On the quasilinear elliptic systems involving critical Hardy–Sobolev and Sobolev exponents

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Abstract

In this paper, a system of quasilinear elliptic equations is investigated, which involves critical exponents and multiple Hardy–type terms. By variational methods and analytic techniques, the existence of positive solutions to the system is established. The conclusions are new even when the Hardy–type terms disappear.

Keywords: quasilinear elliptic system, positive solution, critical exponent, Hardy inequality, variational method.

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1 Introduction

In this paper, we study the following elliptic system:

$$\begin{cases} -\Delta_p u - \mu \dfrac{u^{p-1}}{|x|^p} = \dfrac{\eta_1}{p^*} H_u(u,v) + \dfrac{\eta_2}{p^*(t)} \dfrac{Q_u(u,v)}{|x|^t}, \\ -\Delta_p v - \mu \dfrac{v^{p-1}}{|x|^p} = \dfrac{\eta_1}{p^*} H_v(u,v) + \dfrac{\eta_2}{p^*(t)} \dfrac{Q_v(u,v)}{|x|^t}, \\ u, v > 0, \quad (u,v) \in \mathcal{D} \times \mathcal{D}, \end{cases} \tag{1.1}$$

where $1 < p < N$, $0 \le \mu < \bar{\mu} := \big((N-p)/p\big)^p$, $0 < t < p$, $\eta_1 > 0$, $\eta_2 > 0$, $-\Delta_p \cdot := -\mathrm{div}(|\nabla \cdot|^{p-2} \cdot)$ is the p–Laplace operator, the space $\mathcal{D} := D^{1,p}(\mathbb{R}^N)$ denotes the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to $(\int_{\mathbb{R}^N} |\nabla \cdot|^p \,\mathrm{d}x)^{1/p}$, $\bar{\mu}$ is the best Hardy constant,

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$p^* := Np/(N-p)$ is the critical Sobolev exponent and $p^*(t) := p(N-t)/(N-p)$ is the critical Hardy–Sobolev exponent with $p^*(0) = p^*$. H_u, H_v, Q_u and Q_v are the partial derivatives of the 2–variable C^1–functions $H(u,v)$ and $Q(u,v)$ respectively. The functions H and Q satisfy the following conditions:

($\mathcal{H}$) $H, Q \in C^1(\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$,

$$H_u(u,0) = H_u(0,v) = H_v(u,0) = H_v(0,v) = 0, \quad \forall\, u, v \geq 0,$$

$$Q_u(u,0) = Q_u(0,v) = Q_v(u,0) = Q_v(0,v) = 0, \quad \forall\, u, v \geq 0,$$

$$H(\lambda u, \lambda v) = \lambda^{p^*} H(u,v), \quad \forall\, \lambda \geq 0,\ u, v \geq 0, \quad (p^*\text{–}homogeneity),$$

$$Q(\lambda u, \lambda v) = \lambda^{p^*(t)} Q(u,v), \quad \forall\, \lambda \geq 0, \quad u, v \geq 0, \ (p^*(t)\text{–}homogeneity),$$

and the 1*–homogenous functions* G *and* $\bar{G}$ *are concave, where* G *and* $\bar{G}$ *are defined as follows:*

$$G(\alpha^{p^*}, \beta^{p^*}) = H(\alpha, \beta), \quad \bar{G}(\alpha^{p^*(t)}, \beta^{p^*(t)}) = Q(\alpha, \beta), \quad \forall\, \alpha, \beta \geq 0.$$

The following properties are important and well–known:

($\mathcal{H}'$) *Suppose* $F(s,t)$ *is a* q*–homogeneous differential function with* $q \geq 1$. *Then*
(i) $sF_s(s,t) + tF_t(s,t) = qF(s,t), \quad \forall s, t \in \mathbb{R}$;
(ii) C_F *is attained at some* $(s_0, t_0) \in \mathbb{R}^2$, *where*

$$C_F := \max\{F(s,t) | s, t \in \mathbb{R},\ |s|^q + |t|^q = 1\};$$

(iii) $|F(s,t)| \leq C_F(|s|^q + |t|^q), \quad \forall\, s, t \in \mathbb{R}$;
(iv) $F_s(s,t)$ *and* $F_t(s,t)$ *are* $(q-1)$*–homogeneous.*

In this paper, we work in the product space $\mathcal{D} \times \mathcal{D}$. The corresponding energy functional of (1.1) is defined on $\mathcal{D} \times \mathcal{D}$ by

$$\begin{aligned} I(u,v) := \ & \frac{1}{p} \int_{\mathbb{R}^N} \Big(|\nabla u|^p + |\nabla v|^p - \mu \frac{|u|^p + |v|^p}{|x|^p} \Big) \mathrm{d}x \\ & - \frac{\eta_1}{p^*} \int_{\mathbb{R}^N} H(|u|, |v|) \mathrm{d}x - \frac{\eta_2}{p^*(t)} \int_{\mathbb{R}^N} \frac{Q(|u|, |v|)}{|x|^t} \mathrm{d}x. \end{aligned}$$

Then $I \in C^1(\mathcal{D} \times \mathcal{D}, \mathbb{R})$. A pair of functions $(u,v) \in \mathcal{D} \times \mathcal{D}$ is said to be a solution of (1.1) if $u, v > 0$, and

$$(u,v) \neq (0,0), \qquad \langle I'(u,v), (\varphi, \phi) \rangle = 0, \quad \forall\, (\varphi, \phi) \in \mathcal{D} \times \mathcal{D},$$

where $I'(u,v)$ denotes the Fréchet derivative of I at (u,v).

Problem (1.1) is related to the Hardy and Hardy–Sobolev inequalities [8, 20]):

$$\int_{\mathbb{R}^N} \frac{|u|^p}{|x|^p} \,\mathrm{d}x \ \leq \ \frac{1}{\bar{\mu}} \int_{\mathbb{R}^N} |\nabla u|^p \,\mathrm{d}x \,, \forall\ u \in C_0^\infty(\mathbb{R}^N)\,, \tag{1.2}$$

$$\Big(\int_{\mathbb{R}^N}\frac{|u|^{p^*(t)}}{|x|^t}\,\mathrm{d}x\Big)^{\frac{p}{p^*(t)}}\ \le\ C(p,t)\int_{\mathbb{R}^N}|\nabla u|^p\,\mathrm{d}x\,,\forall\ u\in C_0^\infty(\mathbb{R}^N)\,, \tag{1.3}$$

where $C(p,t)$ is a constant depending on p and t, $1<p<N$ and $0\le t<p$.

By (1.2) the operator $L:=(-\Delta_p\cdot-\mu|\cdot|^{p-2}\cdot/|x|^p)$ is positive for all $\mu<\bar{\mu}$ and therefore the following equivalent norm of $\mathcal{D}$ can be defined:

$$\|u\|:=\Big(\int_{\mathbb{R}^N}\Big(|\nabla u|^p-\mu\,\frac{|u|^p}{|x|^p}\Big)\mathrm{d}x\Big)^{\frac{1}{p}},\quad\forall\,u\in\mathcal{D}.$$

Suppose $(\mathcal{H})$ holds. By $(\mathcal{H}')$, (1.2) and (1.1), the following best Hardy–Sobolev constants are well–defined:

$$S(\mu,t):=\inf_{u\in\mathcal{D}\backslash\{0\}}\frac{\displaystyle\int_{\mathbb{R}^N}\Big(|\nabla u|^p-\mu\,\frac{|u|^p}{|x|^p}\Big)\mathrm{d}x}{\displaystyle\Big(\int_{\mathbb{R}^N}\frac{|u|^{p^*(t)}}{|x|^t}\mathrm{d}x\Big)^{\frac{p}{p^*(t)}}}\,, \tag{1.4}$$

$$S_H(\mu,0):=\inf_{u,v\in\mathcal{D}\backslash\{0\}}\frac{\displaystyle\int_{\mathbb{R}^N}\Big(|\nabla u|^p+|\nabla v|^p-\mu\,\frac{|u|^p+|v|^p}{|x|^p}\Big)\mathrm{d}x}{\displaystyle\Big(\int_{\mathbb{R}^N}H(|u|,|v|)\mathrm{d}x\Big)^{\frac{p}{p^*}}}, \tag{1.5}$$

$$S_Q(\mu,t):=\inf_{u,v\in\mathcal{D}\backslash\{0\}}\frac{\displaystyle\int_{\mathbb{R}^N}\Big(|\nabla u|^p+|\nabla v|^p-\mu\,\frac{|u|^p+|v|^p}{|x|^p}\Big)\mathrm{d}x}{\displaystyle\Big(\int_{\mathbb{R}^N}\frac{Q(|u|,|v|)}{|x|^t}\mathrm{d}x\Big)^{\frac{p}{p^*(t)}}}, \tag{1.6}$$

where $0\le t<p,\quad -\infty<\mu<\bar{\mu}$. It should be mentioned that the strongly coupled terms $\int_{\mathbb{R}^N}H(|u|,|v|)\mathrm{d}x$ and $\int_{\mathbb{R}^N}\frac{Q(|u|,|v|)}{|x|^t}\mathrm{d}x$ are critical in the senses of Sobolev or Hardy–Sobolev embedding. Morais Filho etc. studied the constant $S_H(0,0)$ and proved the existence of solutions for a quasilinear elliptic systems in [17]. Alves etc. studied in [3] the following best constant and found its extremals:

$$A(\sigma,\tau):=\inf_{u,v\in D^{1,2}(\mathbb{R}^N)\backslash\{0\}}\frac{\displaystyle\int_{\mathbb{R}^N}\big(|\nabla u|^2+|\nabla v|^2\big)\mathrm{d}x}{\displaystyle\Big(\int_{\mathbb{R}^N}|u|^\sigma|v|^\tau\mathrm{d}x\Big)^{\frac{2}{2^*}}}, \tag{1.7}$$

where $1<\sigma,\tau<2^*-1,\quad \sigma+\tau=2^*:=2N/(N-2)$. Note that $A(\sigma,\tau)$ in (1.7) is a special case of $S_H(0,0)$. The methods and conclusions in [3] and [17] are very stimulating.

In recent years, much attention has been paid to the semilinear and quasilinear elliptic problems involving the Hardy and Hardy–Sobolev inequalities, and many results were obtained providing us very good insight into the problems (e.g. [1], [5],

[6], [9], [10], [11], [14], [15], [18], [19], [22], [23], [30], [32], [33] and the references therein). In particular, Filippucci etc. studied in [18] the following problem:

$$\begin{cases} -\Delta_p u - \mu \dfrac{u^{p-1}}{|x|^p} = u^{p^*-1} + \dfrac{u^{p^*(s)-1}}{|x|^s}, \\ u \in \mathcal{D}, \quad u > 0 \ \text{ in } \ \mathbb{R}^N, \\ -\infty < \mu < \bar{\mu}, \quad 0 < s < p. \end{cases} \tag{1.8}$$

The main difficulty of studying (1.8) is that the critical Hardy–Sobolev and Sobolev exponents appear simultaneously in the equation and induce more difficulties. By very technic and complicated analysis, the authors of [18] proved the existence of positive solutions to (1.8) by the Mountain–Pass theorem ([4]) and the concentration compactness principle ([26, 27]). The extremals of the best constant $S(\mu, t)$ in (1.4) and some related singular quasilinear elliptic problems were investigated in [1], [18], [19] and [23], and we infer that, for all $0 \le t < p,\ 0 \le \mu < \bar{\mu}$, the best constant $S_{\mu,t}$ is achieved by the implicit extremal function:

$$V^{\varepsilon}_{\mu,t}(x) = \varepsilon^{\frac{p-N}{p}} U_{\mu,t}(\varepsilon^{-1}x)\,, \quad \forall\, \varepsilon > 0\,, \tag{1.9}$$

that satisfies

$$\int_{\mathbb{R}^N} \Big(|\nabla V^{\varepsilon}_{\mu,t}(x)|^p - \mu \frac{|V^{\varepsilon}_{\mu,t}(x)|^p}{|x|^p} \Big) = \int_{\mathbb{R}^N} \frac{|V^{\varepsilon}_{\mu,t}(x)|^{p^*(t)}}{|x|^t} = (S_{\mu,t})^{\frac{N-t}{p-t}},$$

where $U_{\mu,t}(x)$ is some radial function.

Compared with single singular elliptic equations, the singular elliptic systems involving the Hardy and Hardy–Sobolev inequalities have been seldom studied, we can only find several results in [2], [7], [16], [21], [24], [25], [28] and [29], where some nonlinear singular critical systems were investigated, the corresponding best Hardy–Sobolev constants were studied and existence results of solutions were obtained. The main difficulties of studying singular elliptic systems are that, the singularity may occur and the strongly coupled terms may cause more difficulties.

To continue, we define

$$M_H := \max\big\{ H(|\alpha|, |\beta|)^{\frac{p}{p^*}} \,\big|\, \alpha, \beta \in \mathbb{R},\ |\alpha|^p + |\beta|^p = 1 \big\}; \tag{1.10}$$

$$M_Q := \max\big\{ Q(|\alpha|, |\beta|)^{\frac{p}{p^*(t)}} \,\big|\, \alpha, \beta \in \mathbb{R},\ |\alpha|^p + |\beta|^p = 1 \big\}. \tag{1.11}$$

Then there exist $(\alpha_i, \beta_i) \in \mathbb{R}^+ \times \mathbb{R}^+,\ i = 1, 2$, such that M_H and M_Q are achieved respectively, that is,

$$M_H = H(\alpha_1, \beta_1)^{\frac{p}{p^*}}, \quad \alpha_1^p + \beta_1^p = 1, \tag{1.12}$$

$$M_Q = Q(\alpha_2, \beta_2)^{\frac{p}{p^*(t)}}, \quad \alpha_2^p + \beta_2^p = 1. \tag{1.13}$$

In this paper, stimulated by the references mentioned above, we investigate (1.1). The main results of this paper are summarized in the following theorems. To the best of our knowledge, the conclusions are new even in the case $\mu = 0$.

Theorem 1.1. *Suppose that* $0 \le t < p,\ -\infty < \mu < \bar{\mu}$ *and* $(\mathcal{H})$ *holds. Then*
(i) $S_H(\mu, 0) = M_H^{-1} S(\mu, 0),\quad S_Q(\mu, t) = M_Q^{-1} S(\mu, t)$.
(ii) *For all* $0 \le \mu < \bar{\mu}$, $S_H(\mu, 0)$ *has the minimizers* $\big(\alpha_1 V^{\varepsilon}_{\mu,0}(x),\ \beta_1 V^{\varepsilon}_{\mu,0}(x)\big)$, $S_Q(\mu, t)$ *has the minimizers* $\big(\alpha_2 V^{\varepsilon}_{\mu,t}(x),\ \beta_2 V^{\varepsilon}_{\mu,t}(x)\big)$, *where* $V^{\varepsilon}_{\mu,t}(x)$ *are defined as in* (1.9).

Theorem 1.2. *Suppose that* $1 < p < N,\ 0 \le \mu < \bar{\mu},\ 0 < t < p,\ \eta_1 > 0,\ \eta_2 > 0$ *and* $(\mathcal{H})$ *holds. Then the problem* (1.1) has a solution.

Remark 1.1. *The coefficients* $1/p^*$ *and* $1/p^*(t)$ *in* (1.1) *are only used for the convenience of computation and have no particular meanings. By Theorem* 1.1*, the existence of solutions to* (1.1) *is obvious in anyone of the following cases:* (i) $\eta_1 = 0,\ \eta_2 > 0,\ t \ge 0$; (ii) $\eta_1 > 0,\ \eta_2 = 0,\ t \ge 0$; (iii) $t = 0,\ \eta_1 > 0,\ \eta_2 > 0$.

Remark 1.2. *The following problem is an example of* (1.1) *:*

$$\begin{cases} -\Delta_p u - \mu \dfrac{u^{p-1}}{|x|^p} = \dfrac{\sigma_1}{p^*} u^{\sigma_1 - 1} v^{\tau_1} + \dfrac{\sigma_2}{p^*(t)} \dfrac{u^{\sigma_2 - 1} v^{\tau_2}}{|x|^t}, \\ -\Delta_p v - \mu \dfrac{v^{p-1}}{|x|^p} = \dfrac{\tau_1}{p^*} u^{\sigma_1} v^{\tau_1 - 1} + \dfrac{\tau_2}{p^*(t)} \dfrac{u^{\sigma_2} v^{\tau_2 - 1}}{|x|^t}, \\ u, v > 0, \quad (u, v) \in \mathcal{D} \times \mathcal{D}. \end{cases} \tag{1.14}$$

where the parameters satisfy the following condition:

$(\mathcal{H}'')$ $N \ge 3,\ \ 1 < p < N,\ \ 0 \le \mu < \bar{\mu} = \big(\frac{N-p}{p}\big)^p,\ \ 0 < t < p,\ \ \sigma_i, \tau_i > 1,\ i = 1, 2,$
$\sigma_1 + \tau_1 = p^* = \frac{Np}{N-p},\ \ \sigma_2 + \tau_2 = p^*(t) = \frac{p(N-t)}{N-p}.$

Note that (1.14) *involves the critical Hardy–Sobolev and Sobolev exponents and admits a solution by Theorem* 1.2.

This paper is organized as follows. Theorem 1.1 is verified in Section 2, some preliminary results are established in Section 3 and Theorem 1.2 is proved in Section 4. In the following argument, $\|u\| = (\int_{\mathbb{R}^N} (|\nabla u|^p - \mu |u|^p |x|^{-p}) \mathrm{d}x)^{1/p}$ denotes the equivalent norm of the space $\mathcal{D}$, $\|(u, v)\|_{\mathcal{D}\times\mathcal{D}} = (\|u\|^p + \|v\|^p)^{1/p}$ is the norm of the space $\mathcal{D} \times \mathcal{D}$. For all $\varepsilon > 0$ small enough, $O(\varepsilon^t)$ denotes the quantity satisfying $|O(\varepsilon^t)|/\varepsilon^t \le C$, $o(\varepsilon^t)$ means $|o(\varepsilon^t)|/\varepsilon^t \to 0$ as $\varepsilon \to 0$ and $o(1)$ is a generic infinitesimal value. In particular, the quantity $O_1(\varepsilon^t)$ means that there exist the constants $C_1, C_2 > 0$ such that $C_1 \varepsilon^t \le O_1(\varepsilon^t) \le C_2 \varepsilon^t$ as ε small. We always denote positive constants as C and omit $\mathrm{d}x$ in integrals for convenience.

2 The best constants $S_H(\mu, 0)$ and $S_Q(\mu, t)$

In this section, we study $S_H(\mu, 0)$ and $S_Q(\mu, t)$, and verify Theorem 1.1.

Proof of Theorem 1.1. (i) We only show the proof for $S_Q(\mu,t)$. The argument is similar to that of [17], where the best constant $S_H(0,0)$ was studied.

Let $w\in\mathcal{D}\backslash\{0\}$ and (α_2,β_2) be defined as in (1.13). Choosing $(u,v)=(\alpha_2 w,\beta_2 w)$ in (1.6) we have

$$\frac{(|\alpha_2|^p+|\beta_2|^p)\int_{\mathbb{R}^N}\Big(|\nabla w|^p-\mu\frac{|w|^p}{|x|^p}\Big)}{|Q(\alpha_2,\beta_2)|^{\frac{p}{p^*(t)}}\Big(\int_{\mathbb{R}^N}\frac{|w|^{p^*(t)}}{|x|^t}\Big)^{\frac{p}{p^*(t)}}}\geq S_Q(\mu,t). \tag{2.1}$$

Taking the infimum as $w\in\mathcal{D}\setminus\{0\}$ in (2.1), by (1.4) and (1.10)–(1.13) we have

$$M_Q^{-1}S(\mu,t)\geq S_Q(\mu,t). \tag{2.2}$$

For any $u,v\in\mathcal{D}\setminus\{0\}$, by Proposition 1 of [17] we have that

$$\begin{aligned}\int_{\mathbb{R}^N}\frac{Q(|u|,|v|)}{|x|^t}&=\int_{\mathbb{R}^N}Q\big(|x|^{-\frac{t}{p^*(t)}}|u|,|x|^{-\frac{t}{p^*(t)}}|v|\big)\\&\leq Q\big(\|\,|x|^{-\frac{t}{p^*(t)}}u\|_{L^{p^*(t)}(\mathbb{R}^N)},\|\,|x|^{-\frac{t}{p^*(t)}}v\|_{L^{p^*(t)}(\mathbb{R}^N)}\big).\end{aligned} \tag{2.3}$$

Set

$$\theta:=\Big(\|\,|x|^{-\frac{t}{p^*(t)}}u\|^p_{L^{p^*(t)}(\mathbb{R}^N)}+\|\,|x|^{-\frac{t}{p^*(t)}}v\|^p_{L^{p^*(t)}(\mathbb{R}^N)}\Big)^{-\frac{1}{p}}.$$

Then

$$\|\theta|x|^{-\frac{t}{p^*(t)}}u\|^p_{L^{p^*(t)}(\mathbb{R}^N)}+\|\theta|x|^{-\frac{t}{p^*(t)}}v\|^p_{L^{p^*(t)}(\mathbb{R}^N)}=1. \tag{2.4}$$

From (1.11), (1.13), (2.3) and (2.4) it follows that

$$\begin{aligned}&\frac{\int_{\mathbb{R}^N}\Big(|\nabla u|^p+|\nabla v|^p-\mu\frac{|u|^p+|v|^p}{|x|^p}\Big)}{\Big(\int_{\mathbb{R}^N}\frac{Q(|u|,|v|)}{|x|^t}\Big)^{\frac{p}{\alpha+\beta}}}\\
&\geq S(\mu,t)\frac{\Big(\int_{\mathbb{R}^N}\frac{|u|^{p^*(t)}}{|x|^t}\Big)^{\frac{p}{p^*(t)}}+\Big(\int_{\mathbb{R}^N}\frac{|v|^{p^*(t)}}{|x|^t}\Big)^{\frac{p}{p^*(t)}}}{\Big(Q\big(\|\,|x|^{-\frac{t}{p^*(t)}}u\|_{L^{p^*(t)}(\mathbb{R}^N)},\|\,|x|^{-\frac{t}{p^*(t)}}v\|_{L^{p^*(t)}(\mathbb{R}^N)}\big)\Big)^{\frac{p}{p^*(t)}}}\\
&=S(\mu,t)\frac{\|\,|x|^{-\frac{t}{p^*(t)}}u\|^p_{L^{p^*(t)}(\mathbb{R}^N)}+\|\,|x|^{-\frac{t}{p^*(t)}}v\|^p_{L^{p^*(t)}(\mathbb{R}^N)}}{\Big(Q\big(\|\,|x|^{-\frac{t}{p^*(t)}}u\|_{L^{p^*(t)}(\mathbb{R}^N)},\|\,|x|^{-\frac{t}{p^*(t)}}v\|_{L^{p^*(t)}(\mathbb{R}^N)}\big)\Big)^{\frac{p}{p^*(t)}}}\\
&=S(\mu,t)\frac{\|\theta|x|^{-\frac{t}{p^*(t)}}u\|^p_{L^{p^*(t)}(\mathbb{R}^N)}+\|\theta|x|^{-\frac{t}{p^*(t)}}v\|^p_{L^{p^*(t)}(\mathbb{R}^N)}}{\Big(Q\big(\|\theta|x|^{-\frac{t}{p^*(t)}}u\|_{L^{p^*(t)}(\mathbb{R}^N)},\|\theta|x|^{-\frac{t}{p^*(t)}}v\|_{L^{p^*(t)}(\mathbb{R}^N)}\big)\Big)^{\frac{p}{p^*(t)}}}\\
&\geq\frac{1}{|Q(\alpha_2,\beta_2)|^{\frac{p}{p^*(t)}}}S(\mu,t)=M_Q^{-1}S(\mu,t).\end{aligned}$$

Taking the infimum as $u, v \in \mathcal{D} \setminus \{0\}$ we have

$$M_Q^{-1} S(\mu, t) \leq S_Q(\mu, t),$$

which together with (2.2) implies that

$$S_Q(\mu, t) = M_Q^{-1} S(\mu, t).$$

(ii) From (i), (1.5) and (1.6) the desired result follows. □

3 Appropriate Palais–Smale sequence

To find positive solutions of (1.1), we define the functional J on $\mathcal{D} \times \mathcal{D}$ by

$$J(u, v) := \frac{1}{p} \|(u, v)\|^p - \frac{\eta_1}{p^*} \int_{\mathbb{R}^N} H(u_+, v_+) - \frac{\eta_2}{p^*(t)} \int_{\mathbb{R}^N} \frac{Q(u_+, v_+)}{|x|^t},$$

where $w_+ = \max\{w, 0\}$ for all $w \in \mathcal{D}$. Then $J \in C^1(\mathcal{D} \times \mathcal{D}, \mathbb{R})$ according to $(\mathcal{H})$ and a solution of (1.1) is a nontrivial critical point of J. We follow the argument similar to that of [16], where the problem (1.8) was investigated.

Lemma 3.1. (Mountain–Pass lemma, [4]) *Let E be a Banach space and $\Phi \in C^1(E)$. Assume that*

(i) $\Phi(0) = 0$.

(ii) *There exist $\lambda, R > 0$ such that $\Phi(u) \geq \lambda$ for all $u \in E$ with $\|u\|_E = R$.*

(iii) *There exists $v_0 \in E$ such that $\limsup_{t \to \infty} \Phi(tv_0) < 0$.*

Take $t_0 > 0$ such that $\|t_0 v_0\|_E > R$ and $\Phi(t_0 v_0) < 0$. Set

$$\Gamma := \{\gamma \in C([0, 1], E) | \gamma(0) = 0 \text{ and } \gamma(1) = t_0 v_0\}, \qquad c := \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} \Phi(\gamma(t)).$$

Then there exists a Palais–Smale sequence at level c for Φ, that is there exists a sequence $\{u_k\} \subset E$ such that

$$\lim_{k \to \infty} \Phi(u_k) = c, \qquad \lim_{k \to \infty} \Phi'(u_k) = 0 \text{ strongly in } E^{-1}.$$

Lemma 3.2. *Suppose that $(\mathcal{H})$ holds. Set*

$$c^* := \min \left\{ \frac{1}{N} \eta_1^{\frac{p-N}{p}} S_H(\mu, 0)^{\frac{N}{p}}, \ \frac{p-t}{p(N-t)} \eta_2^{\frac{p-N}{p-t}} S_Q(\mu, t)^{\frac{N-t}{p-t}} \right\}.$$

Then for some $c \in (0, c^*)$, there exists a Palais–Smale sequence at level c for J, that is there exists a sequence $\{(u_k, v_k)\} \subset \mathcal{D} \times \mathcal{D}$ such that

$$\lim_{k \to \infty} J(u_k, v_k) = c, \qquad \lim_{k \to \infty} J'(u_k, v_k) = 0 \text{ strongly in } (\mathcal{D} \times \mathcal{D})^{-1}.$$

Proof. We divide the argument into several steps.

Claim 1. The functional J verifies the hypotheses of Lemma 3.1 at any $(u,v) \in \mathcal{D}\times\mathcal{D}$ with $(u_+,v_+) \neq (0,0)$.

In fact, $J \in C^1(\mathcal{D}\times\mathcal{D},\mathbb{R})$, $J(0,0)=0$. From (1.6) it follows that

$$\begin{aligned} J(u,v) &\geq \frac{1}{p}\|(u,v)\|^p - \frac{\eta_1}{p^* S_H(\mu,0)^{\frac{p^*}{p}}}\|(u,v)\|^{p^*} - \frac{\eta_2}{p^*(t) S_Q(\mu,t)^{\frac{p^*(t)}{p}}}\|(u,v)\|^{p^*(t)} \\ &= \Big(C_1 - C_2\|(u,v)\|^{p^*-p} - C_3\|(u,v)\|^{p^*(t)-p}\Big)\|(u,v)\|^p, \end{aligned}$$

where $C_i,\ i=1,2,3$, are positive constants. Then there exist $\lambda, R>0$, such that $J(u,v)\geq\lambda$ for all $(u,v)\in\mathcal{D}\times\mathcal{D}$ with $\|(u,v)\|=R$. Furthermore, for any $(u,v)\in\mathcal{D}\times\mathcal{D}$ with $(u_+,v_+)\neq(0,0)$, we have

$$\lim_{t\to+\infty} J(tu,tv) = -\infty,$$

which implies that there exists $t_{(u,v)}>0$ such that $\|(t_{(u,v)}u, t_{(u,v)}v)\| > R$ and $J(tu,tv)<0$ for all $t>t_{(u,v)}$. Define

$$\Gamma_{(u,v)} := \{\gamma\in C([0,1],\mathcal{D}\times\mathcal{D}) | \gamma(0)=(0,0) \text{ and } \gamma(1)=(t_{(u,v)}u, t_{(u,v)}v)\},$$

$$c_{(u,v)} := \inf_{\gamma\in\Gamma_{(u,v)}} \sup_{t\in[0,1]} J(\gamma(t)).$$

Then the hypotheses of Lemma 3.1 are satisfied and there exists a sequence $\{(u_k,v_k)\}\subset\mathcal{D}\times\mathcal{D}$ such that

$$\lim_{k\to\infty} J(u_k,v_k) = c_{(u,v)}, \qquad \lim_{k\to\infty} J'(u_k,v_k)=0 \text{ strongly in } (\mathcal{D}\times\mathcal{D})^{-1}.$$

In particular, we have that

$$c_{(u,v)} \geq \lambda > 0, \quad \forall\,(u,v)\in\mathcal{D}\times\mathcal{D}\setminus\{(0,0)\}.$$

Claim 2. There exists $(u,v)\in\mathcal{D}\times\mathcal{D}\setminus\{(0,0)\}$ such that $u,v\geq 0$ and

$$c_{(u,v)} < \frac{1}{N}\eta_1^{\frac{p-N}{p}} S_H(\mu,0)^{\frac{N}{p}}.$$

In fact, since $\mu\in[0,\bar{\mu})$, by Theorem 1.1 we can choose $(u,v)=\big(\alpha_1 V^{\varepsilon}_{\mu,0}(x), \beta_1 V^{\varepsilon}_{\mu,0}(x)\big)$, the extremals of $S_H(\mu,0)$. Then

$$\begin{aligned} c_{(u,v)} &\leq \sup_{t\geq0} J(tu,tv) \leq \sup_{t\geq 0} K(t) \\ &= \frac{1}{N}\left(\frac{\|(u,v)\|^p}{\big(\eta_1\int_{\mathbb{R}^N} H(u,v)\big)^{p/p^*}}\right)^{p^*/(p^*-p)} \\ &= \frac{1}{N}\eta_1^{\frac{p-N}{p}} S_H(\mu,0)^{\frac{N}{p}}, \end{aligned}$$

where

$$K(t) := \frac{t^p}{p}\|(u,v)\|^p - \eta_1 \frac{t^{p^*}}{p^*}\int_{\mathbb{R}^N} H(u,v).$$

Let $t_1, t_2 > 0$ be the points where $\sup_{t\geq 0} J(tu,tv)$ and $\sup_{t\geq 0} K(t)$ are attained respectively. Suppose that $J(t_1u, t_1v) = K(t_2)$. Then

$$K(t_1) - \eta_2 \frac{t_1^{p^*(t)}}{p^*(t)}\int_{\mathbb{R}^N} \frac{Q(u,v)}{|x|^t} = K(t_2),$$

which implies that $K(t_2) < K(t_1)$, a contradiction with the definition of t_2. Consequently,

$$c_{(u,v)} \leq \sup_{t\geq 0} J(tu,tv) < \sup_{t\geq 0} K(t) = \frac{1}{N}\eta_1^{\frac{p-N}{p}} S_H(\mu,0)^{\frac{N}{p}}.$$

Claim 3. There exists $(u,v) \in \mathcal{D}\times\mathcal{D}\setminus\{(0,0)\}$ such that $u,v\geq 0$ and

$$0 < c_{(u,v)} < c^*.$$

In fact, by Theorem 1.1 we can choose $(u,v) = \big(\alpha_2 V^{\varepsilon}_{\mu,t}(x), \beta_2 V^{\varepsilon}_{\mu,t}(x)\big) > 0$, the extremals of $S_Q(\mu,t)$. Then arguing as above we can obtain that

$$\begin{aligned} c_{(u,v)} &\leq \sup_{t\geq 0} J(tu,tv) \\ &< \sup_{t\geq 0}\Big(\frac{t^p}{p}\|(u,v)\|^p - \eta_2 \frac{t^{p^*(t)}}{p^*(t)}\int_{\mathbb{R}^N}\frac{Q(u,v)}{|x|^t}\Big) \\ &= \frac{p-t}{p(N-t)}\eta_2^{\frac{p-N}{p-t}} S_Q(\mu,t)^{\frac{N-t}{p-t}}, \end{aligned}$$

which together with claim 2 implies that claim 3 holds.

From Lemma 3.1 and claims 1–3 it follows the conclusions of Lemma 3.2 for a suitable $(u,v)\in\mathcal{D}\times\mathcal{D}$. □

Lemma 3.3. *Let $\{(u_k,v_k)\}\subset\mathcal{D}\times\mathcal{D}$ be a Palais–Smale sequence at the level $c<c^*$ as in Lemma 3.2. If $u_k \rightharpoonup 0$ and $v_k \rightharpoonup 0$ weakly in $\mathcal{D}$ as $k\to\infty$, then there exists $\varepsilon_0>0$ such that for all $\delta>0$, either*

$$\lim_{k\to\infty}\int_{B_\delta(0)} H\big((u_k)_+,(v_k)_+\big) = 0 \quad or \quad \lim_{k\to\infty}\int_{B_\delta(0)} H\big((u_k)_+,(v_k)_+\big) \geq \varepsilon_0.$$

Proof. The argument needs several steps.

Claim 4. For all $\Omega \subset\subset \mathbb{R}^N\setminus\{0\}$, up to a subsequence, we have

$$\lim_{k\to\infty}\int_\Omega \frac{|u_k|^p}{|x|^p} = \lim_{k\to\infty}\int_\Omega \frac{|v_k|^p}{|x|^p} = \lim_{k\to\infty}\int_\Omega \frac{Q(|u_k|,|v_k|)}{|x|^t} = 0, \tag{3.1}$$

$$\lim_{k\to\infty}\int_\Omega |\nabla u_k|^p = \lim_{k\to\infty}\int_\Omega |\nabla v_k|^p = \lim_{k\to\infty}\int_\Omega H(|u_k|,|v_k|) = 0. \tag{3.2}$$

In fact, since $\Omega \subset\subset \mathbb{R}^N \setminus \{0\}$, the embedding $\mathcal{D} \hookrightarrow L^q(\Omega)$ is compact for any $1 \le q < p^*$, $|x|^{-1}$ is bounded on Ω and $p^*(t) < p^*$. Then (3.1) follows from $(\mathcal{H}')$ and we only need to verify (3.2).

Arguing as in Proposition 2 of [18], take $\varphi \in C_0^\infty(\mathbb{R}^N \setminus \{0\})$ such that $0 \le \varphi \le 1$ and $\varphi|_\Omega \equiv 1$. Note that the weak convergence of $\{u_k\}$ and $\{v_k\}$ in $\mathcal{D}$ implies the boundedness. Then

$$\int_{\mathbb{R}^N} |\nabla u_k|^{p-1} |\nabla(\varphi^p)||u_k| \le \|\nabla u_k\|_p^{p-1} \|u_k\|_{L^p(\mathrm{supp}|\nabla\varphi|)} = o(1),$$

$$\int_{\mathbb{R}^N} |\nabla v_k|^{p-1} |\nabla(\varphi^p)||v_k| \le \|\nabla v_k\|_p^{p-1} \|v_k\|_{L^p(\mathrm{supp}|\nabla\varphi|)} = o(1),$$

$$\int_{\mathbb{R}^N} \Big(|\varphi \nabla u_k|^p + |\varphi \nabla v_k|^p \Big) = \int_{\mathbb{R}^N} \Big(|\nabla(\varphi u_k)|^p + |\nabla(\varphi v_k)|^p \Big) + o(1).$$

Furthermore,

$$\begin{aligned} o(1) &= \langle J'(u_k, v_k), (\varphi^p u_k, \varphi^p v_k) \rangle \\ &= \int_{\mathbb{R}^N} \Big(|\varphi \nabla u_k|^p + |\varphi \nabla v_k|^p \Big) - \eta_1 \int_{\mathbb{R}^N} \varphi^p H((u_k)_+, (v_k)_+) \\ &\quad + O\Big(\int_{\mathbb{R}^N} (|\nabla u_k|^{p-1} |\nabla(\varphi^p)||u_k| + |\nabla v_k|^{p-1} |\nabla(\varphi^p)||v_k|) \Big) + o(1) \\ &= \int_{\mathbb{R}^N} \Big(|\varphi \nabla u_k|^p + |\varphi \nabla v_k|^p \Big) - \eta_1 \int_{\mathbb{R}^N} \varphi^p (H((u_k)_+, (v_k)_+) + o(1) \\ &= \int_{\mathbb{R}^N} \Big(|\nabla(\varphi u_k)|^p + |\nabla(\varphi v_k)|^p \Big) - \eta_1 \int_{\mathbb{R}^N} \varphi^p (H((u_k)_+, (v_k)_+) + o(1) \\ &\ge \|\varphi u_k\|^p + \|\varphi v_k\|^p - \eta_1 \int_{\mathbb{R}^N} \varphi^p H((u_k)_+, (v_k)_+) + o(1), \end{aligned}$$

which implies that

$$\begin{aligned} &\|\varphi u_k\|^p + \|\varphi v_k\|^p \\ &\le \eta_1 \int_{\mathbb{R}^N} \varphi^p H((u_k)_+, (v_k)_+) + o(1) \\ &\le \eta_1 \Big(\int_{\mathbb{R}^N} H((u_k)_+, (v_k)_+) \Big)^{(p^*-p)/p^*} \Big(\int_{\mathbb{R}^N} H(|\varphi u_k|, |\varphi v_k|) \Big)^{p/p^*} + o(1) \\ &\le \eta_1 \Big(\int_{\mathbb{R}^N} H((u_k)_+, (v_k)_+) \Big)^{(p^*-p)/p^*} S_H(\mu, 0)^{-1} \|(\varphi u_k, \varphi v_k)\|^p + o(1), \end{aligned}$$

and therefore

$$\left(1 - \eta_1 \Big(\int_{\mathbb{R}^N} H((u_k)_+, (v_k)_+) \Big)^{(p^*-p)/p^*} S_H(\mu, 0)^{-1} \right) \|(\varphi u_k, \varphi v_k)\|^p \le o(1). \tag{3.3}$$

On the other hand,

$$J(u_k, v_k) - \frac{1}{p} \langle J'(u_k, v_k), (u_k, v_k) \rangle = c + o(1)\|(u_k, v_k)\| = c + o(1),$$

which implies that

$$c + o(1) = \eta_1\Big(\frac{1}{p} - \frac{1}{p^*}\Big)\int_{\mathbb{R}^N} H((u_k)_+,(v_k)_+) + \eta_2\Big(\frac{1}{p} - \frac{1}{p^*(t)}\Big)\int_{\mathbb{R}^N} \frac{Q((u_k)_+,(v_k)_+)}{|x|^t}.$$

Consequently,

$$\eta_1 \int_{\mathbb{R}^N} H((u_k)_+,(v_k)_+) \le cN + o(1), \tag{3.4}$$

which together with (3.3) implies that

$$\Big(1 - \eta_1^{\frac{N-p}{N}}(cN)^{p/N} S_H(\mu,0)^{-1}\Big)\|(\varphi u_k,\varphi v_k)\|^p \le o(1).$$

Since $c < c^*$, we have that

$$\lim_{k\to\infty} \|(\varphi u_k,\varphi v_k)\|^p = 0,$$

and therefore

$$\lim_{k\to\infty} \int_{\mathbb{R}^N} H(|\varphi u_k|,|\varphi v_k|) = 0.$$

Then the definition of φ implies that (3.2) holds and claim 4 is proved.

Claim 5. For all $\delta > 0$, define the quantities:

$$\tau = \limsup_{k\to\infty}\int_{B_\delta(0)} H((u_k)_+,(v_k)_+), \qquad \omega = \limsup_{k\to\infty}\int_{B_\delta(0)} \frac{Q((u_k)_+,(v_k)_+)}{|x|^t},$$

$$\gamma = \limsup_{k\to\infty}\int_{B_\delta(0)}\Big(|\nabla u_k|^p + |\nabla v_k|^p - \mu\frac{|u_k|^p + |v_k|^p}{|x|^p}\Big).$$

Then

$$S_H(\mu,0)\,\tau^{\frac{p}{p^*}} \le \gamma, \qquad S_Q(\mu,t)\,\omega^{\frac{p}{p^*(t)}} \le \gamma. \tag{3.5}$$

Furthermore,

$$\gamma \le \eta_1\tau + \eta_2\omega. \tag{3.6}$$

In fact, according to claim 4, τ,ω and γ are well–defined and independent of δ. Take $\varphi \in C_0^\infty(\mathbb{R}^N)$ such that $0 \le \varphi \le 1$ and $\varphi|_{B_\delta(0)} \equiv 1$. Then we have

$$S_H(\mu,0)\Big(\int_{\mathbb{R}^N} H((\phi u_k)_+,(\phi v_k)_+)\Big)^{\frac{p}{p^*}} \le \|(\varphi u_k,\varphi v_k)\|^p.$$

As $k\to\infty$, claim 4 implies that

$$\begin{aligned} &S_H(\mu,0)\Big(\int_{B_\delta(0)} H((u_k)_+,(v_k)_+)\Big)^{\frac{p}{p^*}} \\ &\le \int_{B_\delta(0)}\Big(|\nabla u_k|^p + |\nabla v_k|^p - \mu\frac{|u_k|^p + |v_k|^p}{|x|^p}\Big) + o(1). \end{aligned}$$

Consequently,

$$S_H(\mu,0)\,\tau^{\frac{p}{p^*}} \le \gamma.$$

The second inequality in (3.5) can be verified similarly.

Since $\varphi u_k, \varphi v_k \in \mathcal{D}$ and $\lim_{k\to\infty}\langle J'(u_k,v_k),(\varphi u_k,\varphi v_k)\rangle = 0$, by claim 4 and the definitions of τ,ω and γ, we deduce that $\gamma \le \eta_1\tau + \eta_2\omega$. Claim 5 is verified.

From (3.6) it follows that

$$S_H(\mu,0)\,\tau^{\frac{p}{p^*}} \le \gamma \le \eta_1\tau + \eta_2\omega,$$

which implies that

$$\tau^{\frac{p}{p^*}}\big(S_H(\mu,0) - \eta_1\tau^{\frac{p^*-p}{p^*}}\big) \le \eta_2\omega. \tag{3.7}$$

From (3.4) it follows that

$$\eta_1\tau \le cN < c^*N < \eta_1^{\frac{p-N}{p}} S_H(\mu,0)^{\frac{N}{p}} = \eta_1^{\frac{p-N}{p}} S_H(\mu,0)^{\frac{p^*}{p^*-p}}. \tag{3.8}$$

By (3.7) and (3.8), there exists a constant $C_1 = C_1(\mu,c,\eta_1,\eta_2) > 0$ such that

$$\tau^{\frac{p}{p^*}} \le C_1\omega. \tag{3.9}$$

Similarly, there exists a positive constant $C_2 = C_2(\mu,c,t,\eta_1,\eta_2)$ such that

$$\omega^{\frac{p}{p^*(t)}} \le C_2\tau. \tag{3.10}$$

Then it follows from (3.9) and (3.10) that there exists a positive constant $\varepsilon_0 = \varepsilon_0(N,p,\mu,c,t)$ such that

$$\text{either}\quad \tau=\omega=0 \quad\text{ or }\quad \min\{\tau,\omega\} \ge \varepsilon_0.$$

The proof of Lemma 3.3 is complete. □

4 Existence of positive solutions

Lemma 4.1. *Let $\{(u_k,v_k)\}$ be the sequence defined as in Lemma* 3.3. *Then*

$$\Lambda := \limsup_{k\to\infty}\int_{\mathbb{R}^N} H((u_k)_+,(v_k)_+) > 0. \tag{4.1}$$

Proof. Arguing by contradiction, we assume that

$$\lim_{k\to\infty}\int_{\mathbb{R}^N} H((u_k)_+,(v_k)_+) = 0. \tag{4.2}$$

Since $\lim_{k\to\infty}\langle J'(u_k,v_k),(u_k,v_k)\rangle = 0$, by (4.1) we have

$$\|(u_k,v_k)\|^p = \eta_2\int_{\mathbb{R}^N}\frac{Q((u_k)_+,(v_k)_+)}{|x|^t} + o(1),\quad k\to\infty.$$

Then

$$
\begin{aligned}
&S_Q(\mu,t)\Big(\int_{\mathbb{R}^N}\frac{Q((u_k)_+,(v_k)_+)}{|x|^t}\Big)^{\frac{p}{p^*(t)}}\\
&\le \|(u_k,v_k)\|^p=\eta_2\int_{\mathbb{R}^N}\frac{Q((u_k)_+,(v_k)_+)}{|x|^t}+o(1),
\end{aligned}
$$

$$
\begin{aligned}
&\Big(\int_{\mathbb{R}^N}\frac{Q((u_k)_+,(v_k)_+)}{|x|^t}\Big)^{\frac{p}{p^*(t)}}\\
&\times\bigg(S_Q(\mu,t)-\eta_2\Big(\int_{\mathbb{R}^N}\frac{Q((u_k)_+,(v_k)_+)}{|x|^t}\Big)^{\frac{p^*(t)-p}{p^*(t)}}\bigg)\le o(1).
\end{aligned}
\tag{4.3}
$$

From (3.4) and (4.2) it follows that

$$
\eta_2\int_{\mathbb{R}^N}\frac{Q((u_k)_+,(v_k)_+)}{|x|^t}=\frac{cp(N-t)}{p-t}+o(1)<\frac{c^*p(N-t)}{p-t}+o(1),
$$

which together with (4.3) implies that

$$
\lim_{k\to\infty}\int_{\mathbb{R}^N}\frac{Q((u_k)_+,(v_k)_+)}{|x|^t}=0,
$$

a contradiction with (3.4) and the fact that $c\in(0,c^*)$. □

Lemma 4.2. *Let $\{(u_k,v_k)\}$ be defined as in Lemma 3.3. Then there exists $\varepsilon_1\in(0,\varepsilon_0/2]$, with ε_0 given in Lemma 3.3, such that for all $\varepsilon\in(0,\varepsilon_1)$, there exists a positive sequence $\{r_k\}\subset\mathbb{R}$ such that $\{(\tilde u_k,\tilde v_k)\}:=\big\{\big(r_k^{(N-p)/p}u_k(r_kx),r_k^{(N-p)/p}v_k(r_kx)\big)\big\}\subset\mathcal{D}\times\mathcal{D}$, is again a Palais–Smale sequence of the type given in Lemma 3.3 and satisfies*

$$
\int_{B_1(0)}H((\tilde u_k)_+,(\tilde v_k)_+)=\varepsilon,\qquad\forall\,k\in\mathbb{N}.
\tag{4.4}
$$

Proof. Let ε_0,Λ be defined as in Lemma 3.3 and (4.1) respectively. Set $\varepsilon_1:=\min\{\varepsilon_0/2,\Lambda\}$ and fix $\varepsilon\in(0,\varepsilon_1)$. Up to a subsequence (still denoted by $\{(u_k,v_k)\}$), for any $k\in\mathbb{N}$, there exists $r_k>0$ such that

$$
\int_{B_{r_k}(0)}H((u_k)_+,(v_k)_+)=\varepsilon,\qquad\forall\,k\in\mathbb{N}.
$$

Then the scaling invariance implies that $\{(\tilde u_k,\tilde v_k)\}$ satisfies (4.4) and is also a Palais–Smale sequence of the type given in Lemma 3.3. □

Proof of Theorem 1.2. Since $\{(\tilde u_k,\tilde v_k)\}$ satisfies (4.4) and is also a Palais–Smale sequence, we have that

$$
\begin{aligned}
&C(1+\|(\tilde u_k,\tilde v_k)\|)\\
&\ge J(\tilde u_k,\tilde v_k)-\frac{1}{p^*(t)}\langle J'(\tilde u_k,\tilde v_k),(\tilde u_k,\tilde v_k)\rangle\\
&=\Big(\frac1p-\frac{1}{p^*(t)}\Big)\|(\tilde u_k,\tilde v_k)\|^p+\eta_1\Big(\frac{1}{p(t)}-\frac{1}{p^*}\Big)\int_{\mathbb{R}^N}H((\tilde u_k)_+,(\tilde v_k)_+)\\
&\ge\Big(\frac1p-\frac{1}{p^*(t)}\Big)\|(\tilde u_k,\tilde v_k)\|^p,
\end{aligned}
$$

which implies that $\{(\tilde{u}_k, \tilde{v}_k)\}$ is bounded in $\mathcal{D} \times \mathcal{D}$. Up to a subsequence, there exists $\tilde{u}, \tilde{v} \in \mathcal{D}$ such that

$$\tilde{u}_k \rightharpoonup \tilde{u} \;\; \text{weakly}, \quad \tilde{v}_k \rightharpoonup \tilde{v} \;\; \text{weakly}, \quad k \to \infty.$$

If $\tilde{u} \equiv \tilde{v} \equiv 0$, from Lemma 3.3 it follows that either

$$\lim_{k\to\infty} \int_{B_1(0)} H((\tilde{u}_k)_+, (\tilde{v}_k)_+) = 0 \quad \textit{or} \quad \lim_{k\to\infty} \int_{B_1(0)} H((\tilde{u}_k)_+, (\tilde{v}_k)_+) \geq \varepsilon_0,$$

which contradicts (4.4) as $0 < \varepsilon < \varepsilon_0/2$. Then $(\tilde{u}, \tilde{v}) \not\equiv (0, 0)$. Arguing as in [12] (see also [13, 31, 33]), we deduce that $(\tilde{u}, \tilde{v})$ is a solution of the following problem:

$$\begin{cases} -\Delta_p u - \mu \dfrac{u^{p-1}}{|x|^p} = \dfrac{\eta_1}{p^*} H_u(u_+, v_+) + \dfrac{\eta_2}{p^*(t)} \dfrac{Q_u(u_+, v_+)}{|x|^t}, \\ -\Delta_p v - \mu \dfrac{v^{p-1}}{|x|^p} = \dfrac{\eta_1}{p^*} H_v(u_+, v_+) + \dfrac{\eta_2}{p^*(t)} \dfrac{Q_v(u_+, v_+)}{|x|^t}, \\ (u, v) \in \mathcal{D} \times \mathcal{D}. \end{cases} \tag{4.5}$$

Set $w_- = \max\{-w, 0\}$ for all $w \in \mathcal{D} \setminus \{0\}$. Multiplying the first equation in (4.5) by $\tilde{u}_-$, the second by $\tilde{v}_-$, and integrating, we have that $\|\tilde{u}_-\| = \|\tilde{v}_-\| = 0$, which implies that $\tilde{u}_- = \tilde{v}_- = 0$ and therefore $(\tilde{u}, \tilde{v})$ is a nonnegative nontrivial solution of (4.5). If $\tilde{u} \equiv 0$, by $(\mathcal{H})$ and (4.5) we get $\tilde{v} \equiv 0$. Similarly, $\tilde{v} \equiv 0$ also implies $\tilde{u} \equiv 0$. Then $\tilde{u} \not\equiv 0$ and $\tilde{v} \not\equiv 0$. From the maximum principle it follows that $\tilde{u}, \tilde{v} > 0$ in $\mathbb{R}^N$ and $(\tilde{u}, \tilde{v})$ is a solution of the problem (1.1).

The proof of Theorem 1.2 is complete. □

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