particular, f is parabolic starlike in $|z| < \log 2$.*

The results are all sharp.

Proof. Since $f \in \mathscr{S}^*_e$, $zf'(z)/f(z) \prec e^z$ and hence Lemma 2.1 gives

$$e^{-r} \leq \operatorname{Re}\frac{zf'(z)}{f(z)} \leq e^r, \quad |z| = r < 1$$

which verifies the validity of parts (i) and (ii). The function h given by (1.2) shows that the constants $-\log\alpha$ and $\log\beta$ are best possible.

Also, we can write $zf'(z)/f(z) = e^{w(z)}$ where w is an analytic function in $\mathbb{D}$ with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in \mathbb{D}$. By Schwarz lemma, $|w(z)| \leq |z|$ for all $z \in \mathbb{D}$. If $|z| < \gamma\pi/2$, then

$$|\arg e^{w(z)}| = |\operatorname{Im} w(z)| \leq |w(z)| \leq |z| < \gamma\pi/2.$$

Thus $f \in \mathscr{S}\mathscr{S}^*(\gamma)$ in $|z| < \gamma\pi/2$. At the point $z_0 = i\gamma\pi/2$, the function h given by (1.2) gives

$$\left|\arg\frac{z_0 h'(z_0)}{h(z_0)}\right| = |\arg e^{z_0}| = |\operatorname{Im} z_0| = \frac{\gamma\pi}{2}.$$

This proves (iii).

For the proof of (iv), note that f is k-starlike is $|z| < r$ whenever $\operatorname{Re} e^{w(z)} > k|e^{w(z)} - 1|$. Since $\operatorname{Re} e^{w(z)} > e^{-r}$ and $|e^{w(z)} - 1| < 1 - e^{-r}$, we conclude that the condition $e^{-r} > k(1-e^{-r})$ is sufficient for the inequality $\operatorname{Re} e^{w(z)} > k|e^{w(z)} - 1|$ to hold. Hence solving $e^{-r} > k(1-e^{-r})$, we obtain $r < \log((1+k)/k)$. For the function h given by (1.2) and for $z_0 = -\log((1+k)/k)$, we have

$$\operatorname{Re}\frac{z_0 h'(z_0)}{h(z_0)} = \operatorname{Re} e^{z_0} = \frac{k}{1+k} = k|1-e^{z_0}| = k\left|1 - \frac{z_0 h'(z_0)}{h(z_0)}\right|.$$

This completes the proof of the theorem. ∎

Remark 3.1. Let $f \in \mathscr{S}^*_e$. Then $zf'(z)/f(z) = e^{w(z)}$ where w is Schwarz function. Differentiation gives $1 + zf''(z)/f'(z) = e^{w(z)} + zw'(z)$ so that

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) \geq \operatorname{Re} e^{w(z)} - |zw'(z)|.$$

By using the identity $|w'(z)| \leq (1-|w(z)|^2)/(1-|z|^2)$, we deduce that

$$\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) \geq e^{-|z|} - \frac{|z|}{1-|z|^2}.$$

The function $g(r) = e^{-r} - r/(1-r^2)$ is decreasing in $[0,1)$ and $g(0) = 1$. Hence $\operatorname{Re}(1+zf''(z)/f'(z)) > \alpha$ in $|z| < r(\alpha)$ where $r(\alpha)$ is the real root of the equation $g(r) = \alpha$ in $(0,1)$. In particular, $r(0) \approx 0.478172$.

Now, we will determine the $\mathcal{S}^*_e$-radii for several interesting subclasses of analytic functions. We begin with the fundamental class $\mathcal{S}$ of normalized univalent functions.

Theorem 3.2. *The $\mathcal{S}^*_e$-radius for the class $\mathcal{S}$ is given by $\mathcal{R}_{\mathcal{S}^*_e}(\mathcal{S}) = (e-1)/(e+1) \approx 0.462117$. This radius is sharp.*

Proof. A function $f \in \mathcal{S}$ satisfies the sharp inequality (see [30, Theorem 6.5, p. 168]):

$$\left|\log\frac{zf'(z)}{f(z)}\right| \leq \log\frac{1+|z|}{1-|z|} \quad (z \in \mathbb{D}).$$

If $|z| < (e-1)/(e+1)$, then $|\log(zf'(z)/f(z))| < 1$. Thus $f \in \mathcal{S}^*_e$ in $|z| < (e-1)/(e+1)$. To show that the bound $(e-1)/(e+1)$ cannot be increased, consider the Koebe function $k(z) = z/(1-z)^2$. At the point $z_0 = (e-1)/(e+1)$, a computation shows that

$$\left|\log\frac{z_0k'(z_0)}{k(z_0)}\right| = \left|\log\frac{1+z_0}{1-z_0}\right| = 1.$$

This proves that $(e-1)/(e+1)$ is the $\mathcal{S}^*_e$-radius for the class $\mathcal{S}$ and that the result is sharp. ∎

Since $\mathcal{S}^* \subset \mathcal{S}$ and Koebe function $k(z) = z/(1-z)^2$ is starlike, Theorem 3.2 shows that $\mathcal{R}_{\mathcal{S}^*_e}(\mathcal{S}^*) = (e-1)/(e+1)$. We next determine the $\mathcal{S}^*_e$-radius for the class $\mathcal{S}^*[A,B]$ $(-1 \leq B < A \leq 1)$ with the cases $B \geq 0$ and $B < 0$.

Theorem 3.3. *Let $0 \leq B < A \leq 1$. Then the $\mathcal{S}^*_e$-radius for the class $\mathcal{S}^*[A,B]$ is given by*

$$\mathcal{R}_{\mathcal{S}^*_e}(\mathcal{S}^*[A,B]) = \min\left\{1, \frac{e-1}{Ae-B}\right\}.$$

The result is sharp. In particular, if $1-B \leq (1-A)e$, then $\mathcal{S}^*[A,B] \subset \mathcal{S}^*_e$.

Proof. Let $f \in \mathcal{S}^*[A,B]$. Then $zf'(z)/f(z) \in \mathcal{P}[A,B]$ so that Lemma 2.3 gives

$$\left|\frac{zf'(z)}{f(z)} - \frac{1-ABr^2}{1-B^2r^2}\right| \leq \frac{(A-B)r}{1-B^2r^2}, \quad |z| = r < 1.$$

Since $B \geq 0$, $a := (1-ABr^2)/(1-B^2r^2) \leq 1$. Using Lemma 2.2, the function f satisfies $|\log(zf'(z)/f(z))| \leq 1$ provided

$$\frac{(A-B)r}{1-B^2r^2} \leq \frac{1-ABr^2}{1-B^2r^2} - \frac{1}{e}.$$

This yields $r \leq (e-1)/(Ae-B)$. The result is sharp for the function

$$f(z) = \begin{cases} z(1+Bz)^{\frac{A-B}{B}}, & B \neq 0; \\ ze^{Az}, & B = 0. \end{cases} \tag{3.1}$$

The function $f \in \mathcal{S}^*[A,B]$ and at the point $z_0 = (1-e)/(Ae-B)$, we have

$$\left|\log \frac{z_0 f'(z_0)}{f(z_0)}\right| = \left|\log \frac{1+Az_0}{1+Bz_0}\right| = |\log(1/e)| = 1. \qquad \blacksquare$$

Theorem 3.4. *Let $-1 \le B < A \le 1$, with $B < 0$. Let*

$$R_1 = \min\left\{1, \sqrt{\frac{1+e^2}{(1+e^2)B^2 + 2e(1-AB)}}\right\}, \quad R_2 = \min\left\{1, \frac{e-1}{Ae-B}\right\}$$

and

$$R_3 = \min\left\{1, \frac{2(e-1)}{(A-B)+\sqrt{(A-B)^2+4B(eB-A)(e-1)}}\right\}.$$

Then the $\mathcal{S}_e^*$-radius for the class $\mathcal{S}^*[A,B]$ is given by

$$\mathcal{R}_{\mathcal{S}_e^*}(\mathcal{S}^*[A,B]) = \begin{cases} R_2, & R_2 \le R_1; \\ R_3, & R_2 > R_1. \end{cases}$$

Proof. Let $f \in \mathcal{S}^*[A,B]$. Then Lemma 2.3 shows that the quantity $w = zf'(z)/f(z)$ lies in the disk $|w-a| \le R$, where $a := (1-ABr^2)/(1-B^2r^2) > 1$ and $R := (A-B)r/(1-B^2r^2)$. Also, observe that the numbers R_1, R_2 and R_3 are determined so that $r \le R_1$ if and only if $a \le (e+e^{-1})/2$, $r \le R_2$ if and only if $R \le a - 1/e$ and $r \le R_3$ if and only if $R \le e - a$.

Suppose that $R_2 \le R_1$. Since $r \le R_1$ is equivalent to $a \le (e+e^{-1})/2$, for $0 \le r \le R_2$, it follows that $a \le (e+e^{-1})/2$. From Lemma 2.2 the $\mathcal{S}_e^*$-radius satisfies the inequality $R \le a - 1/e$. This shows that $f \in \mathcal{S}_e^*$ in $|z| \le R_2$.

Next, assume that $R_2 > R_1$. In this case, since $r \ge R_1$ if and only if $a \ge (e+e^{-1})/2$, for $r = R_2$, we have $a \ge (e+e^{-1})/2$. Lemma 2.2 shows that $f \in \mathcal{S}_e^*$ in $|z| \le r$ if $R \le e - a$, or equivalently if $r \le R_3$.

For the function f given by (3.1), $\{zf'(z)/f(z) : |z| < r\} = \{w : |w-a| < R\}$, where a and R are as defined above. This shows that the result is sharp. $\blacksquare$

Corollary 3.1. $\mathcal{R}_{\mathcal{S}_e^*}(\mathcal{K}) = (e-1)/e \approx 0.632121$. *The result is sharp.*

Proof. By the well-known Marx Strohhäcker theorem [27, Theorem 2.6(a), p. 57], $\mathcal{K} \subset \mathcal{S}^*(1/2)$ and $\mathcal{S}^*[0,-1] = \mathcal{S}^*(1/2)$, therefore by Theorem 3.4, the $\mathcal{S}_e^*$-radius for the class $\mathcal{K}$ is at least $(e-1)/e$. The half-plane mapping $l(z) = z/(1-z) \in \mathcal{K}$ satisfies $zl'(z)/l(z) = 1/(1-z)$. In particular, at the point $z_0 = (e-1)/e$, we have

$$\left|\log \frac{z_0 l'(z_0)}{l(z_0)}\right| = |\log e| = 1.$$

This establishes the sharpness of the result. $\blacksquare$

Let $\mathcal{W}$ be the class of functions $f \in \mathcal{A}$ satisfying $f(z)/z \in \mathcal{P}$. MacGregor [23], Ali *et al.* [4] and Mendiratta [26] determined $\mathcal{S}^*$, $\mathcal{S}_L^*$ and $\mathcal{S}_{RL}^*$ radii respectively for the class $\mathcal{W}$. The following result determines the sharp $\mathcal{S}_e^*$-radius for the class $\mathcal{W}$.

Theorem 3.5. *The $\mathcal{S}_e^*$-radius for the class $\mathcal{W}$ is given by*

$$\mathcal{R}_{\mathcal{S}_e^*}(\mathcal{W}) = \frac{e-1}{e+\sqrt{e^2+(e-1)^2}} \approx 0.28956.$$

This bound is best possible.

Proof. Let $f \in \mathcal{W}$. Then the function $g(z) = f(z)/z$ belongs to the class $\mathcal{P}$ and using Lemma 2.3, it is easy to deduce that

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \frac{2r}{1-r^2}.$$

In view of Lemma 2.2, the disk $|w-1| \leq 2r/(1-r^2)$ lies inside $|\log w| \leq 1$ if

$$\frac{2r}{1-r^2} \leq 1 - \frac{1}{e}.$$

This yields $r \leq R := (e-1)/(e+\sqrt{e^2+(e-1)^2})$. Hence $\mathcal{R}_{\mathcal{S}_e^*}(\mathcal{W}) \geq R$. The function $f(z) = z(1+z)/(1-z) \in \mathcal{W}$ and at the point $z_0 = -R$, we obtain

$$\left|\log \frac{z_0 f'(z_0)}{f(z_0)}\right| = \left|\log \frac{1-2R-R^2}{1-R^2}\right| = |\log(1/e)| = 1.$$

Thus $\mathcal{R}_{\mathcal{S}_e^*}(\mathcal{W}) \leq R$. ∎

Motivated by [5, 24, 25], we close this section by determining the $\mathcal{S}_e^*$-radii for several classes of functions $f \in \mathcal{A}$ characterized by its ratio with a certain function g. Let $\mathcal{F}_1$ be the class of functions $f \in \mathcal{A}$ satisfying $f/g \in \mathcal{P}$ for some $g \in \mathcal{W}$.

Theorem 3.6. *The $\mathcal{S}_e^*$-radius for the class $\mathcal{F}_1$ is*

$$\mathcal{R}_{\mathcal{S}_e^*}(\mathcal{F}_1) = \frac{e-1}{2e+\sqrt{4e^2+(e-1)^2}} \approx 0.154269.$$

The result is sharp.

Proof. Let $f \in \mathcal{F}_1$ and define $p, q : \mathbb{D} \to \mathbb{C}$ by $p(z) = g(z)/z$ and $q(z) = f(z)/g(z)$. Then $p, q \in \mathcal{P}$ and using Lemma 2.3, it follows that

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \left|\frac{zp'(z)}{p(z)}\right| + \left|\frac{zq'(z)}{q(z)}\right| \leq \frac{4r}{1-r^2} \quad (|z| = r)$$

In view of Lemma 2.2, $f \in \mathcal{S}_e^*$ provided $4r/(1-r^2) \leq 1 - 1/e$, which gives $r \leq R := (e-1)/(2e+\sqrt{4e^2+(e-1)^2})$. To show that R is the sharp $\mathcal{S}_e^*$-radius for $\mathcal{F}_1$, consider the function $f_0(z) = z(1+z)^2/(1-z)^2$ with $g_0(z) = z(1+z)/(1-z)$. Clearly $f_0 \in \mathcal{F}$ and at the point $z_0 = -R$, a routine calculation shows that

$$\left|\log \frac{z_0 f_0'(z_0)}{f_0(z_0)}\right| = \left|\log \frac{1-4R-R^2}{1-R^2}\right| = |\log(1/e)| = 1. \qquad ∎$$

Let $\mathcal{F}_2$ be the class of functions $f \in \mathcal{A}$ satisfying the inequality

$$\left|\frac{f(z)}{g(z)} - 1\right| < 1 \quad (z \in \mathbb{D}) \tag{3.2}$$

for some $g \in \mathcal{W}$.

Theorem 3.7. *The $\mathcal{S}_e^*$-radius for the class $\mathcal{F}_2$ is given by*

$$\mathcal{R}_{\mathcal{S}_e^*}(\mathcal{F}_2) = \frac{2(e-1)}{3e+\sqrt{9e^2+4(2e-1)(e-1)}} \approx 0.190884.$$

This bound is best possible.

Proof. Let $f \in \mathscr{F}_2$ and define functions $p, q : \mathbb{D} \to \mathbb{C}$ by $p(z) = g(z)/z$ and $q(z) = g(z)/f(z)$. Since the inequality (3.2) is equivalent to $\operatorname{Re} g(z)/f(z) > 1/2$, therefore $p \in \mathscr{P}$ and $q \in \mathscr{P}(1/2)$. Applying Lemma 2.3 to the identity

$$\frac{zf'(z)}{f(z)} = 1 + \frac{zp'(z)}{p(z)} - \frac{zq'(z)}{q(z)}$$

we obtain

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \frac{r(3+r)}{1-r^2}.$$

By Lemma 2.2, the function f satisfies $|\log(zf'(z)/f(z))| \leq 1$ if $r(3+r)/(1-r^2) \leq 1 - 1/e$. This yields $r \leq R := 2(e-1)/(3e + \sqrt{9e^2 + 4(2e-1)(e-1)})$. For sharpness, note that the function $f_0(z) = z(1+z)^2/(1-z)$ belongs to class $\mathscr{F}_2$ with $g_0(z) = z(1+z)/(1-z)$. At the point $z_0 = -R$, we have

$$\left|\log \frac{z_0 f_0'(z_0)}{f_0(z_0)}\right| = \left|\log \frac{1 - 3R - 2R^2}{1 - R^2}\right| = |\log(1/e)| = 1.$$

Hence the result is sharp. ∎

The other related radius problems carried out in [5] can also be performed on the similar lines as that of Theorems 3.6 and 3.7 for the class $\mathscr{S}_e^*$.

Remark 3.2. Let $\mathscr{CS}^*$ be the class of close-to-star functions defined by

$$\mathscr{CS}^* = \left\{ f \in \mathscr{A} : \frac{f}{g} \in \mathscr{P} \text{ and } g \in \mathscr{S}^* \right\}.$$

Then $\mathscr{R}_{\mathscr{S}_e^*}(\mathscr{CS}^*) = (e-1)/(2e + \sqrt{1+3e^2}) \approx 0.167641$. To see this, let $f \in \mathscr{CS}^*$ and $g \in \mathscr{S}^*$ such that $p(z) = f(z)/g(z)$ belongs to $\mathscr{P}$. Then $zg'(z)/g(z) \in \mathscr{P}$ so that

$$\left|\frac{zp'(z)}{p(z)}\right| \leq \frac{2r}{1-r^2} \quad \text{and} \quad \left|\frac{zg'(z)}{g(z)} - \frac{1+r^2}{1-r^2}\right| \leq \frac{2r}{1-r^2},$$

by applying Lemma 2.3. Using these estimates in the identity

$$\frac{zf'(z)}{f(z)} = \frac{zg'(z)}{g(z)} + \frac{zp'(z)}{p(z)}$$

it is easy to see that

$$\left|\frac{zf'(z)}{f(z)} - \frac{1+r^2}{1-r^2}\right| \leq \frac{4r}{1-r^2}.$$

The rest of the proof is similar to Theorem 2.10 and so its details are omitted.

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References

[1] J. W. Alexander, Functions which map the interior of the unit circle upon simple regions, Ann. of Math. (2) **17** (1915), no. 1, 12–22.

[2] R. M. Ali, N. E. Cho, N. K. Jain and V. Ravichandran, Radii of starlikeness and convexity for functions with fixed second coefficient defined by subordination, Filomat **26** (2012), no. 3, 553–561.

[3] R. M. Ali, N. E. Cho, V. Ravichandran and S. S. Kumar, Differential subordination for functions associated with the lemniscate of Bernoulli, Taiwanese J. Math. **16** (2012), no. 3, 1017–1026.

[4] R. M. Ali, N. K. Jain and V. Ravichandran, Radii of starlikeness associated with the lemniscate of Bernoulli and the left-half plane, Appl. Math. Comput. **218** (2012), no. 11, 6557–6565.
[5] R. M. Ali, N. K. Jain and V. Ravichandran, On the radius constants for classes of analytic functions, Bull. Malays. Math. Sci. Soc. (2) **36** (2013), no. 1, 23–38.
[6] R. M. Ali, S. Nagpal and V. Ravichandran, Second-order differential subordination for analytic functions with fixed initial coefficient, Bull. Malays. Math. Sci. Soc. (2) **34** (2011), no. 3, 611–629.
[7] R. M. Ali and V. Ravichandran, Uniformly convex and uniformly starlike functions, Math. Newsletter **21** (2011), no. 1, 16-30.
[8] R. M. Ali, V. Ravichandran and N. Seenivasagan, Coefficient bounds for *p*-valent functions, Appl. Math. Comput. **187** (2007), no. 1, 35–46.
[9] R. M. Ali, V. Ravichandran and N. K. Jain, Convolution of certain analytic functions, J. Anal. **18** (2010), 1–8.
[10] M. K. Aouf, J. Dziok and J. Sokół, On a subclass of strongly starlike functions, Appl. Math. Lett. **24** (2011), no. 1, 27–32.
[11] S. D. Bernardi, Convex and starlike univalent functions, Trans. Amer. Math. Soc. **135** (1969), 429–446.
[12] L. de Branges, A proof of the Bieberbach conjecture, Acta Math. **154** (1985), no. 1-2, 137–152.
[13] A. Gangadharan, V. Ravichandran and T. N. Shanmugam, Radii of convexity and strong starlikeness for some classes of analytic functions, J. Math. Anal. Appl. **211** (1997), no. 1, 301–313.
[14] A. W. Goodman, On uniformly convex functions, Ann. Polon. Math. **56** (1991), no. 1, 87–92.
[15] I. S. Jack, Functions starlike and convex of order α, J. London Math. Soc. (2) **3** (1971), 469–474.
[16] W. Janowski, Some extremal problems for certain families of analytic functions. I, Ann. Polon. Math. **28** (1973), 297–326.
[17] S. Kanas and A. Wisniowska, Conic regions and *k*-uniform convexity, J. Comput. Appl. Math. **105** (1999), no. 1-2, 327–336.
[18] S. S. Kumar V. Kumar, V. Ravichandran and N. E. Cho, Sufficient conditions for starlike functions associated with the lemniscate of Bernoulli, J. Inequal. Appl. **2013**, 2013:176, 13 pp.
[19] Y. Ling and S. Ding, On radii of starlikeness and convexity for convolutions of starlike functions, Internat. J. Math. Math. Sci. **20** (1997), no. 2, 403–404.
[20] A. E. Livingston, On the radius of univalence of certain analytic functions, Proc. Amer. Math. Soc. **17** (1966), 352–357.
[21] W. C. Ma and D. Minda, Uniformly convex functions, Ann. Polon. Math. **57** (1992), no. 2, 165–175.
[22] W. C. Ma and D. Minda, A unified treatment of some special classes of univalent functions, in *Proceedings of the Conference on Complex Analysis (Tianjin, 1992)*, 157–169, Conf. Proc. Lecture Notes Anal., I Int. Press, Cambridge, MA.
[23] T. H. MacGregor, Functions whose derivative has a positive real part, Trans. Amer. Math. Soc. **104** (1962), 532–537.
[24] T. H. MacGregor, The radius of univalence of certain analytic functions, Proc. Amer. Math. Soc. **14** (1963), 514–520.
[25] T. H. MacGregor, The radius of univalence of certain analytic functions. II, Proc Amer. Math. Soc. **14** (1963), 521–524.
[26] R. Mendiratta, S. Nagpal and V. Ravichandran, A subclass of starlike functions associated with left-half of the lemniscate of Bernoulli, preprint.
[27] S. S. Miller and P. T. Mocanu, *Differential Subordinations*, Monographs and Textbooks in Pure and Applied Mathematics, 225, Dekker, New York, 2000.
[28] N. M. Özgür and J. Sokół, On starlike functions connected with k-Fibonacci numbers, Bull. Malays. Math. Sci. Soc. (2), to appear.
[29] E. Paprocki and J. Sokół, The extremal problems in some subclass of strongly starlike functions, Zeszyty Nauk. Politech. Rzeszowskiej Mat. No. 20 (1996), 89–94.
[30] C. Pommerenke, *Univalent functions*, Vandenhoeck & Ruprecht, Göttingen, 1975.
[31] V. Ravichandran, F. Rønning and T. N. Shanmugam, Radius of convexity and radius of starlikeness for some classes of analytic functions, Complex Variables Theory Appl. **33** (1997), no. 1-4, 265–280.
[32] M. S. Robertson, Certain classes of starlike functions, Michigan Math. J. **32** (1985), no. 2, 135–140.
[33] W. Rogosinski, On the coefficients of subordinate functions, Proc. London Math. Soc. (2) **48** (1943), 48–82.
[34] F. Rønning, Uniformly convex functions and a corresponding class of starlike functions, Proc. Amer. Math. Soc. **118** (1993), no. 1, 189–196.

[35] S. Ruscheweyh, A subordination theorem for F-like functions, J. London Math. Soc. (2) **13** (1976), no. 2, 275–280.

[36] St. Ruscheweyh and T. Sheil-Small, Hadamard products of Schlicht functions and the Pólya-Schoenberg conjecture, Comment. Math. Helv. **48** (1973), 119–135.

[37] A. Schild, On starlike functions of order α, Amer. J. Math. **87** (1965), 65–70.

[38] G. M. Shah, On the univalence of some analytic functions, Pacific J. Math. **43** (1972), 239–250.

[39] T. N. Shanmugam and V. Ravichandran, Certain properties of uniformly convex functions, in *Computational methods and function theory 1994 (Penang)*, 319–324, Ser. Approx. Decompos., 5 World Sci. Publ., River Edge, NJ.

[40] J. Sokół, On some subclass of strongly starlike functions, Demonstratio Math. **31** (1998), no. 1, 81–86.

[41] J. Sokół, On sufficient condition to be in a certain subclass of starlike functions defined by subordination, Appl. Math. Comput. **190** (2007), no. 1, 237–241.

[42] J. Sokół, On an application of certain sufficient condition for starlikeness, J. Math. Appl. **30** (2008), 131–135.

[43] J. Sokół, Coefficient estimates in a class of strongly starlike functions, Kyungpook Math. J. **49** (2009), no. 2, 349–353.

[44] J. Sokół, Radius problems in the class $\mathscr{S}^*_L$, Appl. Math. Comput. **214** (2009), no. 2, 569–573.

[45] J. Sokół and J. Stankiewicz, Radius of convexity of some subclasses of strongly starlike functions, Zeszyty Nauk. Politech. Rzeszowskiej Mat. No. 19 (1996), 101–105.

[46] N. Tuneski, On the quotient of the representations of convexity and starlikeness, Math. Nachr. **248/249** (2003), 200–203.

[47] B. A. Uralegaddi, M. D. Ganigi and S. M. Sarangi, Univalent functions with positive coefficients, Tamkang J. Math. **25** (1994), no. 3, 225–230.

[48] B. Kowalczykand and A. Lecko, Fekete-Szeg problem for a certain subclass of close-to-convex functions, Bull. Malays. Math. Sci. Soc. (2), to appear.