On the universal α-central extension of the semi-direct product of Hom-Leibniz algebras

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Abstract We introduce Hom-actions, semidirect product and establish the equivalence between split extensions and the semi-direct product extension of Hom-Leibniz algebras. We analyze the functorial properties of the universal (α)-central extensions of (α)-perfect Hom-Leibniz algebras. We establish under what conditions an automorphism or a derivation can be lifted in an α-cover and we analyze the universal α-central extension of the semi-direct product of two α-perfect Hom-Leibniz algebras.

Key words: universal (α)-central extension, Hom-action, semi-direct product, derivation.

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1 Introduction

Hom-Lie algebras were introduced in [11] as Lie algebras whose Jacobi identity is twisted by means of a map. This fact occurs in different applications in models of quantum phenomena or in analysis of complex systems and processes exhibiting complete or partial scaling invariance.

From the introductory paper, the investigation of several kinds of Hom-structures is in progress (for instance, see [1, 2, 3, 4, 15, 17, 21, 22] and references given therein). Naturally, the non-skew-symmetric version of Hom-Lie algebras, the so called Hom-Leibniz algebras, was considered as well (see [2, 7, 9, 12, 16, 17, 19]). A Hom-Leibniz algebra is a triple $(L, [-,-], \alpha_L)$ consisting of a $\mathbb{K}$-vector space L, a bilinear map $[-,-] : L \times L \to L$ and a homomorphism of $\mathbb{K}$-vector spaces $\alpha_L : L \to L$ satisfying the Hom-Leibniz identity:

$$[\alpha_L(x), [y, z]] = [[x, y], \alpha_L(z)] - [[x, z], \alpha_L(y)]$$

for all $x, y, z \in L$. When $\alpha_L = Id$, the definition of Leibniz algebra [13] is recovered. If the bracket is skew-symmetric, then we recover the definition of Hom-Lie algebra [11].

Lie and Leibniz algebras have found important applications in Mathematics and Physics, in particular degenerations, contractions and deformations (see [8, 18] and references given therein). The analysis of these properties in the Hom-Lie setting [10] have led to deal with universal central extensions.

The goal of the present paper is to continue with the investigations on universal (α)-central extensions of (α)-perfect Hom-Leibniz algebras initiated in [7]. In concrete, we consider the extension of results about the universal central extension of the semi-direct product of Leibniz algebra in [6] to the framework of Hom-Leibniz algebras.

To do so, we organize the paper as follows: an initial section recalling the background material on Hom-Leibniz algebras. We introduce the concepts of Hom-action and semi-direct product and we prove a new result (Lemma 2.11) that establishes the equivalence between split extensions and the semi-direct product extension. Section 3 is devoted to analyze the functorial properties of the universal (α)-central extensions of (α)-perfect Hom-Leibniz algebras. In section 4 we establish under what conditions an automorphism or a derivation can be lifted in an α-cover (a central extension $f : (L', \alpha_{L'}) \to (L, \alpha_L)$ where $(L', \alpha_{L'})$ is α-perfect ($L' = [\alpha_{L'}(L'), \alpha_{L'}(L')]$). Final section is devoted to analyze the relationships between the universal α-central extension of the semi-direct product of two α-perfect Hom-Leibniz algebras, such that one of them Hom-acts over the other one, and the semi-direct product of the universal α-central extensions of both of them.

2 Preliminaries on Hom-Leibniz algebras

In this section we introduce necessary material on Hom-Leibniz algebras which will be used in subsequent sections.

Definition 2.1 *[17] A Hom-Leibniz algebra is a triple $(L, [-,-], \alpha_L)$ consisting of a $\mathbb{K}$-vector space L, a bilinear map $[-,-] : L \times L \to L$ and a $\mathbb{K}$-linear map $\alpha_L : L \to L$ satisfying:*

$$[\alpha_L(x), [y, z]] = [[x, y], \alpha_L(z)] - [[x, z], \alpha_L(y)] \quad (\mathrm{Hom-Leibniz\ identity}) \qquad (1)$$

for all $x, y, z \in L$.

A Hom-Leibniz algebra $(L, [-,-], \alpha_L)$ is said to be multiplicative [19] if the $\mathbb{K}$-linear map α_L preserves the bracket, that is, if $\alpha_L[x, y] = [\alpha_L(x), \alpha_L(y)]$, for all $x, y \in L$.

Example 2.2

a) Taking $\alpha = Id$ in Definition 2.1 we obtain the definition of Leibniz algebra [13]. Hence Hom-Leibniz algebras include Leibniz algebras as a full subcategory, thereby motivating the name "Hom-Leibniz algebras" as a deformation of Leibniz algebras twisted by a homomorphism. Moreover it is a multiplicative Hom-Leibniz algebra.

b) Hom-Lie algebras [11] are Hom-Leibniz algebras whose bracket satisfies the condition $[x, x] = 0$, for all x. So Hom-Lie algebras can be considered as a full subcategory of Hom-Leibniz algebras category. For any multiplicative Hom-Leibniz algebra $(L, [-,-], \alpha_L)$ it is associated the Hom-Lie algebra $(L_{\mathrm{Lie}}, [-,-], \widetilde{\alpha})$, where $L_{\mathrm{Lie}} = L/L^{\mathrm{ann}}$, the bracket is the canonical bracket induced on the quotient and $\widetilde{\alpha}$ is the homomorphism naturally induced by α. Here $L^{\mathrm{ann}} = \langle \{[x, x] : x \in L\} \rangle$.

c) Let $(D, \dashv, \vdash, \alpha_D)$ be a Hom-dialgebra. Then $(D, \dashv, \vdash, \alpha_D)$ is a Hom-Leibniz algebra with respect to the bracket $[x, y] = x \dashv y - y \vdash x$, for all $x, y \in A$ [20].

d) Let $(L, [-,-])$ be a Leibniz algebra and $\alpha_L : L \to L$ a Leibniz algebra endomorphism. Define $[-,-]_\alpha : L \otimes L \to L$ by $[x, y]_\alpha = [\alpha(x), \alpha(y)]$, for all $x, y \in L$. Then $(L, [-,-]_\alpha, \alpha_L)$ is a multiplicative Hom-Leibniz algebra.

e) Abelian or commutative Hom-Leibniz algebras are $\mathbb{K}$-vector spaces L with trivial bracket and any linear map $\alpha_L : L \to L$.

Definition 2.3 *A homomorphism of Hom-Leibniz algebras $f : (L, [-,-], \alpha_L) \to (L', [-,-]', \alpha_{L'})$ is a $\mathbb{K}$-linear map $f : L \to L'$ such that*

a) $f([x, y]) = [f(x), f(y)]'$,

b) $f \circ \alpha_L(x) = \alpha_{L'} \circ f(x)$,

for all $x, y \in L$.

A homomorphism of multiplicative Hom-Leibniz algebras is a homomorphism of the underlying Hom-Leibniz algebras.

In the sequel we refer to Hom-Leibniz algebra as a multiplicative Hom-Leibniz algebra and we shall use the shortened notation (L, α_L) when there is not confusion with the bracket operation.

Definition 2.4 *Let $(L, [-,-], \alpha_L)$ be a Hom-Leibniz algebra. A Hom-Leibniz subalgebra (H, α_H) is a linear subspace H of L, which is closed for the bracket and invariant by α_L, that is,*

a) $[x, y] \in H$, for all $x, y \in H$, *b)* $\alpha_L(x) \in H$*, for all* $x \in H$ $(\alpha_H = \alpha_{L|})$*.*

A Hom-Leibniz subalgebra (H, α_H) *of* (L, α_L) *is said to be a two-sided Hom-ideal if* $[x, y], [y, x] \in H$*, for all* $x \in H, y \in L$*.*

If (H, α_H) *is a two-sided Hom-ideal of* (L, α_L)*, then the quotient* L/H *naturally inherits a structure of Hom-Leibniz algebra with respect to the endomorphism* $\widetilde{\alpha} : L/H \to L/H, \widetilde{\alpha}(\bar{l}) = \overline{\alpha_L(l)}$*, which is said to be the quotient Hom-Leibniz algebra.*

So we have defined the category $\mathsf{Hom-Leib}$ (respectively, $\mathsf{Hom-Leib}_{\mathrm{mult}}$) whose objects are Hom-Leibniz (respectively, multiplicative Hom-Leibniz) algebras and whose morphisms are the homomorphisms of Hom-Leibniz (respectively, multiplicative Hom-Leibniz) algebras. There is an obvious inclusion functor $inc : \mathsf{Hom-Leib}_{\mathrm{mult}} \to \mathsf{Hom-Leib}$. This functor has as left adjoint the multiplicative functor $(-)_{\mathrm{mult}} : \mathsf{Hom-Leib} \to \mathsf{Hom-Leib}_{\mathrm{mult}}$ which assigns to a Hom-Leibniz algebra $(L, [-,-], \alpha_L)$ the multiplicative Hom-Leibniz algebra $(L/I, [-,-], \tilde{\alpha})$, where I is the two-sided ideal of L spanned by the elements $\alpha_L[x, y] - [\alpha_L(x), \alpha_L(y)]$, for all $x, y \in L$.

Definition 2.5 *Let* (H, α_H) *and* (K, α_K) *be two-sided Hom-ideals of a Hom-Leibniz algebra* $(L, [-,-], \alpha_L)$*. The commutator of* (H, α_H) *and* (K, α_K)*, denoted by* $([H, K], \alpha_{[H,K]})$*, is the Hom-Leibniz subalgebra of* (L, α_L) *spanned by the brackets* $[h, k], h \in H, k \in K$*.*

Obviously, $[H, K] \subseteq H \cap K$ and $[K, H] \subseteq H \cap K$. When $H = K = L$, we obtain the definition of derived Hom-Leibniz subalgebra. Let us observe that, in general, $([H, K], \alpha_{[H,K]})$ is not a Hom-ideal, but if $H, K \subseteq \alpha_L(L)$, then $([H, K], \alpha_{[H,K]})$ is a two-sided ideal of $(\alpha_L(L), \alpha_{L|})$. When $\alpha = Id$, the classical notions are recovered.

Definition 2.6 *Let* $(L, [-,-], \alpha_L)$ *be a Hom-Leibniz algebra. The subspace* $Z(L) = \{x \in L \mid [x, y] = 0 = [y, x], \text{for all } y \in L\}$ *is said to be the center of* $(L, [-,-], \alpha_L)$*.*

When $\alpha_L : L \to L$ *is a surjective homomorphism, then* $Z(L)$ *is a Hom-ideal of* L*.*

2.1 Hom-Leibniz actions

Definition 2.7 *Let* (L, α_L) *and* (M, α_M) *be Hom-Leibniz algebras. A (right) Hom-action of* (L, α_L) *over* (M, α_M) *consists of two bilinear maps,* $\lambda : L \otimes M \to M$*,* $\lambda(l \otimes m) = l \centerdot m$*, and* $\rho : M \otimes L \to M$*,* $\rho(m \otimes l) = m \centerdot l$*, satisfying the following identities:*

a) $\alpha_M(m) \centerdot [x, y] = (m \centerdot x) \centerdot \alpha_L(y) - (m \centerdot y) \centerdot \alpha_L(x)$*;*

b) $\alpha_L(x) \centerdot (m \centerdot y) = (x \centerdot m) \centerdot \alpha_L(y) - [x, y] \centerdot \alpha_M(m)$*;*

c) $\alpha_L(x) \centerdot (y \centerdot m) = [x, y] \centerdot \alpha_M(m) - (x \centerdot m) \centerdot \alpha_L(y)$;

d) $\alpha_L(x) \centerdot [m, m'] = [x \centerdot m, \alpha_M(m')] - [x \centerdot m', \alpha_M(m)]$;

e) $[\alpha_M(m), m' \centerdot x] = [m, m'] \centerdot \alpha_L(x) - [m \centerdot x, \alpha_M(m')]$;

f) $[\alpha_M(m), x \centerdot m'] = [m \centerdot x, \alpha_M(m')] - [m, m'] \centerdot \alpha_L(x)$;

g) $\alpha_M(x \centerdot m) = \alpha_L(x) \centerdot \alpha_M(m)$;

h) $\alpha_M(m \centerdot x) = \alpha_M(m) \centerdot \alpha_L(x)$;

for all $x, y \in L$ *and* $m, m' \in M$.

When (M, α_M) *is an abelian Hom-Leibniz algebra, that is the bracket on* M *is trivial, then the Hom-action is called Hom-representation.*

Example 2.8

a) *Let* M *be a representation of a Leibniz algebra* L *[14]. Then* (M, Id_M) *is a Hom-representation of the Hom-Leibniz algebra* (L, Id_L).

b) *Let* (K, α_K) *be a Hom-Leibniz subalgebra of a Hom-Leibniz algebra* (L, α_L) *(even* $(K, \alpha_K) = (L, \alpha_L)$*) and* (H, α_H) *a two-sided Hom-ideal of* (L, α_L). *There exists a Hom-action of* (K, α_K) *over* (H, α_H) *given by the bracket in* (L, α_L).

c) *An abelian sequence of Hom-Leibniz algebras is an exact sequence of Hom-Leibniz algebras* $0 \to (M, \alpha_M) \xrightarrow{i} (K, \alpha_K) \xrightarrow{\pi} (L, \alpha_L) \to 0$, *where* (M, α_M) *is an abelian Hom-Leibniz algebra,* $\alpha_K \circ i = i \circ \alpha_M$ *and* $\pi \circ \alpha_K = \alpha_L \circ \pi$.

An abelian sequence induces a Hom-representation structure of (L, α_L) *over* (M, α_M) *by means of the actions given by* $\lambda : L \otimes M \to M, \lambda(l, m) = [k, m], \pi(k) = l$ *and* $\rho : M \otimes L \to M, \rho(m, l) = [m, k], \pi(k) = l$.

Definition 2.9 *Let* (M, α_M) *and* (L, α_L) *be Hom-Leibniz algebras together with a Hom-action of* (L, α_L) *over* (M, α_M). *Its semi-direct product* $(M \rtimes L, \widetilde{\alpha})$ *is the Hom-Leibniz algebra with underlying* $\mathbb{K}$*-vector space* $M \oplus L$, *endomorphism* $\widetilde{\alpha} : M \rtimes L \to M \rtimes L$ *given by* $\widetilde{\alpha}(m, l) = (\alpha_M(m), \alpha_L(l))$ *and bracket*

$$[(m_1, l_1), (m_2, l_2)] = ([m_1, m_2] + \alpha_L(l_1) \centerdot m_2 + m_1 \centerdot \alpha_L(l_2), [l_1, l_2]).$$

Let (M, α_M) and (L, α_L) be Hom-Leibniz algebras with a Hom-action of (L, α_L) over (M, α_M), then we can construct the sequence

$$0 \to (M, \alpha_M) \xrightarrow{i} (M \rtimes L, \widetilde{\alpha}) \xrightarrow{\pi} (L, \alpha_L) \to 0 \tag{2}$$

where $i : M \to M \rtimes L, i(m) = (m, 0)$, and $\pi : M \rtimes L \to L, \pi(m, l) = l$. Moreover, this sequence splits by $\sigma : L \to M \rtimes L, \sigma(l) = (0, l)$, that is, σ satisfies $\pi \circ \sigma = Id_L$ and $\widetilde{\alpha} \circ \sigma = \sigma \circ \alpha_L$.

Definition 2.10 *Let (M, α_M) and (L, α_L) be Hom-Leibniz algebras such that there is a Hom-action of (L, α_L) over (M, α_M). Two extensions of (L, α_L) by (M, α_M), $0 \to (M, \alpha_M) \xrightarrow{i} (K, \alpha_K) \xrightarrow{\pi} (L, \alpha_L) \to 0$ and $0 \to (M, \alpha_M) \xrightarrow{i'} (K', \alpha_{K'}) \xrightarrow{\pi'} (L, \alpha_L) \to 0$, are said to be equivalent if there exists a homomorphism of Hom-Leibniz algebras $\varphi : (K, \alpha_K) \to (K', \alpha_{K'})$ making the following diagram commutative.*

$$\begin{tikzcd}
0 \arrow[r] & (M,\alpha_M) \arrow[r, "i"] \arrow[d, equal] & (K,\alpha_K) \arrow[r, "\pi"] \arrow[d, "\varphi"'] & (L,\alpha_L) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & (M,\alpha_M) \arrow[r, "i'"] & (K',\alpha_{K'}) \arrow[r, "\pi'"] & (L,\alpha_L) \arrow[r] & 0
\end{tikzcd}$$

Lemma 2.11 *Let (C, Id_C) and (A, α_A) be Hom-Leibniz algebras together with a Hom-action of (C, Id_C) over (A, α_A).*

A sequence of Hom-Leibniz algebras $0 \to (A, \alpha_A) \xrightarrow{i} (B, \alpha_B) \xrightarrow{\pi} (C, Id_C) \to 0$ is split if and only if it is equivalent to the semi-direct sequence $0 \to (A, \alpha_A) \xrightarrow{j} (A \rtimes C, \widetilde{\alpha}) \xrightarrow{p} (C, Id_C) \to 0$.

Proof. If $0 \to (A, \alpha_A) \xrightarrow{i} (B, \alpha_B) \xrightarrow{\pi} (C, Id_C) \to 0$ is split by $s : (C, Id_C) \to (B, \alpha_B)$, then the Hom-action of (C, Id_C) over (A, α_A) is given by

$$c \centerdot a = [s(c), i(a)]\,; \quad a \centerdot c = [i(a), s(c)]$$

With this Hom-action of (C, Id_C) over (A, α_A) we can construct the following split extension:

$$\begin{tikzcd}
0 \arrow[r] & (A,\alpha_A) \arrow[r, "j"] & (A \rtimes C,\widetilde{\alpha}) \arrow[r, shift left, "p"] & (C,Id_C) \arrow[l, shift left, "\sigma"] \arrow[r] & 0
\end{tikzcd}$$

where $j : A \to A \rtimes C$, $j(a) = (a, 0)$, $p : A \rtimes C \to C$, $p(a, c) = c$ and $\sigma : C \to A \rtimes C$, $\sigma(c) = (0, c)$. Moreover the Hom-action of (C, Id_C) over (A, α_A) induced by this extension coincides whit the initial one:

$$c \star a = [\sigma\,(c)\,, j(a)] = [(0, c)\,, (a, 0)] = ([0, a] + Id_C\,(c) \centerdot a + 0 \centerdot 0, [c, 0]) = (c \centerdot a, 0) \equiv c \centerdot a$$

Finally, both extensions are equivalent since the homomorphism of Hom-Leibniz algebras $\varphi : (A \rtimes C, \widetilde{\alpha}) \to (B, \alpha_B)$, $\varphi\,(a, c) = i(a) + s(c)$, makes commutative the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & (A,\alpha_A) \arrow[r, "j"] \arrow[d, equal] & (A \rtimes C,\widetilde{\alpha}) \arrow[r, shift left, "p"] \arrow[d, dashed, "\varphi"] & (C,Id_C) \arrow[l, shift left, "\sigma"] \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & (A,\alpha_A) \arrow[r, "i"] & (B,\alpha_B) \arrow[r, shift left, "\pi"] & (C,Id_C) \arrow[l, shift left, "s"] \arrow[r] & 0
\end{tikzcd} \tag{3}$$

For the converse, if both extensions are equivalent, i.e. there exists a homomorphism of Hom-Leibniz algebras $\varphi : (A \rtimes C, \widetilde{\alpha}) \to (B, \alpha_B)$ making commutative diagram (3), then $s : (C, Id_C) \to (B, \alpha_B)$ given by $s(c) = \varphi(0, c)$, is a homomorphism that splits the extension. $\square$

Definition 2.12 *Let (M, α_M) be a Hom-representation of a Hom-Leibniz algebra (L, α_L). A derivation of (L, α_L) over (M, α_M) is a $\mathbb{K}$-linear map $d : L \to M$ satisfying:*

a) $d[l_1, l_2] = \alpha_L(l_1) \bullet d(l_2) + d(l_1) \bullet \alpha_L(l_2)$

b) $d \circ \alpha_L = \alpha_M \circ d$

for all $l_1, l_2 \in L$.

Example 2.13

a) The $\mathbb{K}$-linear map $\theta : M \rtimes L \to M, \theta(m, l) = m$, is a derivation, where (M, α_M) is a Hom-representation of $(M \rtimes L, \widetilde{\alpha})$ via π.

b) When $(M, \alpha_M) = (L, \alpha_L)$ is considered as a representation following Example 2.8 b), then a derivation consists of a $\mathbb{K}$-linear map $d : L \to L$ such that $d[l_1, l_2] = [\alpha_L(l_1), d(l_2)] + [d(l_1), \alpha_L(l_2)]$ and $d \circ \alpha_L = \alpha_L \circ d$.

Proposition 2.14 *Let (M, α_M) be a Hom-representation of a Hom-Leibniz algebra (L, α_L). For every homomorphism of Hom-Leibniz algebras $f : (X, \alpha_X) \to (L, \alpha_L)$ and every f-derivation $d : (X, \alpha_X) \to (M, \alpha_M)$ there exists a unique homomorphism of Hom-Leibniz algebras $h : (X, \alpha_X) \to (M \rtimes L, \widetilde{\alpha})$, such that the following diagram is commutative*

$$\begin{tikzcd}
 & (X, \alpha_X) \arrow[dl, "d"'] \arrow[d, "h"] \arrow[dr, "f"] & \\
(M, \alpha_M) \arrow[r, shift left, "i"] & (M \rtimes L, \widetilde{\alpha}) \arrow[l, shift left, "\theta"] \arrow[r, "\pi"] & (L, \alpha_L)
\end{tikzcd}$$

Conversely, every homomorphism of Hom-Leibniz algebras $h : (X, \alpha_X) \to (M \rtimes L, \widetilde{\alpha})$, determines a homomorphism of Hom-Leibniz algebras $f = \pi \circ h : (X, \alpha_X) \to (L, \alpha_L)$ and any f-derivation $d = \theta \circ h : (X, \alpha_X) \to (M, \alpha_M)$.

Proof. The homomorphism $h : X \to M \rtimes L, h(x) = (d(x), f(x))$ satisfies all the conditions. $\square$

Corollary 2.15 *The set of all derivations from (L, α_L) to (M, α_M) is in one-to-one correspondence with the set of Hom-Leibniz algebra homomorphisms $h : (L, \alpha_L) \to (M \rtimes L, \widetilde{\alpha})$ such that $\pi \circ h = Id_L$.*

3 Functorial properties

In this section we analyze functorial properties of the universal (α)-central extensions of (α)-perfect Hom-Leibniz algebras. For detailed motivation, constructions and characterizations we refer to [7].

Definition 3.1 *A short exact sequence of Hom-Leibniz algebras* $(K) : 0 \to (M, \alpha_M) \xrightarrow{i} (K, \alpha_K) \xrightarrow{\pi} (L, \alpha_L) \to 0$ *is said to be central if* $[M, K] = 0 = [K, M]$. *Equivalently,* $M \subseteq Z(K)$.

We say that (K) *is* α*-central if* $[\alpha_M(M), K] = 0 = [K, \alpha_M(M)]$. *Equivalently,* $\alpha_M(M) \subseteq Z(K)$.

A central extension $(K) : 0 \to (M, \alpha_M) \xrightarrow{i} (K, \alpha_K) \xrightarrow{\pi} (L, \alpha_L) \to 0$ *is said to be universal if for every central extension* $(K') : 0 \to (M', \alpha_{M'}) \xrightarrow{i'} (K', \alpha_{K'}) \xrightarrow{\pi'} (L, \alpha_L) \to 0$ *there exists a unique homomorphism of Hom-Leibniz algebras* $h : (K, \alpha_K) \to (K', \alpha_{K'})$ *such that* $\pi' \circ h = \pi$.

We say that the central extension $(K) : 0 \to (M, \alpha_M) \xrightarrow{i} (K, \alpha_K) \xrightarrow{\pi} (L, \alpha_L) \to 0$ *is universal* α*-central if for every* α*-central extension* $(K) : 0 \to (M', \alpha_{M'}) \xrightarrow{i'} (K', \alpha_{K'}) \xrightarrow{\pi'} (L, \alpha_L) \to 0$ *there exists a unique homomorphism of Hom-Leibniz algebras* $h : (K, \alpha_K) \to (K', \alpha_{K'})$ *such that* $\pi' \circ h = \pi$.

Remark 3.2 *Obviously, every universal* α*-central extension is a universal central extension. Note that in the case* $\alpha_M = Id_M$*, both notions coincide.*

A perfect ($L = [L, L]$) Hom-Leibniz algebra (L, α_L) admits universal central extension, which is $(\mathfrak{uce}(L), \widetilde{\alpha})$, where $\mathfrak{uce}(L) = \frac{L \otimes L}{I_L}$ and I_L is the subspace of $L \otimes L$ spanned by the elements of the form $-[x_1, x_2] \otimes \alpha_L(x_3) + [x_1, x_3] \otimes \alpha_L(x_2) + \alpha_L(x_1) \otimes [x_2, x_3], x_1, x_2, x_3 \in L$; every class $x_1 \otimes x_2 + I_L$ is denoted by $\{x_1, x_2\}$, for all $x_1, x_2 \in L$. $\mathfrak{uce}(L)$ is endowed with a structure of Hom-Leibniz algebra with respect to the bracket $[\{x_1, x_2\}, \{y_1, y_2\}] = \{[x_1, x_2], [y_1, y_2]\}$ and the endomorphism $\widetilde{\alpha} : \mathfrak{uce}(L) \to \mathfrak{uce}(L)$ defined by $\widetilde{\alpha}(\{x_1, x_2\}) = \{\alpha_L(x_1), \alpha_L(x_2)\}$. By construction, $u_L : (\mathfrak{uce}(L), \widetilde{\alpha}) \to (L, \alpha_L)$, given by $u_L\{x_1, x_2\} = [x_1, x_2]$, gives rise to the universal central extension $0 \to (HL_2^{\alpha}(L), \widetilde{\alpha}_{|}) \to (\mathfrak{uce}(L), \widetilde{\alpha}) \xrightarrow{u_L} (L, \alpha_L) \to 0$.

A Hom-Leibniz algebra (L, α_L) is said to be α-perfect if $L = [\alpha_L(L), \alpha_L(L)]$. Theorem 5.5 in [7] shows that a Hom-Leibniz algebra (L, α_L) is α-perfect if and only if it admits a universal α-central extension, which is $(\mathfrak{uce}_{\alpha}^{\mathrm{Leib}}(L), \overline{\alpha})$, where $\mathfrak{uce}_{\alpha}^{\mathrm{Leib}}(L) = \frac{\alpha_L(L) \otimes \alpha_L(L)}{I_L}$ and I_L is the vector subspace spanned by the elements of the form $-[x_1, x_2] \otimes \alpha_L(x_3) + [x_1, x_3] \otimes \alpha_L(x_2) + \alpha_L(x_1) \otimes [x_2, x_3]$, for all $x_1, x_2, x_3 \in L$. We denote by $\{\alpha_L(x_1), \alpha_L(x_2)\}$ the equivalence class of $\alpha_L(x_1) \otimes \alpha_L(x_2) + I_L$. $\mathfrak{uce}_{\alpha}^{\mathrm{Leib}}(L)$ is endowed with a structure of Hom-Leibniz algebra with respect to the bracket $[\{\alpha_L(x_1), \alpha_L(x_2)\}, \{\alpha_L(y_1), \alpha_L(y_2)\}] = \{[\alpha_L(x_1), \alpha_L(x_2)], [\alpha_L(y_1), \alpha_L(y_2)]\}$ and the endomorphism $\overline{\alpha} : \mathfrak{uce}_{\alpha}^{\mathrm{Leib}}(L) \to \mathfrak{uce}_{\alpha}^{\mathrm{Leib}}(L)$ defined by $\overline{\alpha}(\{\alpha_L(x_1), \alpha_L(x_2)\})$

$= \{\alpha_L^2(x_1), \alpha_L^2(x_2)\}$. The homomorphism of Hom-Leibniz algebras $U_\alpha : \mathfrak{uce}_\alpha^{\rm Leib}(L) \to L$ given by $U_\alpha(\{\alpha_L(x_1), \alpha_L(x_2)\}) = [\alpha_L(x_1), \alpha_L(x_2)]$ gives rise to the universal α-central extension $0 \to (Ker(U_\alpha), \overline{\alpha}_|) \to (\mathfrak{uce}_\alpha^{\rm Leib}(L), \overline{\alpha}) \stackrel{U_\alpha}{\to} (L, \alpha_L) \to 0$. See [7] for details.

Definition 3.3 *A perfect Hom-Leibniz algebra (L, α_L) is said to be centrally closed if its universal central extension is*

$$0 \to 0 \to (L, \alpha_L) \stackrel{\sim}{\to} (L, \alpha_L) \to 0$$

i.e. $HL_2^\alpha(L) = 0$ and $(\mathfrak{uce}_{\rm Leib}(L), \widetilde{\alpha}) \cong (L, \alpha_L)$.

A Hom-Leibniz algebra (L, α_L) is said to be superperfect if $HL_1^\alpha(L) = HL_2^\alpha(L) = 0$.

Corollary 3.4 *If $0 \to (Ker(U_\alpha), \alpha_{K_|}) \to (K, \alpha_K) \stackrel{U_\alpha}{\to} (L, \alpha_L) \to 0$ is the universal $\alpha-$central extension of an α-perfect Hom-Leibniz algebra (L, α_L), then (K, α_K) it is centrally closed.*

Proof. By Corollary 4.12 *a)* in [7], $HL_1^\alpha(K) = HL_2^\alpha(K) = 0$.

$HL_1^\alpha(K) = 0$ if and only if (K, α_K) is perfect. By Theorem 4.11 *c)* in [7] it admits a universal central extension $0 \to (HL_2^\alpha(K), \widetilde{\alpha}_|) \to (\mathfrak{uce}(K), \widetilde{\alpha}) \stackrel{u}{\to} (K, \alpha_K) \to 0$. Since $HL_2^\alpha(K) = 0$, then u is an isomorphism. □

Lemma 3.5 *Let $\pi : (K, \alpha_K) \twoheadrightarrow (L, \alpha_L)$ be a central extension where (L, α_L) is a perfect Hom-Leibniz algebra. Then the following statements hold:*

a) *$K = [K, K] + Ker(\pi)$ and $\overline{\pi} : ([K, K], \alpha_{K|}) \twoheadrightarrow (L, \alpha_L)$ is an epimorphism where $([K, K], \alpha_{[K,K]})$ is a perfect Hom-Leibniz algebra.*

b) *$\pi(Z(K)) \subseteq Z(L)$ y $\alpha_L(Z(L)) \subseteq \pi(Z(K))$.*

Proof.
a) It suffices to consider the following commutative diagram:

$$\begin{tikzcd}
\left(Ker(\pi) \cap [K,K], \alpha_{Ker(\pi)\cap[K,K]}\right) \arrow[r, tail] \arrow[d, tail] & \left([K,K], \alpha_{[K,K]}\right) \arrow[r, two heads, "\overline{\pi}"] \arrow[d, tail] & \left([L,L], \alpha_{[L,L]}\right) \arrow[d, equal] \\
\left(Ker(\pi), \alpha_{K|}\right) \arrow[r, tail] \arrow[d, two heads] & (K, \alpha_K) \arrow[r, two heads, "\pi"] \arrow[d, two heads] & (L, \alpha_L) \arrow[d, two heads] \\
* \arrow[r, tail] & (K/[K,K], \overline{\alpha_K}) \arrow[r, two heads] & (L/[L,L], \overline{\alpha_L})
\end{tikzcd}$$

b) Direct checking □

Definition 3.6 *A Hom-Leibniz algebra (L, α_L) is said to be simply connected if every central extension $\tau : (F, \alpha_F) \twoheadrightarrow (L, \alpha_L)$ splits uniquely as the product of Hom-Leibniz algebras $(F, \alpha_F) = \big(Ker\,(\tau)\,, \alpha_{F|}\big) \times (L, \alpha_L)$.*

Proposition 3.7 *For a perfect Hom-Leibniz algebra (L, α_L), the following statements are equivalent:*

a) (L, α_L) is simply connected.

b) (L, α_L) is centrally closed.

If $u : (L, \alpha_L) \twoheadrightarrow (M, \alpha_M)$ is a central extension, then:

c) Statement a) (respectively, statement b)) implies that $u : (L, \alpha_L) \twoheadrightarrow (M, \alpha_M)$ is a universal central extension.

d) If in addition $u : (L, \alpha_L) \twoheadrightarrow (M, \alpha_M)$ is a universal α-central extension, then statements a) and b) hold.

Proof. a) $\Rightarrow$ b) Let $0 \to (Ker\,(u_\alpha) = HL_2^\alpha\,(L)\,, \widetilde{\alpha}) \to (\mathfrak{uce}_\alpha\,(L)\,, \widetilde{\alpha}) \stackrel{u_\alpha}{\to} (L, \alpha_L) \to 0$ be the universal central extension of (L, α_L), then it is split. Consequently there exists an isomorphism $\mathfrak{uce}_\alpha\,(L) \cong L$ and $H_2^\alpha\,(L) = 0$.

b) $\Rightarrow$ a) The universal central extension of (L, α_L) is $0 \to 0 \to (L, \alpha_L) \stackrel{\sim}{\to} (L, \alpha_L) \to 0$. Consequently every central extension splits uniquely thanks to the universal property.

c) Let $u : (L, \alpha_L) \twoheadrightarrow (M, \alpha_M)$ be a central extension. By Theorem 4.11 *b)* in [7], it is universal if (L, α_L) is perfect and every central extension of (L, α_L) splits.

(L, α_L) is perfect by hypothesis and by statement *a)*, it is simply connected, which means that every central extension splits.

d) If $u : (L, \alpha_L) \twoheadrightarrow (M, \alpha_M)$ is a universal $\alpha-$central extension, then by Theorem 4.1. *a)* in [7] every central extension (L, α_L) splits. Consequently (L, α_L) is simply connected, equivalently, it is centrally closed. □

Now we are going to study functorial properties of the universal central extensions.

Consider a homomorphism of perfect Hom-Leibniz algebras $f : (L', \alpha_{L'}) \to (L, \alpha_L)$. This homomorphism induces a $\mathbb{K}$-linear map $f \otimes f : L' \otimes L' \to L \otimes L$ given by $(f \otimes f)\,(x_1 \otimes x_2) = f\,(x_1) \otimes f\,(x_2)$, that maps the submodule $I_{L'}$ to the submodule I_L, hence $f \otimes f$ induces a $\mathbb{K}$-linear map $\mathfrak{uce}(f) : \mathfrak{uce}(L') \to \mathfrak{uce}(L)$, given by $\mathfrak{uce}(f)\,\{x_1, x_2\} = \{f(x_1), f(x_2)\}$, which is a homomorphism of Hom-Leibniz algebras as well.

Moreover, the following diagram is commutative:

$$\begin{tikzcd}
HL_2^\alpha(L') \arrow[d] & HL_2^\alpha(L) \arrow[d] \\
(\mathfrak{uce}(L'), \widetilde{\alpha}') \arrow[r, "\mathfrak{uce}(f)"] \arrow[d, two heads, "u_{L'}"'] & (\mathfrak{uce}(L), \widetilde{\alpha}) \arrow[d, two heads, "u_L"] \\
(L', \alpha_{L'}) \arrow[r, "f"] & (L, \alpha_L)
\end{tikzcd} \tag{4}$$

From diagram (4), the existence of a covariant right exact functor $\mathfrak{uce}$: $\mathsf{Hom}-\mathsf{Leib}^{\mathrm{perf}} \to \mathsf{Hom}-\mathsf{Leib}^{\mathrm{perf}}$ between the category of perfect Hom-Leibniz algebras is derived. Consequently, an automorphism f of (L, α_L) gives rise to an automorphism $\mathfrak{uce}(f)$ of $(\mathfrak{uce}(L), \widetilde{\alpha})$. Commutativity of diagram (4) implies that $\mathfrak{uce}(f)$ leaves $HL_2^\alpha(L)$ invariant. So the Hom-group homomorphism (see [5, section 5])

$$\begin{array}{rcl} \mathrm{Aut}(L, \alpha_L) & \to & \{g \in \mathrm{Aut}(\mathfrak{uce}(L), \widetilde{\alpha}) : f(HL_2^\alpha(L)) = HL_2^\alpha(L)\} \\ f & \mapsto & \mathfrak{uce}(f) \end{array}$$

is obtained.

By means of similar considerations as the previous ones, an analogous analysis with respect to the functorial properties of α-perfect Hom-Leibniz algebras can be done. Namely, consider a homomorphism of $\alpha-$perfect Hom-Leibniz algebras $f : (L', \alpha_{L'}) \to (L, \alpha_L)$. Let I_L the vector subspace of $\alpha_L(L) \otimes \alpha_L(L)$ spanned by the elements of the form $-[x_1, x_2] \otimes \alpha_L(x_3) + [x_1, x_3] \otimes \alpha_L(x_2) + \alpha_L(x_1) \otimes [x_2, x_3], x_1, x_2, x_2 \in L$, respectively $I_{L'}$. f induces a $\mathbb{K}$-linear map $f \otimes f$: $(\alpha_{L'}(L') \otimes \alpha_{L'}(L'), \alpha_{L' \otimes L'}) \to (\alpha_L(L) \otimes \alpha({}_L L), \alpha_{L \otimes L})$, given by $(f \otimes f)(\alpha_{L'}(x'_1) \otimes \alpha_{L'}(x'_2)) = \alpha_L(f(x'_1)) \otimes \alpha_L(f(x'_2))$ such that $(f \otimes f)(I_{L'}) \subseteq I_L$. Consequently, it induces a homomorphism of Hom-Leibniz algebras $\mathfrak{uce}_\alpha(f) : (\mathfrak{uce}_{\alpha'}(L'), \overline{\alpha'}) \to (\mathfrak{uce}_\alpha(L), \overline{\alpha})$ given by $\mathfrak{uce}_\alpha(f)\{\alpha_{L'}(x'_1), \alpha_{L'}(x'_2)\} = \{\alpha_L(f(x'_1)), \alpha_L(f(x'_2))\}$ such that the following diagram is commutative

$$\begin{tikzcd}
Ker(U_{\alpha'}) \arrow[d] & Ker(U_\alpha) \arrow[d] \\
(\mathfrak{uce}_\alpha(L'), \overline{\alpha'}) \arrow[r, "\mathfrak{uce}_\alpha(f)"] \arrow[d, two heads, "U_{\alpha'}"'] & (\mathfrak{uce}_\alpha(L), \overline{\alpha}) \arrow[d, two heads, "U_\alpha"] \\
(L', \alpha_{L'}) \arrow[r, "f"] & (L, \alpha_L)
\end{tikzcd} \tag{5}$$

From diagram (5) one derives the existence of a covariant right exact functor $\mathfrak{uce}_\alpha : \mathsf{Hom}-\mathsf{Leib}^{\alpha-\mathrm{perf}} \to \mathsf{Hom}-\mathsf{Leib}^{\alpha-\mathrm{perf}}$ between the α-perfect Hom-Leibniz algebras category. Consequently, an automorphism f of (L,α_L) gives rise to an automorphism $\mathfrak{uce}_\alpha(f)$ of $(\mathfrak{uce}_\alpha(L),\overline{\alpha})$. Commutativity of diagram (5) implies that $\mathfrak{uce}_\alpha(f)$ leaves $Ker(U_\alpha)$ invariant. So the homomorphism of Hom-groups

$$\begin{array}{rcl} \mathrm{Aut}(L,\alpha_L) & \to & \{g\in \mathrm{Aut}(\mathfrak{uce}_\alpha(L),\overline{\alpha}) : f(Ker(U_\alpha) = Ker(U_\alpha)\} \\ f & \mapsto & \mathfrak{uce}_\alpha(f) \end{array}$$

is obtained.

Now we consider a derivation d of the α-perfect Hom-Leibniz algebra (L,α_L). The linear map $\varphi:\alpha_L(L)\otimes\alpha_L(L)\to\alpha_L(L)\otimes\alpha_L(L)$ given by $\varphi\left(\alpha_L(x_1)\otimes\alpha_L(x_2)\right)=d(\alpha_L(x_1))\otimes\alpha_L^2\left(x_2\right)+\alpha_L^2\left(x_1\right)\otimes d\left(\alpha_L(x_2)\right)$, leaves invariant the vector subspace I_L of $\alpha_L(L)\otimes\alpha_L(L)$ spanned by the elements of the form $-[x_1,x_2]\otimes\alpha_L(x_3)+[x_1,x_3]\otimes\alpha_L(x_2)+\alpha_L(x_1)\otimes[x_2,x_3], x_1,x_2,x_2\in L$. Hence it induces a linear map $\mathfrak{uce}_\alpha(d):\left(\mathfrak{uce}_\alpha\left(L\right),\overline{\alpha}\right)\to\left(\mathfrak{uce}_\alpha\left(L\right),\overline{\alpha}\right)$, given by $\mathfrak{uce}_\alpha(d)(\{\alpha_L(x_1),\alpha_L(x_2)\})=\{d(\alpha_L(x_1)),\alpha_L^2\left(x_2\right)\}+\{\alpha_L^2\left(x_1\right),d\left(\alpha_L(x_2)\right)\}$, that makes commutative the following diagram:

$$\begin{tikzcd} \left(\mathfrak{uce}_\alpha\left(L\right),\overline{\alpha}\right) \arrow[r, "uce_\alpha(d)"] \arrow[d, two heads, "U_\alpha"'] & \left(\mathfrak{uce}_\alpha\left(L\right),\overline{\alpha}\right) \arrow[d, two heads, "U_\alpha"] \\ (L,\alpha_L) \arrow[r, "d"] & (L,\alpha_L) \end{tikzcd} \tag{6}$$

Consequently, a derivation d of (L,α_L) gives rise to a derivation $\mathfrak{uce}_\alpha(d)$ of $(\mathfrak{uce}_\alpha(L),\overline{\alpha})$. The commutativity of diagram (6) implies that $\mathfrak{uce}_\alpha(d)$ maps $Ker(U_\alpha)$ on itself.

Hence, it is obtained the homomorphism of Hom-$\mathbb{K}$-vector spaces

$$\begin{array}{rcl} \mathfrak{uce}_\alpha:\mathrm{Der}(L,\alpha_L) & \to & \{\delta\in \mathrm{Der}(\mathfrak{uce}_\alpha(L),\overline{\alpha}) : \delta(Ker(U_\alpha)\subseteq Ker(U_\alpha)\} \\ d & \mapsto & \mathfrak{uce}_\alpha(d) \end{array}$$

whose kernel belongs to the subalgebra of derivations of (L,α_L) such that vanish on $[\alpha_L(L),\alpha_L(L)]$.

The functorial properties of $\mathfrak{uce}_\alpha(-)$ relative to the derivations are described by the following result.

Lemma 3.8 *Let $f:(L',\alpha_{L'})\to(L,\alpha_L)$ be a homomorphism of α-perfect Hom-Leibniz algebras. Consider $d\in Der(L)$ and $d'\in Der(L')$ such that $f\circ d'=d\circ f$, then $\mathfrak{uce}_\alpha(f)\circ\mathfrak{uce}_\alpha(d')=\mathfrak{uce}_\alpha(d)\circ\mathfrak{uce}_\alpha(f)$.*

Proof. Routine checking. □

4 Lifting automorphisms and derivations

In this section we analyze under what conditions an automorphism or a derivation can be lifted to an α-cover. We restrict the study to α-covers since we must compose central extensions in the constructions. This fact does not allow to obtain more general results, mainly due to Lemma 4.10 in [7].

Definition 4.1 *A central extension of Hom-Leibniz algebras* $f : (L', \alpha_{L'}) \twoheadrightarrow (L, \alpha_L)$*, where* $(L', \alpha_{L'})$ *is an* α*-perfect Hom-Leibniz algebra, is said to be an* α*-cover.*

Lemma 4.2 *If* $f : (L', \alpha_{L'}) \to (L, \alpha_L)$ *is a surjective homomorphism of Hom-Leibniz algebras and* $(L', \alpha_{L'})$ *is* $\alpha-$*perfect, then* (L, α_L) *is* α*-perfect as well.*

Let $f : (L', \alpha_{L'}) \twoheadrightarrow (L, \alpha_L)$ be an α-cover. Thanks to Lemma 4.2 (L, α_L) is an α-perfect Hom-Leibniz algebra as well. By Theorem 5.5 in [7], everyone admits universal α-central extension. Having in mind the functorial properties given in diagram (5), we can construct the following diagram:

$$\begin{tikzcd}
Ker(U_{\alpha'}) \arrow[d] & Ker(U_{\alpha}) \arrow[d] \\
(\mathfrak{uce}_{\alpha}(L'), \overline{\alpha'}) \arrow[r, "\mathfrak{uce}_{\alpha}(f)"] \arrow[d, "U_{\alpha'}"'] & (\mathfrak{uce}_{\alpha}(L), \overline{\alpha}) \arrow[d, "U_{\alpha}"] \\
(L', \alpha_{L'}) \arrow[r, "f"] & (L, \alpha_L)
\end{tikzcd}$$

Since $U_{\alpha'} : \left(\mathfrak{uce}_{\alpha'}(L'), \overline{\alpha'}\right) \twoheadrightarrow (L', \alpha_{L'})$ is a universal α-central extension, then by Remark 3.2, it is a universal central extension as well. Since $f : (L', \alpha_{L'}) \twoheadrightarrow (L, \alpha_L)$ is a central extension and $U_{\alpha'} : \left(\mathfrak{uce}_{\alpha'}(L'), \overline{\alpha'}\right) \twoheadrightarrow (L', \alpha_{L'})$ is a universal central extension, then by Proposition 4.15 in [7] the extension $f \circ U_{\alpha'} : (\mathfrak{uce}_{\alpha'}(L'), \widetilde{\alpha}') \to (L, \alpha_L)$ is α-central which is universal in the sense of Definition 4.13 in [7].

On the other hand, since $U_{\alpha} : (\mathfrak{uce}_{\alpha}(L), \overline{\alpha}) \twoheadrightarrow (L, \alpha_L)$ is a universal α-central extension, then there exists a unique homomorphism $\varphi : (\mathfrak{uce}_{\alpha}(L), \overline{\alpha}) \twoheadrightarrow \left(\mathfrak{uce}_{\alpha'}(L'), \overline{\alpha'}\right)$ such that $f \circ U_{\alpha'} \circ \varphi = U_{\alpha}$.

Moreover $\varphi \circ \mathfrak{uce}_{\alpha}(f) = Id$ since the following diagram is commutative

$$\begin{tikzcd}
0 \arrow[r] & \left(Ker(f \circ U_{\alpha'}), \overline{\alpha'_{|}}\right) \arrow[r] & \left(\mathfrak{uce}_{\alpha'}(L'), \overline{\alpha'}\right) \arrow[r, "f \circ U_{\alpha'}"] \arrow[d, shift right, "\varphi \circ \mathfrak{uce}_{\alpha}(f)"'] \arrow[d, shift left, "Id"] & (L, \alpha_L) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \left(Ker(f \circ U_{\alpha'}), \overline{\alpha'_{|}}\right) \arrow[r] & \left(\mathfrak{uce}_{\alpha'}(L'), \overline{\alpha'}\right) \arrow[r, "f \circ U_{\alpha'}"] & (L, \alpha_L) \arrow[r] & 0
\end{tikzcd}$$

and $f \circ U_{\alpha'}$ is an α-central extension which is universal in the sense of Definition 4.13 in [7].

Conversely, $\mathfrak{uce}_\alpha(f) \circ \varphi = Id$ since the following diagram is commutative

$$\begin{tikzcd}
0 \arrow[r] & \left(Ker(U_\alpha), \overline{\alpha_|}\right) \arrow[r] & (\mathfrak{uce}_\alpha(L), \overline{\alpha}) \arrow[r, "U_\alpha"] \arrow[d, shift right, "\mathfrak{uce}_\alpha(f)\circ\varphi"'] \arrow[d, shift left, "Id"] & (L, \alpha_L) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \left(Ker(U_\alpha), \overline{\alpha_|}\right) \arrow[r] & (\mathfrak{uce}_\alpha(L), \overline{\alpha}) \arrow[r, "U_\alpha"] & (L, \alpha_L) \arrow[r] & 0
\end{tikzcd}$$

whose horizontal rows are central extensions and $(\mathfrak{uce}_\alpha(L), \overline{\alpha})$ is α-perfect, then Lemma 5.4 in [7] guarantees the uniqueness of the vertical homomorphism.

Consequently $\mathfrak{uce}_\alpha(f)$ is an isomorphism and from now on we will use the notation $\mathfrak{uce}_\alpha(f)^{-1}$ instead of φ.

On the other hand, $U_{\alpha'} \circ \mathfrak{uce}_\alpha(f)^{-1} : (\mathfrak{uce}_\alpha(L), \overline{\alpha}) \twoheadrightarrow (L', \alpha_{L'})$ is an $\alpha-$cover. In the sequel, we will denote its kernel by

$$C := Ker(U_{\alpha'} \circ \mathfrak{uce}_\alpha(f)^{-1}) = \mathfrak{uce}_\alpha(f)\left(Ker\left(U_{\alpha'}\right)\right).$$

Theorem 4.3 *Let $f : (L', \alpha_{L'}) \twoheadrightarrow (L, \alpha_L)$ be an $\alpha-$cover.*

For any $h \in Aut(L, \alpha_L)$, there exists a unique $\theta_h \in Aut(L', \alpha_{L'})$ such that the following diagram is commutative:

$$\begin{tikzcd}
(L', \alpha_{L'}) \arrow[r, two heads, "f"] \arrow[d, "\theta_h"'] & (L, \alpha_L) \arrow[d, "h"] \\
(L', \alpha_{L'}) \arrow[r, two heads, "f"] & (L, \alpha_L)
\end{tikzcd} \tag{7}$$

if and only if the automorphism $\mathfrak{uce}_\alpha(h)$ of $(\mathfrak{uce}_\alpha(L), \overline{\alpha})$ satisfies $\mathfrak{uce}_\alpha(h)(C) = C$. In this case, it is uniquely determined by diagram (7) and $\theta_h(Ker(f)) = Ker(f)$.

Moreover, the map

$$\begin{array}{rcl}
\Theta : \{h \in Aut(L, \alpha_L) : \mathfrak{uce}_\alpha(h)(C) = C\} & \to & \{g \in Aut(L', \alpha_{L'}) : g(Ker(f)) = Ker(f)\} \\
h & \mapsto & \theta_h
\end{array}$$

is a group isomorphism.

Proof. Let $h \in Aut(L, \alpha_L)$ and assume that there exists a $\theta_h \in Aut(L', \alpha_{L'})$ such that diagram (7) is commutative.

Then $h \circ f : (L', \alpha_{L'}) \to (L, \alpha_L)$ is an $\alpha-$cover, hence θ_h is a homomorphism from the α-cover $h \circ f$ to the α-cover f which is unique by Remark 5.3 *b)* and Lemma 4.7 in [7].

By application of the functor $\mathfrak{uce}_\alpha(-)$ to diagram (7), one obtains the following commutative diagram:

$$\begin{tikzcd}
\left(\mathfrak{uce}_\alpha(L'), \overline{\alpha_{L'}}\right) \arrow[r, two heads, "\mathfrak{uce}_\alpha(f)"] \arrow[d, tail, two heads, "\mathfrak{uce}_\alpha(\theta_h)"'] & (\mathfrak{uce}_\alpha(L), \overline{\alpha_L}) \arrow[d, tail, two heads, "\mathfrak{uce}_\alpha(h)"] \\
(\mathfrak{uce}_\alpha(L'), \overline{\alpha_{L'}}) \arrow[r, two heads, "\mathfrak{uce}_\alpha(f)"] & (\mathfrak{uce}_\alpha(L), \overline{\alpha_L})
\end{tikzcd}$$

Hence $\mathfrak{uce}_\alpha(h)\,(C) = \mathfrak{uce}_\alpha(h) \circ \mathfrak{uce}_\alpha(f)\,(Ker\,(U_{\alpha'})) = \mathfrak{uce}_\alpha(f) \circ \mathfrak{uce}_\alpha(\theta_h)\,(Ker\,(U_{\alpha'})) = \mathfrak{uce}_\alpha(f)\,(Ker\,(U_{\alpha'})) = C$.

Conversely, from diagram (5), we have that $U_\alpha = f \circ U_{\alpha'} \circ \mathfrak{uce}_\alpha(f)^{-1}$, hence we obtain the following diagram:

$$\begin{tikzcd}
C \arrow[r, hook] \arrow[d] & (\mathfrak{uce}_\alpha(L), \overline{\alpha}) \arrow[r, two heads, "U_{\alpha'} \circ \mathfrak{uce}_\alpha(f)^{-1}"] \arrow[d, "\mathfrak{uce}_\alpha(h)"] & (L', \alpha_{L'}) \arrow[r, two heads, "f"] \arrow[d, dashed, "\theta_h"] & (L, \alpha_L) \arrow[d, "h"] \\
C \arrow[r, hook] & (\mathfrak{uce}_\alpha(L), \overline{\alpha}) \arrow[r, two heads, "U_{\alpha'} \circ \mathfrak{uce}_\alpha(f)^{-1}"] & (L', \alpha_{L'}) \arrow[r, two heads, "f"] & (L, \alpha_L)
\end{tikzcd}$$

If $\mathfrak{uce}_\alpha(h)\,(C) = C$, then $U_{\alpha'} \circ \mathfrak{uce}_\alpha(f)^{-1} \circ \mathfrak{uce}_\alpha(h)\,(C) = U_{\alpha'} \circ \mathfrak{uce}_\alpha(f)^{-1}(C) = 0$, then there exists a unique $\theta_h : (L', \alpha_{L'}) \to (L', \alpha_{L'})$ such that $\theta_h \circ U_{\alpha'} \circ \mathfrak{uce}_\alpha(f)^{-1} = U_{\alpha'} \circ \mathfrak{uce}_\alpha(f)^{-1} \circ \mathfrak{uce}_\alpha(h)$.

On the other hand, $h \circ f \circ U_{\alpha'} \circ \mathfrak{uce}_\alpha(f)^{-1} = f \circ U_{\alpha'} \circ \mathfrak{uce}_\alpha(f)^{-1} \circ \mathfrak{uce}_\alpha(h) = f \circ U_{\alpha'} \circ \mathfrak{uce}_\alpha(\theta_h) \circ \mathfrak{uce}_\alpha(f)^{-1} = f \circ \theta_h \circ U_{\alpha'} \circ \mathfrak{uce}_\alpha(f)^{-1}$, then $h \circ f = f \circ \theta_h$.

In conclusion, θ_h is uniquely determined by diagram (7) and moreover $\theta_h(Ker(f)) = Ker(f)$.

By the previous arguments, it is easy to check that Θ is a well-defined map, it is a monomorphism thanks to the uniqueness of θ_h and it is an epimorphism, since every $g \in Aut\,(L', \alpha_{L'})$ with $g\,(Ker(f)) = Ker(f)$, induces a unique homomorphism $h : (L, \alpha_L) \to (L, \alpha_L)$ such that $h \circ f = f \circ g$. Then $g = \theta_h$ and $\mathfrak{uce}_\alpha\,(h)\,(C) = C$. □

Corollary 4.4 *If (L, α_L) is an α-perfect Hom-Leibniz algebra, then the map*

$$\begin{array}{rcl} Aut(L, \alpha_L) & \to & \{g \in Aut(\mathfrak{uce}_\alpha(L), \overline{\alpha}) : g(Ker(U_\alpha)) = Ker(U_\alpha)\} \\ h & \mapsto & \mathfrak{uce}_\alpha(h) \end{array}$$

is a group isomorphism.

Proof. By application of Theorem 4.3 to the $\alpha-$cover $U_\alpha : (\mathfrak{uce}_\alpha\,(L)\,, \overline{\alpha}) \twoheadrightarrow (L, \alpha_L)$, it is enough to have in mind that under these conditions $C = 0$ and $\mathfrak{uce}_\alpha(f)(0) = 0$. □

Now we analyze under what conditions a derivation of an α-perfect Hom-Leibniz algebra can be lifted to an α-cover.

Theorem 4.5 *Let $f : (L', \alpha_{L'}) \twoheadrightarrow (L, \alpha_L)$ be an $\alpha-$cover. Denote by $C = \mathfrak{uce}_\alpha(f)\,(Ker\,(U_{\alpha'})) \subseteq Ker(U_\alpha)$. Then the following statements hold:*

a) For any $d \in Der\,(L, \alpha_L)$ there exists a $\delta_d \in Der\,(L', \alpha_{L'})$ such that the following diagram is commutative

$$\begin{tikzcd}
(L', \alpha_{L'}) \arrow[r, two heads, "f"] \arrow[d, "\delta_d"'] & (L, \alpha_L) \arrow[d, "d"] \\
(L', \alpha_{L'}) \arrow[r, two heads, "f"] & (L, \alpha_L)
\end{tikzcd} \tag{8}$$

if and only if the derivation $\mathfrak{uce}_\alpha(d)$ *of* $(\mathfrak{uce}_\alpha(L), \overline{\alpha_L})$ *satisfies* $\mathfrak{uce}_\alpha(d)(C) \subseteq C$.

In this case, δ_d *is uniquely determined by (8) and* $\delta_d(Ker(f)) \subseteq Ker(f)$.

b) The map

$$\begin{array}{rcl} \Delta : \{d \in Der(L, \alpha_L) : \mathfrak{uce}_\alpha(d)(C) \subseteq C\} & \to & \{\rho \in Der(L', \alpha_{L'}) : \rho(Ker(f)) \subseteq Ker(f)\} \\ d & \mapsto & \delta_d \end{array}$$

is an isomorphism of Hom-vector spaces (see [19, 2.2]).

c) For the α*-cover* $U_\alpha : (\mathfrak{uce}_\alpha(L), \overline{\alpha_L}) \twoheadrightarrow (L, \alpha_L)$*, the map*

$$\mathfrak{uce}_\alpha : Der(L, \alpha_L) \to \{\delta \in Der(\mathfrak{uce}_\alpha(L), \overline{\alpha_L}) : \delta(Ker(U_\alpha)) \subseteq Ker(U_\alpha)\}$$

is an isomorphism of Hom-vector spaces.

Proof. a) Let $d \in Der(L, \alpha_L)$ and assume the existence of a $\delta_d \in Der(L', \alpha_{L'})$ such that diagram(8) is commutative. Then, by Lemma 3.8, we obtain the following commutative diagram:

$$\begin{tikzcd} (\mathfrak{uce}_\alpha(L'), \overline{\alpha_{L'}}) \arrow[r, "\mathfrak{uce}_\alpha(f)"] \arrow[d, "\mathfrak{uce}_\alpha(\delta_d)"'] & (\mathfrak{uce}_\alpha(L), \overline{\alpha_L}) \arrow[d, "\mathfrak{uce}_\alpha(d)"] \\ (\mathfrak{uce}_\alpha(L'), \overline{\alpha_{L'}}) \arrow[r, "\mathfrak{uce}_\alpha(f)"] & (\mathfrak{uce}_\alpha(L), \overline{\alpha_L}) \end{tikzcd}$$

Hence, having in mind the properties derived from diagram (6), we obtain: $\mathfrak{uce}_\alpha(d)(C) = \mathfrak{uce}_\alpha(d) \circ \mathfrak{uce}_\alpha(f)(Ker(U_{\alpha'})) = \mathfrak{uce}_\alpha(f) \circ \mathfrak{uce}_\alpha(\delta_d)(Ker(U_{\alpha'})) \subseteq \mathfrak{uce}_\alpha(f)(Ker(U_{\alpha'})) = C$.

Conversely, from diagram (5) we have that $U_\alpha = f \circ U_{\alpha'} \circ \mathfrak{uce}_\alpha(f)^{-1}$ and consider the following diagram:

$$\begin{tikzcd} C \arrow[r, hook] \arrow[d] & (\mathfrak{uce}_\alpha(L), \overline{\alpha_L}) \arrow[r, two heads, "U_{\alpha'} \circ \mathfrak{uce}_\alpha(f)^{-1}"] \arrow[d, "\mathfrak{uce}_\alpha(d)"] & (L', \alpha_{L'}) \arrow[r, two heads, "f"] \arrow[d, dashed, "\delta_d"] & (L, \alpha_L) \arrow[d, "d"] \\ C \arrow[r, hook] & (\mathfrak{uce}_\alpha(L), \overline{\alpha_L}) \arrow[r, two heads, "U_{\alpha'} \circ \mathfrak{uce}_\alpha(f)^{-1}"] & (L', \alpha_{L'}) \arrow[r, two heads, "f"] & (L, \alpha_L) \end{tikzcd}$$

Since $\mathfrak{uce}_\alpha(d)(C) \subseteq C$, then $U_{\alpha'} \circ \mathfrak{uce}_\alpha(f)^{-1} \circ \mathfrak{uce}_\alpha(d)(C) \subseteq U_{\alpha'} \circ \mathfrak{uce}_\alpha(f)^{-1}(C) = U_{\alpha'}(Ker(U_{\alpha'})) = 0$. Hence there exists a unique $\mathbb{K}$-linear map $\delta_d : (L', \alpha_{L'}) \to (L', \alpha_{L'})$ such that $\delta_d \circ U_{\alpha'} \circ \mathfrak{uce}_\alpha(f)^{-1} = U_{\alpha'} \circ \mathfrak{uce}_\alpha(f)^{-1} \circ \mathfrak{uce}_\alpha(d)$.

On the other hand $d \circ f \circ U_{\alpha'} \circ \mathfrak{uce}_\alpha(f)^{-1} = d \circ U_\alpha \circ \mathfrak{uce}_\alpha(f) \circ \mathfrak{uce}_\alpha(f)^{-1} = U_\alpha \circ \mathfrak{uce}_\alpha(d) = f \circ U_{\alpha'} \circ \mathfrak{uce}_\alpha(f)^{-1} \circ \mathfrak{uce}_\alpha(d)$, since $\delta_d \circ U_{\alpha'} \circ \mathfrak{uce}_\alpha(f)^{-1} = U_{\alpha'} \circ \mathfrak{uce}_\alpha(f)^{-1} \circ \mathfrak{uce}_\alpha(d)$, then $d \circ f = f \circ \delta_d$.

Finally, a direct checking shows that δ_d is a derivation of L', which is uniquely determined by diagram (8) and $\delta_d(Ker(f)) \subseteq Ker(f)$.

b) The map Δ is a homomorphism of Hom-vector spaces by construction, which is injective by the uniqueness of δ_d, and surjective, since for every $\rho \in Der\,(L', \alpha_{L'})$ such that $\rho\,(Ker(f)) \subseteq Ker(f)$ there exists the following diagram commutative:

$$\begin{tikzcd}
Ker(f) \arrow[r, tail] \arrow[d] & (L', \alpha_{L'}) \arrow[r, two heads, "f"] \arrow[d, "\rho"] & (L, \alpha_L) \arrow[d, dashed, "d"] \\
Ker(f) \arrow[r, tail] & (L', \alpha_{L'}) \arrow[r, two heads, "f"] & (L, \alpha_L)
\end{tikzcd}$$

where $d : (L, \alpha_L) \to (L, \alpha_L)$ is a derivation satisfying $\mathfrak{uce}_\alpha\,(d)\,(C) = \mathfrak{uce}_\alpha\,(d) \circ \mathfrak{uce}_\alpha(f)\,(Ker(U_{\alpha'})) = \mathfrak{uce}_\alpha\,(f) \circ \mathfrak{uce}_\alpha(\rho)\,(Ker(U_{\alpha'})) \subseteq \mathfrak{uce}_\alpha(f)\,(Ker(U_{\alpha'})) = C$.

Finally, the uniqueness of δ_d implies that $\Delta\,(d) = \delta_d = \rho$.

c) It is enough to write the statement *b)* for the $\alpha-$cover $U_\alpha : (\mathfrak{uce}_\alpha(L), \overline{\alpha_L}) \twoheadrightarrow (L, \alpha_L)$. Now $C = \mathfrak{uce}_\alpha(U_\alpha)\,(Ker\,(U_\alpha)) = 0$, and Δ is the map $\mathfrak{uce}_\alpha$ derived from diagram (6). $\square$

5 Universal α-central extension of the semi-direct product

Consider a split extension of α-perfect Hom-Leibniz algebras

$$\begin{tikzcd}
0 \arrow[r] & (M, \alpha_M) \arrow[r, "t"] & (G, \alpha_G) \arrow[r, shift left, "p"] & (Q, Id_Q) \arrow[l, shift left, "s"] \arrow[r] & 0
\end{tikzcd}$$

where, by Lemma 2.11, $(G, \alpha_G) \cong (M, \alpha_M) \rtimes (Q, Id_Q)$, whose Hom-action of (Q, Id_Q) on (M, α_M) is given by $q \centerdot m = [s(q), t(m)]$ and $m \centerdot q = [t(m), s(q)], q \in Q, m \in M$. Moreover we will assume, when it is needed, that the previous action is symmetric, i.e. $q \centerdot m + m \centerdot q = 0, q \in Q, m \in M$.

An example of the above situation is the direct product $(G, \alpha_G) = (M, \alpha_M) \times (Q, Id_Q) = (M \times Q, \alpha_M \times Id_Q)$, where (M, α_M) is an α-perfect Hom-Leibniz algebra and (Q, Id_Q) is a perfect Hom-Leibniz algebra.

Applying the functorial properties of $\mathfrak{uce}_\alpha(-)$ given in diagram (5) and having in mind that (Q, Id_Q) is perfect is equivalent to Q is perfect, we have the following commutative diagram:

$$\begin{tikzcd}
& Ker(U_\alpha^M) \arrow[d, tail] & Ker(U_\alpha^G) \arrow[d, tail] & HL_2(Q) \arrow[d, tail] & \\
& (\mathfrak{uce}_\alpha(M), \overline{\alpha_M}) \arrow[r, "\tau"] \arrow[d, two heads, "U_\alpha^M"] & (\mathfrak{uce}_\alpha(G), \overline{\alpha_G}) \arrow[r, shift left, "\pi"] \arrow[d, two heads, "U_\alpha^G"] & (\mathfrak{uce}(Q), Id_{\mathfrak{uce}(Q)}) \arrow[l, shift left, "\sigma"] \arrow[d, two heads, "u_Q"] & \\
0 \arrow[r] & (M, \alpha_M) \arrow[r, "t"] & (G, \alpha_G) \arrow[r, shift left, "p"] & (Q, Id_Q) \arrow[l, shift left, "s"] \arrow[r] & 0
\end{tikzcd}$$

Here $\tau=\mathfrak{uce}_\alpha(t),\pi=\mathfrak{uce}_\alpha(p),\sigma=\mathfrak{uce}_\alpha(s)$.

The sequence

$$(\mathfrak{uce}_\alpha(M),\overline{\alpha_M}) \xrightarrow{\tau} (\mathfrak{uce}_\alpha(G),\overline{\alpha_G}) \underset{\sigma}{\overset{\pi}{\rightleftarrows}} (\mathfrak{uce}(Q),Id_{\mathfrak{uce}(Q)})$$

is split, since $p\circ s=Id_Q$, then $\mathfrak{uce}_\alpha(p)\circ\mathfrak{uce}_\alpha(s)=\mathfrak{uce}_\alpha(Id_Q)$, i.e. $\pi\circ\sigma=Id_{\mathfrak{uce}(Q)}$. Obviously π is surjective and there exists a Hom-action of $(\mathfrak{uce}(Q),Id_{\mathfrak{uce}(Q)})$ on $(Ker(\pi),\overline{\alpha_{G|}})$ induced by the section σ, which is given by:

$\lambda:\mathfrak{uce}(Q)\otimes Ker(\pi)\to Ker(\pi)$,

$\lambda\left(\{q_1,q_2\}\otimes\{\alpha_G(g_1),\alpha_G(g_2)\}\right)=\{q_1,q_2\}\centerdot\{\alpha_G(g_1),\alpha_G(g_2)\}=$
$[\sigma\{q_1,q_2\},i\{\alpha_G(g_1),\alpha_G(g_2)\}]=[\{s(q_1),s(q_2)\},\{\alpha_G(g_1),\alpha_G(g_2)\}]=$
$\{s[q_1,q_2],\alpha_G[g_1,g_2]\}$

$\rho:Ker(\pi)\otimes\mathfrak{uce}(Q)\to Ker(\pi)$,

$\rho\left(\{\alpha_G(g_1),\alpha_G(g_2)\}\otimes\{q_1,q_2\}\right)=\{\alpha_G(g_1),\alpha_G(g_2)\}\centerdot\{q_1,q_2\}=$
$[i\{\alpha_G(g_1),\alpha_G(g_2)\},\sigma\{q_1,q_2\}]=[\{\alpha_G(g_1),\alpha_G(g_2)\},\{s(q_1),s(q_2)\}]=$
$\{\alpha_G[g_1,g_2],s[q_1,q_2]\}$

By Lemma 2.11, the split exact sequence

$$0\longrightarrow(Ker(\pi),\overline{\alpha_{G|}}) \xrightarrow{i} (\mathfrak{uce}_\alpha(G),\overline{\alpha_G}) \underset{\sigma}{\overset{\pi}{\rightleftarrows}} (\mathfrak{uce}(Q),Id_{\mathfrak{uce}(Q)})\longrightarrow 0$$

is equivalent to the semi-direct product sequence, i.e.

$$(\mathfrak{uce}_\alpha(G),\overline{\alpha_G})\cong\left(Ker(\pi),\overline{\alpha_{G|}}\right)\rtimes\left(\mathfrak{uce}_\alpha(Q),Id_{\mathfrak{uce}_\alpha(Q)}\right)$$

Let $q\in Q$ and $\alpha_M(m_1),\alpha_M(m_2)\in\alpha_M(M)$, then in $(\mathfrak{uce}_\alpha(G),\overline{\alpha_G})$ the following identities hold:

$\{\alpha_G(s(q)),[t(\alpha_M(m_1)),t(\alpha_M(m_2))]\}=\{[s(q),t(\alpha_M(m_1))],\alpha_G(t(\alpha_M(m_2)))\}$
$-\{[s(q),t(\alpha_M(m_2))],\alpha_G(t(\alpha_M(m_1)))\}$

and

$\{[t(\alpha_M(m_1)),t(\alpha_M(m_2))],\alpha_G(s(q))\}=\{\alpha_G(t(\alpha_M(m_1))),[t(\alpha_M(m_2)),s(q)]\}$
$+\{[t(\alpha_M(m_1)),s(q)],\alpha_G(t(\alpha_M(m_2)))\}$.

These equalities together with the α-perfection of (M,α_M) imply:

$\{s(Q),M\}=\{\alpha_G(s(Q)),[\alpha_M(M),\alpha_M(M)]\}\subseteq\{\alpha_M(M),\alpha_M^2(M)\}\subseteq$
$\{\alpha_M(M),\alpha_M(M)\}$

and

$\{M,s(Q)\}=\{[\alpha_M(M),\alpha_M((M)],\alpha_G(s(Q))\}\subseteq\{\alpha_M^2(M),\alpha_M(M)\}+$
$\{\alpha_M(M),\alpha_M^2(M)\}\subseteq\{\alpha_M(M),\alpha_M(M)\}$.

Moreover

$$\tau(\mathfrak{uce}_\alpha(M),\overline{\alpha_M})\equiv\left(\{\alpha_M(M),\alpha_M(M)\},\overline{\alpha_{G|}}\right)\qquad(9)$$

since we have the following identification: $\tau\{\alpha_M(m_1),\alpha_M(m_2)\}=\{t(\alpha_M(m_1)),t(\alpha_M(m_2))\}\equiv\{\alpha_M(m_1),\alpha_M(m_2)\}$, and

$$\sigma(\mathfrak{uce}(Q))=\{s(Q),s(Q)\}=\{\alpha_G(s(Q)),\alpha_G(s(Q))\}$$

since $\sigma(\{q_1,q_2\}) = \{s(q_1),s(q_2)\} = \{\alpha_G(s(q_1)),\alpha_G(s(q_2))\}$.

On the other hand, for every $\alpha_G(g) \in G$, there exists an $\alpha_M(m) \in \alpha_M(M)$ such that $\alpha_G(g) = s\left(p\left(\alpha_G(g)\right)\right) + \alpha_M(m)$. Hence

$$(\mathfrak{uce}_\alpha(G), \overline{\alpha_G}) = (\{s(Q), s(Q)\} + \{\alpha_M(M), \alpha_M(M)\}, \overline{\alpha_G}) \tag{10}$$

Proposition 5.1

$$\left(Ker(\pi), \overline{\alpha_{G|}}\right) = \left(\{\alpha_M(M), \alpha_M(M)\}, \overline{\alpha_{G|}}\right) = \tau\left(\mathfrak{uce}_\alpha(M), \overline{\alpha_M}\right).$$

Proof. Let $\{g_1,g_2\} \in Ker(\pi)$. From (10), $\{g_1,g_2\} = \{s(q_1), s(q_2)\} + \{\alpha_M(m_1), \alpha_M(m_2)\} \in \mathfrak{uce}_\alpha(G)$. Then $\overline{0} = \pi\{g_1,g_2\} = \{p(s(q_1)), p(s(q_2))\} + \{p(\alpha_M(m_1)), p(\alpha_M(m_2))\} = \{q_1,q_2\}$, i.e. $q_1 \otimes q_2 \in I_Q$. Consequently, $\sigma\{q_1,q_2\} = \{s(q_1), s(q_2)\} = 0$ since $s(q_1) \otimes s(q_2) \in \sigma(I_Q) \subseteq I_G$. So any element in the kernel has the form $\{\alpha_M(m_1), \alpha_M(m_2)\}$. The reverse inclusion is obvious.

Second equality was proved in (9). □

On the other hand $\sigma\left(\mathfrak{uce}(Q), Id_{\mathfrak{uce}(Q)}\right) = (\{s(Q), s(Q)\}, \overline{\alpha_G})$.

Since $\pi \circ \sigma = Id_{\mathfrak{uce}(Q)}$, then $(\mathfrak{uce}_\alpha(G), \overline{\alpha_G}) = \left(Ker(\pi), \overline{\alpha_{G|}}\right) \rtimes \sigma\left(\mathfrak{uce}(Q), Id_{\mathfrak{uce}(Q)}\right)$. Moreover σ is an isomorphism between $\left(\mathfrak{uce}(Q), Id_{\mathfrak{uce}(Q)}\right)$ and $\sigma\left(\mathfrak{uce}(Q), Id_{\mathfrak{uce}(Q)}\right)$.

These facts imply:

1. $(\mathfrak{uce}_\alpha(G), \overline{\alpha_G}) = \tau\left(\mathfrak{uce}_\alpha(M), \overline{\alpha_M}\right) \rtimes \sigma\left(\mathfrak{uce}(Q), Id_{\mathfrak{uce}(Q)}\right)$.

2. $\sigma\left(\mathfrak{uce}(Q), Id_{\mathfrak{uce}(Q)}\right) \cong \left(\mathfrak{uce}(Q), Id_{\mathfrak{uce}(Q)}\right)$.

From **1.**, an element of $(\mathfrak{uce}_\alpha(G), \overline{\alpha_G})$ can be written as $(\tau(m), \sigma(q))$, for $m \in \left(\mathfrak{uce}_\alpha(M), \overline{\alpha_M}\right)$ and $q \in \left(\mathfrak{uce}(Q), Id_{\mathfrak{uce}(Q)}\right)$ with a suitable choice. Such an element belongs to $Ker(U_\alpha^G)$ if and only if $U_G^\alpha(\tau(m), \sigma(q)) = 0$, i.e. $m \in Ker(U_\alpha^M)$ and $q \in HL_2(Q)$.

From these facts we can derive that

3. $\left(Ker(U_\alpha^G), \overline{\alpha_{G|}}\right) \cong \tau\left(Ker(U_\alpha^M), \overline{\alpha_{M|}}\right) \oplus \sigma\left(HL_2(Q), Id_{\mathfrak{uce}(Q)|}\right)$.

Since there exists a symmetric Hom-action of (Q, Id_Q) on (M, α_M), then there is a Hom-action of $\left(\mathfrak{uce}(Q), Id_{\mathfrak{uce}(Q)}\right)$ on $(\mathfrak{uce}_\alpha(M), \overline{\alpha_M})$ given by:

$$\begin{array}{rcl} \lambda : \mathfrak{uce}(Q) \otimes \mathfrak{uce}_\alpha(M) & \to & \mathfrak{uce}_\alpha(M) \\ \{q_1,q_2\} \otimes \{\alpha_M(m_1), \alpha_M(m_2)\} & \mapsto & \{q_1,q_2\} \centerdot \{\alpha_M(m_1), \alpha_M(m_2)\} = \\ & & \{[q_1,q_2] \centerdot \alpha_M(m_1), \alpha_M^2(m_2)\} - \\ & & \{[q_1,q_2] \centerdot \alpha_M(m_2), \alpha_M^2(m_1)\} \end{array}$$

and

$$\begin{array}{rcl} \rho : \mathfrak{uce}_\alpha(M) \otimes \mathfrak{uce}(Q) & \to & \mathfrak{uce}_\alpha(M) \\ \{\alpha_M(m_1), \alpha_M(m_2)\} \otimes \{q_1,q_2\} & \mapsto & \{\alpha_M(m_1), \alpha_M(m_2)\} \centerdot \{q_1,q_2\} = \\ & & \{\alpha_M(m_1) \centerdot [q_1,q_2], \alpha_M^2(m_2)\} - \\ & & \{\alpha_M^2(m_1), [q_1,q_2] \centerdot \alpha_M(m_2)\} \end{array}$$

Then we can define the following homomorphism of Hom-Leibniz algebras:

$$\tau \rtimes \sigma : \left(\mathfrak{uce}_{\alpha}\left(M\right), \overline{\alpha_M}\right) \rtimes \left(\mathfrak{uce}\left(Q\right), Id_{\mathfrak{uce}(Q)}\right) \to \left(\mathfrak{uce}_{\alpha}\left(G\right), \overline{\alpha_G}\right) \cong$$

$$\tau\left(\mathfrak{uce}_{\alpha}\left(M\right), \overline{\alpha_M}\right) \rtimes \sigma\left(\mathfrak{uce}\left(Q\right), Id_{\mathfrak{uce}(Q)}\right)$$

$$\left(\{\alpha_M(m_1), \alpha_M(m_2)\}, \{q_1, q_2\}\right) \mapsto \left(\{t\left(\alpha_M(m_1)\right), t\left(\alpha_M(m_2)\right)\}, \{s\left(q_1\right), s\left(q_2\right)\}\right)$$

Moreover $\tau \rtimes \sigma$ is an epimorphisms since

$$(\mathfrak{uce}_{\alpha}(G), \overline{\alpha_G}) \cong \tau\left(\mathfrak{uce}_{\alpha}\left(M\right), \overline{\alpha_M}\right) \rtimes \sigma\left(\mathfrak{uce}\left(Q\right), Id_{\mathfrak{uce}(Q)}\right).$$

By the relations coming from the action induced by the split extension

$\tau\left(\{q_1, q_2\} \centerdot \{\alpha_M(m_1), \alpha_M(m_2)\}\right) = \left[\{s\left(q_1\right), s\left(q_2\right)\}, \{t\left(\alpha_M(m_1)\right), t\left(\alpha_M(m_2)\right)\}\right]$

and

$\tau\left(\{\alpha_M(m_1), \alpha_M(m_2)\} \centerdot \{q_1, q_2\}\right) = \left[\{t\left(\alpha_M(m_1)\right), t\left(\alpha_M(m_2)\right)\}, \{s\left(q_1\right), s\left(q_2\right)\}\right]$

one derives that:

$t \circ U_{\alpha}^M\left(\{q_1, q_2\} \centerdot \{\alpha_M(m_1), \alpha_M(m_2)\}\right) = [q_1, q_2] \centerdot [\alpha_M(m_1), \alpha_M(m_2)]$,

and

$t \circ U_{\alpha}^M\left(\{\alpha_M(m_1), \alpha_M(m_2)\} \centerdot \{q_1, q_2\}\right) = [\alpha_M(m_1), \alpha_M(m_2)] \centerdot [q_1, q_2]$.

4. Now we define the surjective homomorphism of Hom-Leibniz algebras

$$\begin{array}{rcl} \Phi := \left(t \circ U_{\alpha}^M\right) \rtimes \left(s \circ u_Q\right) : \left(\mathfrak{uce}_{\alpha}\left(M\right) \rtimes \mathfrak{uce}\left(Q\right), \overline{\alpha_M} \rtimes Id_{\mathfrak{uce}(Q)}\right) & \to & (G, \alpha_G) \\ \left(\{\alpha_M(m_1), \alpha_M(m_2)\}, \{q_1, q_2\}\right) & \mapsto & \left(t\left[\alpha_M(m_1), \alpha_M(m_2)\right], s\left[q_1, q_2\right]\right) \end{array}$$

that makes commutative the following diagram:

$$\begin{tikzcd} \left(\mathfrak{uce}_{\alpha}\left(M\right) \rtimes \mathfrak{uce}\left(Q\right), \overline{\alpha_M} \rtimes Id_{\mathfrak{uce}(Q)}\right) \arrow[rr, two heads, "\tau \rtimes \sigma"] \arrow[dr, dashed, "\Phi"'] & & \left(\mathfrak{uce}_{\alpha}\left(G\right), \overline{\alpha_G}\right) \arrow[dl, two heads, "U_{\alpha}^G"] \\ & (G, \alpha_G) & \end{tikzcd} \tag{11}$$

Now we prove that

$$\mathfrak{uce}\left(Q\right) \centerdot Ker(U_{\alpha}^M) \oplus Ker(U_{\alpha}^M) \centerdot \mathfrak{uce}\left(Q\right) \subseteq Ker(\tau) \subseteq Ker(U_{\alpha}^M)$$

Second inclusion is obvious since $t \circ U_{\alpha}^M = U_{\alpha}^G \circ \tau$ and t is injective.
From the commutativity of the following diagram

$$\begin{tikzcd} & \left(Ker(U_{\alpha}^M), \overline{\alpha_{M|}}\right) \arrow[r, dashed] \arrow[d, tail] & \left(Ker(U_{\alpha}^G), \overline{\alpha_{G|}}\right) \arrow[d, tail] \\ Ker(\tau) \arrow[r, tail] \arrow[ur, dashed] & (\mathfrak{uce}_{\alpha}(M), \overline{\alpha_M}) \arrow[r, "\tau"] \arrow[d, two heads, "U_{\alpha}^M"] & (\mathfrak{uce}_{\alpha}(G), \overline{\alpha_G}) \arrow[d, two heads, "U_{\alpha}^G"] \\ & (M, \alpha_M) \arrow[r, "t"] & (G, \alpha_G) \end{tikzcd}$$

we have that $U_\alpha^G \circ \tau \left(Ker(U_\alpha^M)\right) = t \circ U_\alpha^M \left(Ker(U_\alpha^M)\right) = 0$, then $\tau \left(Ker(U_\alpha^M)\right) \subseteq Ker(U_\alpha^G) \subseteq Z\left(\mathfrak{uce}_\alpha(G)\right)$, so,

$\tau \left(\mathfrak{uce}\left(Q\right) \bullet Ker(U_\alpha^M)\right) = \left[\sigma\left(\mathfrak{uce}\left(Q\right)\right), \tau\left(Ker(U_\alpha^M)\right)\right] = 0$

and

$\tau \left(Ker(U_\alpha^M) \bullet \mathfrak{uce}\left(Q\right)\right) = \left[\tau\left(Ker(U_\alpha^M)\right), \sigma\left(\mathfrak{uce}\left(Q\right)\right)\right] = 0$

Consequently, $\mathfrak{uce}\left(Q\right) \bullet Ker(U_\alpha^M) \oplus Ker(U_\alpha^M) \bullet \mathfrak{uce}\left(Q\right) \subseteq Ker(\tau)$.

On the other hand, we observe that $\left(\mathfrak{uce}\left(Q\right) \bullet Ker(U_\alpha^M) \oplus Ker(U_\alpha^M) \bullet \mathfrak{uce}\left(Q\right), \overline{\alpha_{M|}}\right)$ is a two-sided ideal of $\left(\mathfrak{uce}_\alpha\left(M\right), \overline{\alpha_M}\right)$. Then the Hom-action of $\left(\mathfrak{uce}\left(Q\right), Id_Q\right)$ on $\left(\mathfrak{uce}_\alpha\left(M\right), \overline{\alpha_M}\right)$ induces a Hom-action of $\left(\mathfrak{uce}\left(Q\right), Id_Q\right)$ on

$$\left(\overline{\mathfrak{uce}_\alpha\left(M\right)}, \overline{\overline{\alpha_M}}\right) = \left(\frac{\mathfrak{uce}_\alpha\left(M\right)}{\mathfrak{uce}\left(Q\right) \bullet Ker(U_\alpha^M) \oplus Ker(U_\alpha^M) \bullet \mathfrak{uce}\left(Q\right)}, \overline{\overline{\alpha_M}}\right).$$

Since τ vanishes on $\mathfrak{uce}\left(Q\right) \bullet Ker(U_\alpha^M) \oplus Ker(U_\alpha^M) \bullet \mathfrak{uce}\left(Q\right)$, then it induces $\overline{\tau} : \overline{\mathfrak{uce}_\alpha\left(M\right)} \to \tau\left(\mathfrak{uce}_\alpha\left(M\right)\right)$. This fact is illustrated in the following diagram where the notation $I = \mathfrak{uce}\left(Q\right) \bullet Ker(U_\alpha^M) \oplus Ker(U_\alpha^M) \bullet \mathfrak{uce}\left(Q\right)$ is employed:

$$\begin{tikzcd}
\left(I, \overline{\alpha_{M|}}\right) \arrow[d, tail] \arrow[ddr, bend left, "0"] & & \\
\left(\mathfrak{uce}_\alpha(M), \overline{\alpha_M}\right) \arrow[rr, "\tau"] \arrow[dr, two heads] \arrow[dd, two heads] & & \left(\mathfrak{uce}_\alpha(G), \overline{\alpha_G}\right) \\
& \tau\left(\mathfrak{uce}_\alpha(M), \overline{\alpha_M}\right) \arrow[ur] & \\
\left(\overline{\mathfrak{uce}_\alpha\left(M\right)}, \overline{\overline{\alpha_M}}\right) \arrow[ur, dashed, "\overline{\tau}"] & &
\end{tikzcd}$$

Now we can construct the following commutative diagram:

$$\begin{tikzcd}
I \arrow[r, tail] \arrow[d, tail] & I \rtimes 0 \arrow[r, two heads] \arrow[d, tail] & 0 \arrow[d] \\
Ker\left(\tau \rtimes \sigma\right) \arrow[r, tail] \arrow[d, two heads] & \left(\mathfrak{uce}_\alpha\left(M\right) \rtimes \mathfrak{uce}\left(Q\right), \overline{\alpha_M} \rtimes Id_{\mathfrak{uce}(Q)}\right) \arrow[r, two heads, "\tau \rtimes \sigma"] \arrow[d, two heads] & \left(\mathfrak{uce}_\alpha\left(G\right), \overline{\alpha_G}\right) \arrow[d, equal] \\
\frac{Ker(\tau \rtimes \sigma)}{I} \arrow[r, tail] & \left(\overline{\mathfrak{uce}_\alpha\left(M\right)} \rtimes \mathfrak{uce}\left(Q\right), \overline{\overline{\alpha_M}} \rtimes Id_{\mathfrak{uce}(Q)}\right) \arrow[r, two heads, "\Psi"] & \left(\mathfrak{uce}_\alpha\left(G\right), \overline{\alpha_G}\right)
\end{tikzcd}$$

whose bottom row is a central extension. Moreover $\left(\mathfrak{uce}_\alpha\left(G\right), \overline{\alpha_G}\right)$ is an α-perfect Hom-Leibniz algebra, then by Theorem 5.5 in [7], it admits a universal α-central extension and, by Corollary 3.4, $\mathfrak{uce}_\alpha\left(G\right)$ is centrally closed, i.e. $\mathfrak{uce}\left(\mathfrak{uce}_\alpha\left(G\right)\right) \cong \mathfrak{uce}_\alpha\left(G\right)$.

Having in mind the following diagram,

$$\begin{tikzcd}
\left(\overline{\mathfrak{uce}_\alpha\left(M\right)} \rtimes \mathfrak{uce}\left(Q\right), \overline{\alpha_M} \rtimes Id_{\mathfrak{uce}(Q)}\right) \arrow[r, two heads, "\Psi"] \arrow[d, "\Psi"] \arrow[dd, bend right=60, "Id"'] & \left(\mathfrak{uce}_\alpha\left(G\right), \overline{\alpha_G}\right) \arrow[d, equal] \\
\left(\mathfrak{uce}_\alpha\left(G\right), \overline{\alpha_G}\right) \arrow[r, tail, two heads, "Id"] \arrow[d, "\mu"] & \left(\mathfrak{uce}_\alpha\left(G\right), \overline{\alpha_G}\right) \arrow[d, equal] \\
\left(\overline{\mathfrak{uce}_\alpha\left(M\right)} \rtimes \mathfrak{uce}\left(Q\right), \overline{\alpha_M} \rtimes Id_{\mathfrak{uce}(Q)}\right) \arrow[r, two heads, "\Psi"] & \left(\mathfrak{uce}_\alpha\left(G\right), \overline{\alpha_G}\right)
\end{tikzcd}$$

where $Id : \left(\mathfrak{uce}_\alpha\left(G\right), \overline{\alpha_G}\right) \to \left(\mathfrak{uce}_\alpha\left(G\right), \overline{\alpha_G}\right)$ is a universal central extension since $\left(\mathfrak{uce}_\alpha\left(G\right), \overline{\alpha_G}\right)$ is centrally closed and Ψ is a central extension, then there exists a unique homomorphism of Hom-Leibniz algebras $\mu : \left(\mathfrak{uce}_\alpha\left(G\right), \overline{\alpha_G}\right) \to \left(\overline{\mathfrak{uce}_\alpha\left(M\right)} \rtimes \mathfrak{uce}\left(Q\right), \overline{\alpha_M} \rtimes Id_{\mathfrak{uce}(Q)}\right)$ such that $\Psi \circ \mu = Id$.

Since $\Psi \circ \mu \circ \Psi = Id \circ \Psi = \Psi = \Psi \circ Id$ and $\overline{\mathfrak{uce}_\alpha\left(M\right)} \rtimes \mathfrak{uce}\left(Q\right)$ is α-perfect, then Lemma 5.4 in [7] implies that $\mu \circ \Psi = Id$. Consequently, Ψ is an isomorphism, then $Ker(\Psi) = \frac{Ker(\tau \rtimes \sigma)}{I} = 0$, so $Ker\left(\tau \rtimes \sigma\right) \subseteq I$.

The above discussion can be summarized in:

5. $Ker\left(\tau \rtimes \sigma\right) \cong \mathfrak{uce}\left(Q\right) \centerdot Ker(U_\alpha^M) \oplus Ker(U_\alpha^M) \centerdot \mathfrak{uce}\left(Q\right)$

We summarize the above results in the following

Theorem 5.2 *Consider a split extension of α-perfect Hom-Leibniz algebras*

$$0 \longrightarrow (M, \alpha_M) \xrightarrow{t} (G, \alpha_G) \underset{s}{\overset{p}{\rightleftarrows}} (Q, Id_Q) \longrightarrow 0$$

where the induced Hom-action of (Q, Id_Q) on (M, α_M) is symmetric. Then the following statements hold:

1. $(\mathfrak{uce}_\alpha(G), \overline{\alpha_G}) = \tau\left(\mathfrak{uce}_\alpha\left(M\right), \overline{\alpha_M}\right) \rtimes \sigma\left(\mathfrak{uce}\left(Q\right), Id_{\mathfrak{uce}(Q)}\right)$.

2. $\sigma\left(\mathfrak{uce}\left(Q\right), Id_{\mathfrak{uce}(Q)}\right) \cong \left(\mathfrak{uce}\left(Q\right), Id_{\mathfrak{uce}(Q)}\right)$.

3. $\left(Ker(U_\alpha^G), \overline{\alpha_G}_{|}\right) \cong \tau\left(Ker(U_\alpha^M), \overline{\alpha_M}_{|}\right) \oplus \sigma\left(HL_2(Q), {Id_{\mathfrak{uce}(Q)}}_{|}\right)$.

4. *The homomorphism of Hom-Leibniz algebras*

$$\Phi : \left(\mathfrak{uce}_\alpha\left(M\right) \rtimes \mathfrak{uce}\left(Q\right), \overline{\alpha_M} \rtimes Id_{\mathfrak{uce}(Q)}\right) \to (G, \alpha_G)$$

given by $\Phi\left(\left\{\alpha_M(m_1), \alpha_M(m_2)\right\}, \left\{q_1, q_2\right\}\right) = \left(t\left[\alpha_M(m_1), \alpha_M(m_2)\right], s\left[q_1, q_2\right]\right)$ is an epimorphism that makes commutative diagram (11) and its kernel is $Ker(U_\alpha^M) \oplus HL_2(Q)$.

5. $Ker\,(\tau \rtimes \sigma) \cong \mathfrak{uce}\,(Q) \centerdot Ker(U_\alpha^M) \oplus Ker(U_\alpha^M) \centerdot \mathfrak{uce}\,(Q)$

Remark 5.3 *Let us observe that the hypothesis of symmetric Hom-action is not needed in statements* **1., 2.** *and* **3.** *in Theorem 5.2, so they are valid in general.*

Theorem 5.4 *The following statements are equivalent:*

a) $\Phi = (t \circ U_\alpha^M) \rtimes (s \circ u_Q) : \left(\mathfrak{uce}_\alpha\,(M) \rtimes \mathfrak{uce}\,(Q), \overline{\alpha_M} \rtimes Id_{\mathfrak{uce}_\alpha(Q)}\right) \to (G, \alpha_G)$ *is a central extension, hence is an* α*-cover.*

b) The Hom-action of $(\mathfrak{uce}(Q), Id_Q)$ *on* $(Ker(U_\alpha^M), \overline{\alpha_{M|}})$ *is trivial.*

c) $\tau \rtimes \sigma$ *is an isomorphism. Consequently* $\mathfrak{uce}_\alpha\,(M) \rtimes \mathfrak{uce}\,(Q)$ *is the universal* α*-central extension of* (G, α_G)*.*

d) τ *is injective.*

In particular, for the direct product $(G, \alpha_G) = (M, \alpha_M) \times (Q, Id_Q)$ *the following isomorphism holds:*

$$\left(\mathfrak{uce}_\alpha\,(M \times Q), \overline{\alpha_M \times Id_Q}\right) \cong \left(\mathfrak{uce}_\alpha\,(M) \times \mathfrak{uce}\,(Q), \overline{\alpha_M} \times Id_{\mathfrak{uce}(Q)}\right)$$

Proof. a) $\Longleftrightarrow$ *b)*

If $\Phi : \left(\mathfrak{uce}_\alpha\,(M) \rtimes \mathfrak{uce}_\alpha\,(Q), \overline{\alpha_M} \rtimes Id_{\mathfrak{uce}_\alpha(Q)}\right) \to (G, \alpha_G)$ is a central extension and having in mind that $Ker(\Phi) = Ker(U_\alpha^M) \oplus HL_2(Q)$, then the Hom-action of $(\mathfrak{uce}(Q), Id_Q)$ on $(Ker(U_\alpha^M), \overline{\alpha_{M|}})$ is trivial and vice versa.

Moreover $(\mathfrak{uce}_\alpha\,(M) \rtimes \mathfrak{uce}\,(Q), \overline{\alpha_M} \rtimes Id_{\mathfrak{uce}(Q)})$ is $\alpha-$perfect since the Hom-action is trivial.

b) $\Longleftrightarrow$ *c)*

By statement **5.** in Theorem 5.2 we know that $Ker\,(\tau \rtimes \sigma) \cong \mathfrak{uce}\,(Q) \centerdot Ker(U_\alpha^M) \oplus Ker(U_\alpha^M) \centerdot \mathfrak{uce}\,(Q)$, then $\tau \rtimes \sigma$ is injective if and only if the Hom-action is trivial.

From this fact and having in mind diagram (11), immediately follows that $\mathfrak{uce}_\alpha\,(M) \rtimes \mathfrak{uce}\,(Q)$ is the universal α-central extension of (G, α_G).

c) $\Longleftrightarrow$ *d)*

It suffices to have in mind the identification of τ with $\tau \rtimes \sigma$ given by $\tau\,\{\alpha_M\,(m_1), \alpha_M\,(m_2)\} = (\tau \rtimes \sigma)\,(\{\alpha_M\,(m_1), \alpha_M\,(m_2)\}, 0)$, since $Ker(\tau) \cong Ker\,(\tau \rtimes \sigma)$, then the equivalence is obvious.

Finally, in case of the direct product $(G, \alpha_G) = (M, \alpha_M) \times (Q, Id_Q)$ the Hom-action of (Q, Id_Q) on (M, α_M) is trivial, then the Hom-action of $(\mathfrak{uce}(Q), Id_{\mathfrak{uce}(Q)})$ on $(\mathfrak{uce}_\alpha(M), \overline{\alpha_M})$ is trivial as well and, consequently, $(\mathfrak{uce}_\alpha(M) \rtimes \mathfrak{uce}(Q), \overline{\alpha_M} \rtimes Id_{\mathfrak{uce}(Q)}) = (\mathfrak{uce}_\alpha(M) \rtimes \mathfrak{uce}(Q), \overline{\alpha_M} \times Id_{\mathfrak{uce}(Q)})$.

The proof is finished by application of statement *c)* to this particular case. $\square$

Remark 5.5 *Note that when the Hom-Leibniz algebras are considered as Leibniz algebras, i.e. the endomorphisms* α *are identities, then the results in this section recover the corresponding results for Leibniz algebras given in [6].*

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