Global existence and blowing up of solutions for some non-linear wave equations*

Wei Xiao †

(School of Science, Chang'An University, Xi'an, 710064, P. R. China)

Abstract: In this paper, an initial boundary value problem for a system of semi-linear hyperbolic equations with damped term in a bounded domain is considered. We prove the global existence,uniqueness and blow up of solutions and give some estimates for the lifespan of solutions.

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1 Introduction

In the paper, we study the behavior of local solutions of the following super-linear hyperbolic equations with (possibly strong) damping

$$\begin{cases} u_{ktt} - \Delta u_k - \omega \Delta u_{kt} + \mu u_{kt} + f_k(u) = a_k |u_k|^{p-2} u_k & in\ [0,T]\times\Omega \\ u(0,x) = \varphi(x) \qquad u_t(0,x) = \psi(x) & in\ \Omega \qquad\qquad k=1,2. \\ u(t,x) = 0 & on\ [0,T]\times\partial\Omega \end{cases} \tag{1}$$

where Ω is an open bounded Lipschitz subset of R^n $(n \geq 1), T > 0$, $u = (u_1, u_2)$ is unknown, a_k is constant and $a_k > 0$, $p > 2$, f_k is a known continuously differentiable function . We study the behavior of solutions to (1) in the phase space $H_0^1(\Omega) \times H_0^1(\Omega)$. Since stationary solutions play a crucial role in the description of the evolution of (1), several tools from critical point theory turn out to be quite useful for our purposes.

In particular, we consider the mountain pass energy level d (see [1]), the Nehari manifold N (see [20]) of the stationary problem associated to (1) and the two unbounded sets N_+ (inside N) and N_- (outside N). All these tools are defined in detail in Section 2.

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†Email:xiaowei1802002@163.com

A first attempt to tackle it with these tools was made by Sattinger ([26]) who developed the so called potential well theory in order to study the problem with no damping (that is $\omega = \mu = 0$). In the wave equation blow-up literature, there have been significant progress made by Merle and Zaag [17,18] and [19] for the semilinear wave equation

$$u_{tt} = \Delta u + |u|^{p-1}u$$

where $p > 1$ and $p \leq 1 + \frac{4}{N-1}$ if $N \geq 2$. Subsequently, equations with damping terms have been considered by many authors. For equations with (possibly nonlinear) weak damping we refer to ([8] [21] [22] [24] [29] [30]). Much less is known for equations with strong damping; see the paper([9] [12] [23] [27]), but still many problems remain unsolved. It is our purpose to shed some further light on damped wave equations of the kind in both the cases of weak ($\omega = 0$) and strong ($\omega > 0$) damping. As recently done by the first author in ([7] [9]), we will exploit further the properties of the Nehari manifold. However we mention that, by exploiting a completely different method, the existence of solutions with arbitrarily high initial energy has been also obtained in ([11]) for weakly damped wave equations on the whole R^n. Cazenave ([2]) proved bounded-ness of global solutions for $\omega = \mu = 0$ while Esquivel-Avila ([5] [6]) recovered the same result for $\omega = 0$ and $\mu > 0$ and showed that this property may fail in presence of a nonlinear dissipation term.

Motivated by the papers ([3] [4] [9] [13] [15] [16] [31]), in the present paper we consider problem (1). We shall discuss the existence, uniqueness and blow-up properties of solutions in for a system (1) in a bounded domain Ω in R^n. The method mainly used here are the Galerkin method(see [14]), the fixed point method, the potential well method and the concave method.

The paper is organized as follows. In section 2, we first recall the notations used throughout this paper and the main assumptions. In section 3, we present the main result of the paper. From section 4 to 6, we provide the proofs of the results.

2 Notations and assumptions

First we give the notations used throughout this paper. Let $W^{m,q}(\Omega)$ be the usual Sobolev space and its norm is denoted by $\|\cdot\|_{m,q}$. Specially, $W^{m,2}(\Omega)$ and $W^{0,q}(\Omega)$ will be marked by $H^m(\Omega)$ and $L^q(\Omega)$, respectively. Moreover, the norm of $L^q(\Omega)$ is denoted by $\|\cdot\|_q$ and when $q = 2$, the corresponding norm will be written as $\|\cdot\|$ simply.We denote by $(\cdot,\cdot)$ the inner product of $L^2(\Omega)$. It is well known that $H_0^1(\Omega)$ is the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|u\|_{H_0^1} = \|\nabla u\|$. Let X be a Banach space, then $L^q([0,T],X)$ and $C^k([0,T],X)$ stand for the Banach space of the strongly measurable X-valued functions $u : [0,T] \to X$ with $\|u(\cdot)\|_X \in L^q([0,T])$ and $C^k([0,T])$, respectively, where $k \geq 0$ and $\|\cdot\|_X$ is the norm defined on X.

Moreover, we denote by $\langle\cdot,\cdot\rangle$ the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$. When $\omega > 0$ (resp. $\omega = 0$) for all $v, w \in H_0^1(\Omega)$ (resp. for all $v, w \in L^2(\Omega)$), we put

$$(v,w)_* = \omega \int_\Omega \nabla v \cdot \nabla w + \mu \int_\Omega vw, \|v\|_* = (v,v)_*^{1/2};$$

$\|\cdot\|_*$ is an equivalent norm over $H_0^1(\Omega)$ (resp.$L^2(\Omega)$).

We may consider the C^1 functionals $I_k, J_k : H_0^1(\Omega) \to R$ defined by

$$I_k(u) = \|\nabla u\|_2^2 - a_k\|u\|_p^p \qquad \textit{and} \qquad J_k(u) = \frac{1}{2}\|\nabla u\|_2^2 - \frac{a_k}{p}\|u\|_p^p$$

The mountain pass value of J_k (also known as potential well depth) is defined as

$$d_k = \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \max_{\lambda \geq 0} J_k(\lambda u)$$

All nontrivial stationary solutions belong to the so-called Nehari manifold (see [20] and also [28]) defined by

$$N_k = \{u \in H_0^1(\Omega)\setminus\{0\} : I_k(u) = 0\}.$$

It is easy to show that each half line starting from the origin of $H_0^1(\Omega)$ intersects exactly once the manifold N_k and that N_k separates the two unbounded sets

$$N_{k+} = \{u \in H_0^1(\Omega) : I_k(u) > 0\} \cup \{0\}$$

$$N_{k-} = \{u \in H_0^1(\Omega) : I_k(u) < 0\} \cup \{0\}$$

We also consider the (closed) sub-levels of J_k

$$J_k^a = \{u \in H_0^1(\Omega) : J_k(u) \leq a\} \quad (a \in R)$$

and we introduce the stable set W_k and the unstable set U_k defined by

$$W_k = J_k^d \cap N_{k+} \qquad \textit{and} \qquad U_k = J_k^d \cap N_{k-}$$

Here $d = \min\{d_1, d_2\}$, d is the mountain pass level.

It is readily seen (see [28]) that the mountain pass level d_k may also be characterized as

$$d_k = \inf_{u \in N_k} J_k(u)$$

Finally, we consider the energy functional $E(t)$

$$E(t) = \sum_{k=1}^{2}(\frac{1}{2}\|u_{kt}(t)\|^2 + \frac{1}{2}\|\nabla u_k\|_2^2 - \frac{a_k}{p}\|u_k\|_p^p) + \int_\Omega F(u(t))dx.$$

where $F(u) = \int_0^{u_1} f_1(s, u_2)ds + \int_0^{u_2} f_2(0, s)ds$

If $\frac{\partial f_1}{\partial u_2} = \frac{\partial f_2}{\partial u_1}$ hold, then

$$E(t) + \sum_{k=1}^{2}\int_s^t \|u_{kt}(\tau)\|_*^2 d\tau = E(s), \qquad 0 \le s < t < T_{max} \tag{$*$}$$

Define

$$S = \{u \mid u \in C^0([0,T], H_0^1(\Omega)) \cap C^1([0,T], L^2(\Omega)) \cap C^2([0,T], H^{-1}(\Omega)), u_t \in L^2([0,T], H_0^1(\Omega))\}$$

Definition 1. A function $u = (u_1, u_2)$ is called a weak solution of the initial boundary value problem (1), if $u_k(x,t) \in S$ for $k = 1, 2$, and satisfy

$$\langle u_{ktt}, \eta\rangle + \int_\Omega \nabla u_k(t)\nabla\eta + \omega\int_\Omega \nabla u_{kt}(t)\nabla\eta + \mu\int_\Omega u_k(t)\eta + \int_\Omega f_k(u)\eta = \int_\Omega a_k|u_k|^{p-2}u_k\eta$$

holds for $\forall\eta \in H_0^1(\Omega)$ and a.e.$t \in [0,T]$

Now we make the following assumptions.

(A1)

$$\varphi(x) = (\varphi_1(x), \varphi_2(x)) \in H_0^1(\Omega) \times H_0^1(\Omega), \psi(x) = (\psi_1(x), \psi_2(x)) \in L^2(\Omega) \times L^2(\Omega) \tag{2}$$

$$\omega \ge 0, \mu > -\omega\lambda_1 \tag{3}$$

λ_1 being the first eigenvalue of the operator $-\Delta$ under homogeneous Dirichlet boundary conditions.

(A2)

$$2 < p \le \begin{cases} \frac{2n}{n-2} & \omega > 0 \\ \frac{2n-2}{n-2} & \omega = 0 \end{cases} \quad if\ n \ge 3, \quad and \quad 2 < p < \infty,\ if\ \ n = 1, 2. \tag{4}$$

(A3) $f_k : R^2 \to R$ is continuously differentiable such that for each $u = (u_1, u_2) \in H_0^1(\Omega) \times H_0^1(\Omega)$, $u_k f_k(u) \in L^1(\Omega), k = 1, 2$, $F(u) \in L^1(\Omega)$.

(A4) $f_k : H_0^1(\Omega) \times H_0^1(\Omega) \to L^2(\Omega), k = 1, 2$, satisfies a local Lipschitz condition, i.e., for any $\delta > 0$, there exists a positive constant $C(\delta)$ such that
$\|f_k(u) - f_k(v)\| \leq C(\delta)\|u - v\|_{H_0^1 \times H_0^1}$ for $u, v \in H_0^1 \times H_0^1$ with
$\|u\|_{H_0^1 \times H_0^1} \leq \delta$, $\|v\|_{H_0^1 \times H_0^1} \leq \delta$.

(A5)$\dfrac{\partial f_1}{\partial u_2} = \dfrac{\partial f_2}{\partial u_1}$.

(A6) $u_1 f_1 + u_2 f_2 \geq F(u) \geq 0, \forall u_1, u_2 \in R$.

(A7) There exists a positive constant $n_0 \geq 1/p$ such that
$\max\{u_1 f_1 + u_2 f_2, 0\} \leq \dfrac{1}{n_0} F(u), \forall u_1, u_2 \in R$.

Remark1: consider a particular problem (1)in $R^n (n \leq 3)$ with $f_1(u_1, u_2) = \gamma u_1 u_2^2$, $f_2(u_1, u_2) = \gamma u_1^2 u_2$, γ is a constant. If $\gamma \geq 0$, we see that f_i and F satisfy the assumptions (A2-A6). Moreover let $n_0 = \frac{1}{4}$, take $p = 3$ whether $n \leq 2$ or $n = 3$, we see that p, f_i and F satisfy assumptions (A2-A5) and (A7).

3 The main results

Theorem 1. Let the assumptions (A1)-(A5) be fulfilled. Then problem (1) admits a unique weak solution (u_1, u_2) defined on $[0, T_{max})$, and at least one of the following statements is valid:
(1) $T_{max} = \infty$;
(2) $T_{max} < \infty$, and

$$\lim_{t \to T_{max}} \sum_{k=1}^{2} \|u_{kt}(t)\|_2^2 + \|\nabla u_k\|_2^2 = \infty$$

$$E(t) + \sum_{k=1}^{2} \int_0^t \|u_{kt}(\tau)\|_*^2 d\tau = E(0)$$

Definition 2. Let $T_{max} = \sup\{T > 0 : v = v(x, t) \text{ exists on } [0, T)\}$. If $T_{max} < \infty$, we say that the solution to (1) blows up and that T_{max} is the blow up time. If $T_{max} = \infty$, we say that the solution is global.

Theorem 2. Assume that (A1)-(A6) hold and let (u_1, u_2) be the unique local solution to (1). In addition, assume that there exists $\bar{t} \in [0, T_{max})$ such that

$$u_k(\bar{t}) \in W_k \text{ and } E(\bar{t}) \leq d.$$

where $d = \min\{d_1, d_2\}$. Then $T_{max} = \infty$. And for every $t > \bar{t}$,

$$\|\nabla u_k(t)\|_2^2 + \|u_{kt}(t)\|_2^2 \leq \frac{\Theta(\omega, \mu)}{t}$$

where

$$\Theta(\omega, \mu) = \begin{cases} C_\mu(1 + \frac{1}{\omega} + \omega) & \omega > 0 \\ C(1 + \frac{1}{\mu} + \mu) & \omega = 0 \end{cases} \tag{5}$$

and C is independent of μ , whereas C_μ only depends on μ.

Theorem 3. Let the assumptions (A1-A5) and (A7) hold. If $I_1(\varphi_1) < 0$ and $I_2(\varphi_2) < 0$, $E(0) < \min\{d_1, d_2\}$. Then the solution (u_1, u_2) blows up in finite time, i.e., there exists T_{max} such that

$$\lim_{t \to T-} \sum_{k=1}^{2} [\|u_{kt}(t)\|^2 + \|\nabla u_k(t)\|^2] = \infty$$

and an upper bound for T_{max} is estimated

$$T_{max} \leq \frac{1}{2\theta^2\beta} \left\{ \left[\left(\sum \{2\theta(\varphi_k, \psi_k) - \|\nabla \varphi_k\|^2\} \right)^2 + 4\theta^2\beta \sum \|\varphi_k\| \right]^{\frac{1}{2}} + \{\|\nabla \varphi_k\|^2 - 2\theta(\varphi_k, \psi_k)\} \right\}$$

where $\beta = 2(\sum_{k=1}^{2} d_k - E(0))$, $\theta = \dfrac{p-2}{4}$.

4 Proof of Theorem 1

In the paper, we restrict ourselves to the case $\omega > 0$, $\mu \neq 0$ and $n \geq 3$.

For a given $T > 0$, consider the space $H = C([0, T], H_0^1(\Omega)) \cap C^1([0, T], L^2(\Omega))$ endowed with the norm

$$\|u\|_H^2 = \max_{t \in [0,T]} (\|\nabla u(t)\|_2^2 + \|u_t(t)\|_2^2)$$

Lemma 4.1. For every $T > 0$, every $f \in H$ and every initial data $(\varphi, \psi) \in H_0^1(\Omega) \times L^2(\Omega)$, there exists a unique $v \in H \cap C^2([0, T], H^{-1}(\Omega))$ such that $v_t \in L^2([0, T], H_0^1(\Omega))$ which solves the linear problem

$$\begin{cases} v_{tt} - \Delta v - \omega \Delta v_t + \mu v_t = f(t,x) & in\ [0,T] \times \Omega \\ v(0,x) = \varphi(x) \qquad v_t(0,x) = \psi(x) & in\ \Omega \\ v = 0 & on\ [0,T] \times \partial\Omega \end{cases} \tag{6}$$

Proof: For every $h \geq 1$, let $W_h = Span\{w_1, ..., w_h\}$, where $\{w_j\}$ is the orthogonal complete system of eigenfunctions of $-\triangle$ in $H_0^1(\Omega)$ such that $\|w_j\|_2 = 1$ for all j. Then, $\{w_j\}$ is orthogonal and complete in $L^2(\Omega)$ and in $H_0^1(\Omega)$; denote by $\{\lambda_j\}$ the related eigenvalues. Let $\varphi^h(t) = \sum_{j=1}^{h}(\int_\Omega \nabla\varphi \nabla w_j) w_j$ and $\psi^h(t) = \sum_{j=1}^{h}(\int_\Omega \psi w_j) w_j$ so that $\varphi^h \in W_h$, $\psi^h \in W_h$, $\varphi^h \to \varphi$ in $H_0^1(\Omega)$ and $\psi^h \to \psi$ in $L^2(\Omega)$ as $h \to \infty$. For all $h \geq 1$ we seek h functions $\gamma_1^h, \cdots, \gamma_h^h \in C^2[0,T]$ such that

$$v_h(t) = \sum_{j=1}^{h} (\gamma_j^h(t)) w_j \tag{7}$$

solves the problem

$$\begin{cases} \int_\Omega [\ddot{v}_h(t) - \Delta v_h(t) - \omega \Delta \dot{v}_h(t) + \mu \dot{v}_h(t)] \eta dx = \int_\Omega f(t,x) \eta(x) dx \\ v_h(0) = \varphi^h \qquad \dot{v}_h(0) = \psi^h \end{cases} \tag{8}$$

for every $\eta \in W_h$ and $t \geq 0$. For $j = 1, \cdots, h$, taking $\eta = w_j$ in (8) yields the following Cauchy problem for a linear ordinary differential equation with unknown γ_j^h:

$$\begin{cases} \ddot{\gamma}_j^h(t) + (\omega\lambda_j + \mu)\dot{\gamma}_j^h(t) + \lambda_j \gamma_j^h(t) = \Psi_j(t) \\ \gamma_j^h = \int_\Omega \varphi w_j dx \qquad \dot{\gamma}_j^h = \int_\Omega \psi w_j dx \end{cases} \tag{9}$$

where $\Psi_j(t) = \int_\Omega f(t,x) w_j(x) dx \in C[0,T]$. For all j , the above Cauchy problem yields a unique global solution $\gamma_j^h \in C^2[0,T]$. In particular, (7) implies that $\dot{v}_h(t) \in H_0^1(\Omega)$ for every $t \in [0,T]$ so that Sobolev inequality entails
$\|\dot{v}_h\|_{2^*} \leq C \|\nabla \dot{v}_h\|_2$, where $2^* = \dfrac{n(p-2)}{2}$, for every $t \in [0,T]$. Taking $\eta = \dot{v}_h$ into (9), and integrating over $[0,t] \subset [0,T]$, we obtain

$$\|\nabla v_h(t)\|_2^2 + \|\dot{v}_h(t)\|_2^2 + 2\int_0^t \|\dot{v}_h(\tau)\|_*^2 d\tau = \|\nabla\varphi^h\|_2^2 + \|\psi^h\|_2^2 + 2\int_0^t \int_\Omega f(\tau,x)\dot{v}_h(\tau) d\tau \tag{10}$$

for every $h \geq 1$. We estimate the last term in the right-hand side thanks to Holder, Sobolev and Young inequalities

$$2\int_0^t \int_\Omega f(\tau, x)\dot{v}_h(\tau)d\tau \leq CT + \int_0^t \|\dot{v}_h(\tau)\|_*^2 d\tau \tag{11}$$

Recalling that from (10) and (11) we obtain

$$\|v_h\|_H^2 + \int_0^T \|\dot{v}_h(\tau)\|_*^2 d\tau \leq C(T)$$

for every $h \geq 1$, where $C(T) > 0$ is independent of h. By this uniform estimate and using (8), we have:

$\{v_h\}$ is bounded in $L^\infty([0,T], H_0^1(\Omega))$

$\{\dot{v}_h\}$ is bounded in $L^\infty([0,T], L^2(\Omega)) \cap L^2([0,T], H_0^1(\Omega))$

$\{\ddot{v}_h\}$ is bounded in $L^2([0,T], H^{-1}(\Omega))$

Therefore, up to a subsequence, we may pass to the limit in (8) and obtain a weak solution v of (6) with the above regularity. Since $v \in H^1([0,T], H_0^1(\Omega))$, we get $v \in C([0,T], H_0^1(\Omega))$. Moreover, since $\dot{v} \in L^\infty([0,T], L^2(\Omega)) \cap L^2([0,T], H_0^1(\Omega))$ and $\ddot{v} \in L^2([0,T], H^{-1}(\Omega))$,we have $\dot{v} \in C([0,T], L^2(\Omega))$. Finally, from (6) we get $\ddot{v} \in C([0,T], H^{-1}(\Omega))$. The existence of v solving (6) is so proved.

Uniqueness follows arguing for contradiction: if v and w were two solutions of (6) which share the same initial data, by subtracting the equations and testing with $v_t - w_t$, instead of (10) we would get

$$\|\nabla v(t) - \nabla w(t)\|_2^2 + \|v_t(t) - w_t(t)\|_2^2 + 2\int_0^t \|v_t(\tau) - w_t(\tau)\|_*^2 d\tau = 0$$

which immediately yields $w \equiv v$. The proof of the lemma is now complete.

Take (φ, ψ) satisfying (2), let $R^2 = 2(\|\nabla\varphi\|_2^2 + \|\psi\|_2^2)$ and for any $T > 0$ consider

$$\Gamma_T = \{u \in H : u(0) = \varphi, u_t(0) = \psi \ and \ \|u\|_H \leq R\}$$

By Lemma 4.1, for any $u \in \Gamma_T \times \Gamma_T$, we may define $v = \Phi(u)$, being v the unique solution to problem

$$\begin{cases} (v_{ktt} - \Delta v_k - \omega\Delta v_{kt} + \mu v_{kt} = a_k|u_k|^{p-2}u_k - f_k(u) & in\ [0,T]\times\Omega \\ v(0,x) = \varphi(x) \qquad v_t(0,x) = \psi(x) & in\ \Omega \qquad\qquad k = 1,2; \\ u(t,x) = 0 & on\ [0,T]\times\partial\Omega \end{cases} \tag{12}$$

We claim that, for a suitable $T > 0$, Φ is a contractive map satisfying $\Phi(\Gamma_T \times \Gamma_T) \subseteq \Gamma_T \times \Gamma_T$. Given $u \in \Gamma_T$, the corresponding solution $v = \Phi(u)$ satisfies for all $t \in (0, T]$ the energy identity :

$$\begin{aligned} & \|\nabla v_k(t)\|_2^2 + \|v_{kt}(t)\|_2^2 + 2\int_0^t \|v_{kt}(\tau)\|_*^2 d\tau \\ = \quad & \|\nabla \varphi\|_2^2 + \|\psi\|_2^2 + 2\int_0^t \int_\Omega f_k(u) v_t(\tau) d\tau + 2\int_0^t \int_\Omega a_k |u_k|^{p-2} u_k v_t(\tau) d\tau \end{aligned} \tag{13}$$

For the last term, we argue in the same spirit (although slightly differently) as for (11) and we get

$$\begin{aligned} & 2\int_0^t \int_\Omega a_k |u_k|^{p-2} u_k v_t(\tau) d\tau \leq C \int_0^T \|u_k\|_{2^*}^{p-1} \|v_t\|_{2^*} d\tau \\ & \leq C \int_0^T \|u_k\|_*^{p-1} \|v_t\|_* d\tau \leq CTR^{2(p-1)} + 2\int_0^T \|v_t\|_*^2 d\tau \end{aligned} \tag{14}$$

and

$$2\int_0^t \int_\Omega f_k(u) v_t(\tau) d\tau \leq C \int_0^t \|f_k(u)\|_2 \|v_t(\tau)\|_2 d\tau \leq CRT + 2\int_0^T \|v_t\|_*^2 d\tau$$

for all $t \in (0, T]$. Combining (13) with (14) and taking the maximum over $[0, T]$ gives

$$\|v\|_{H\times H}^2 \leq \frac{1}{2} R^2 + CTR^{2(p-1)}$$

Choosing T sufficiently small, we get

$$\|v\|_{H\times H}^2 \leq R$$

which shows that $\Phi(\Gamma_T \times \Gamma_T) \subseteq \Gamma_T \times \Gamma_T$. Now,take w_1 and w_2 in $\Gamma_T \times \Gamma_T$. subtracting the two equations (6) for $v_1 = \Phi(w_1)$ and $v_2 = \Phi(w_2)$, and setting $v = v_1 - v_2$ we obtain for all $\eta \in H_0^1(\Omega)$ and a.e. $t \in (0, T]$

$$\begin{aligned} & \langle v_{ktt}, \eta \rangle + \int_\Omega \nabla v_k(t) \nabla \eta + \omega \int_\Omega \nabla v_{kt}(t) \nabla \eta + \mu \int_\Omega v_k(t) \eta \\ & = \int_\Omega a_k (|v_{1k}|^{p-2} v_{1k} - |v_{2k}|^{p-2} v_{2k}) \eta - (\int_\Omega f_k(v_1) - f_k(v_2) \eta) \end{aligned} \tag{15}$$

by Lagrange Theorem and (A4). Therefore, by taking $\eta = v_{kt}$ in (15) and arguing as above, we obtain

$$\|\Phi(w_1)-\Phi(w_2)\|^2_{H\times H}=\|v\|^2_{H\times H}\le CR^{2p-4}T\|w_1-w_2\|^2_{H\times H}\le\delta\|w_1-w_2\|^2_{H\times H}$$

for some $\delta<1$ provided T is sufficiently small. This proves the claim. By the Contraction Mapping Principle, there exists a unique (weak) solution to (1) defined on $[0,T]$.

Exploiting a standard continuation principle (see [25]), and multiplying by u_{kt} both sides of (1) and then integrating by parts over $[0,t)\times\Omega$, by use of the assumptions (A1) - (A5), we get that Theorem 1.

5 Proof of Theorem 2

In the following we consider the case $\omega>0$. Without loss of generality, we may assume that $\bar{t}=0$. we know that the energy map $E(t)$ is decreasing. Then, if condition

$$u_k(\bar{t})\in W_k \; and \; E(\bar{t})\le d \tag{16}$$

holds true, we have $u_k(t)\in W_k$ and $E(t)\le d$ for every $t\in(0,T_{max})$. Indeed, if it was not the case, there would exist $t_*>0$ such that $u_k(t_*)\in N_k$. By the variational characterization of d, $d\le J_k(u_k(t_*))\le E(t_*)<d$, a contradiction to (16). As a further consequence of (16), a simple computation entails

$$J_k(u_k(t))\ge\frac{p-2}{2p}\|\nabla u_k(t)\|^2_2 \tag{17}$$

for every $t\in(0,T_{max})$, For all $t\in(0,T_{max})$ by (*) we obtain

$$\sum_{k=1}^{2}\frac{1}{2}\|u_{kt}\|^2_2+J_k(u_k(t))+\int_0^t\|u_{kt}(\tau)\|^2_*d\tau=E(0)\le d<\infty$$

Therefore, by virtue of (17) the Continuation Principle yields $T_{max}=\infty$ and

$$\sum_{k=1}^{2}\|\nabla u_k\|^2_2+\|u_{kt}\|^2_2\le C\quad\forall t\in[0,\infty)$$

$$\sum_{k=1}^{2}\int_0^t\|\nabla u_{kt}(\tau)\|^2_2d\tau\le\frac{C}{\omega}\quad\forall t\in[0,\infty) \tag{18}$$

Hence, by Poincaré inequality, we get

$$\int_0^t \|u_{kt}(\tau)\|_2^2 d\tau \le \frac{C}{\omega} \quad for\ every\ t \in [0,\infty) \tag{19}$$

We integrate over $[0,t]$

$$\frac{d}{dt}((1+t)E(t)) \le E(t)$$

and recall that by (see [10]) there holds

$$J_k(u_k(t)) \le C I_k(u_k(t)) \quad for\ every\ t \in [0,\infty)$$

we reach the inequality

$$(1+t)E(t) \le d + \frac{1}{2}\sum_{k=1}^{2}\int_0^t \|u_{kt}(\tau)\|_2^2 d\tau + C\sum_{k=1}^{2}\int_0^t \|I_k(u_{kt}(\tau))\| d\tau \tag{20}$$

for every $t \in [0,\infty)$. Observe also that, by direct computation, there hold

$$\langle u_{ktt}, u_k\rangle = \frac{d}{dt}\int u_{kt}u_k - \|u_{kt}\|_2^2 \tag{21}$$

for a.e. $t \in [0,\infty)$. Moreover, by testing the equation with u_k, we obtain

$$\langle u_{ktt}, u_k\rangle + \|\nabla u_k\|^2 + (u_k, u_{kt})_* + \int f_k u_k = a_k\|u_k\|_p^p$$

for a.e. $t \in [0,\infty)$. Using (21), this yields

$$\sum_{k=1}^{2}\frac{d}{dt}(\int u_{kt}u_k + \frac{1}{2}\|u_k\|_*) + \int F(u) = \sum_{k=1}^{2}\|u_{kt}\|_2^2 - I_k(u_k) \tag{22}$$

By integrating (22) on $[0,t]$ and by (18) and (19), we have

$$\begin{aligned}\sum_{k=1}^{2}\int_0^t I_k(u_k(\tau))d\tau \le \sum_{k=1}^{2}\int_0^t \|u_{kt}\|_2^2 + \|\varphi_k\|_2\|\psi_k\|_2 + \|u_{kt}\|_2\|u_k\|_2 \\ +\frac{1}{2}(\|\varphi_k\|_*^2 - \|u_k\|_*^2) - \int_0^t\int_\Omega F(u)dxdt \le C + \frac{C}{\omega} + C\omega\end{aligned} \tag{23}$$

for every $t \in [0,\infty)$. Then, by combining the above inequalities, from (20) we get

$$E(t) \le C(1 + \frac{1}{\omega} + \omega)\frac{1}{t} \quad for\ every\ t \in [0,\infty)$$

Consequently, by (17) we immediately obtain

$$\|\nabla u_k(t)\|_2^2 + \|u_{kt}(t)\|_2^2 \leq \frac{\Theta(\omega, \mu)}{t} \quad \text{for every } t \in [0, \infty)$$

where Θ is the map defined in (5).

The proof in the case $\omega = 0$ (and $\mu \geq 0$) is similar and follows by obvious modifications of inequalities (19) and (23).

6 Proof of Theorem 3

Lemma 6.1. Suppose that $F(s) \geq 0$ with $s = (s_1, s_2) \in R^1 \times R^1$. Let (u_1, u_2) defined on $[0, T_{max})$ be a weak solution of problem (1). For each $t \in [0, T_{max})$

If for all $k \in \{1, 2\}$, $I_k(u_k(t)) \geq 0$, then

$$\sum J_k(u_k(t)) \geq \sum \frac{p-2}{2p} \|\nabla u_k(t)\|^2; \quad \sum \frac{p-2}{2p} \|\nabla u_k(t)\|^2 \leq E(t) \tag{24}$$

.

If for all $k \in \{1, 2\}$, $I_k(u_k(t)) < 0$, then

$$\frac{p-2}{2p} \|\nabla u_k(t)\|^2 > d_k \tag{25}$$

.

Proof: If $I_k(u_k(t)) \geq 0$ for each $t \in [0, T_{max})$, we obtain

$$E(t) \geq \sum J_k(u_k(t)) = \frac{1}{2} \sum \|\nabla u_k(t)\|^2 - \sum \frac{a_k}{p} \|u_k(t)\|_p^p \geq \frac{p-2}{2p} \sum \|\nabla u_k(t)\|^2$$

Let $I_k(u_k(t)) < 0$ for each $t \in [0, T_{max})$, there exists $t_0 \in [0, T_{max})$ such that

$$\frac{p-2}{2p} \|\nabla u_1(t_0)\|^2 \leq d_1 \quad or \quad \frac{p-2}{2p} \|\nabla u_2(t_0)\|^2 \leq d_2$$

we then obtain

$$a_1 \|u_1(t_0)\|_p^p \leq \|\nabla u_1(t_0)\|^2 \quad or \quad a_2 \|u_2(t_0)\|_p^p \leq \|\nabla u_2(t_0)\|^2$$

A contradiction with $I_k(u_k) < 0$. In fact, since $d_1 \leq \max_{\lambda > 0} J_1(\lambda u_1(t_0)) = \frac{1}{2}\lambda^2 \|\nabla u_1(t_0)\|^2 - \frac{a_1}{p}\lambda^p \|u_1\|_p^p$, taking differentiation with respect to λ in order to seek for a maximal point, we see that $d_1 \leq \frac{p-2}{2p} \frac{\|\nabla u_1(t_0)\|^{2p/(p-2)}}{(a_1 \|u_1(t_0)\|_p)^{2p/(p-2)}}$. Now a simple computation leads to this inequality.

Lemma 6.2. Suppose that $F(s) \geq 0$ with $s = (s_1, s_2) \in R^1 \times R^1$. Let (u_1, u_2) defined on $[0, T_{max})$ be a weak solution of problem (1). We have the following results.For any $0 \leq t < T_{max}$.

If for all $k \in \{1, 2\}$, $I_k(\varphi_k) \geq 0$, $E(0) < \min\{d_1, d_2\}$, then

$$\sum J(u_k(t)) \geq \sum \frac{p-2}{2p}\|\nabla u_k(t)\|^2, \quad \sum \frac{p-2}{2p}\|\nabla u_k(t)\|^2 \leq E(t) \tag{26}$$

If for all $k \in \{1, 2\}$, $I_k(\varphi_k) < 0$, $E(0) < \min\{d_1, d_2\}$,then

$$\frac{p-2}{2p}\|\nabla u_k(t)\|^2 > d_k \tag{27}$$

Proof: Since $I_k(\varphi_k) \geq 0$, by Lemma 6.1, we get that

$$E(0) \geq \sum J_k(\varphi_k) \geq \sum \frac{p-2}{2p}\|\nabla \varphi_k\|^2$$

Define

$$T^* = \sup\{t \in [0, T_{max}] : E(0) \geq \sum \frac{p-2}{2p}\|\nabla u_k(s)\|^2,\ 0 \leq s < t\}$$

If $T^* < T_{max}$, then we have

$$\sum \frac{p-2}{2p}\|\nabla u_k(T^*)\|^2 = E(0)$$

and

$$\sum \frac{p-2}{2p}\|\nabla u_k(t)\|^2 > E(0),\ \forall t \in (T^*, T_{max}) \tag{28}$$

Since $E(0) < \min\{d_1, d_2\}$, we get that $\frac{p-2}{2p}\|\nabla u_k(T^*)\|^2 < d_k$. By the continuity of $\|\nabla u_k(.)\|$, there exists an interval $(T^*, \hat{T}) \subset (T^*, T_{max})$ such that $I_k(u_k(t)) > 0$, $\forall t \in (T^*, \hat{T})$. Again using Lemma 6.1, it yields $\sum \frac{p-2}{2p}\|\nabla u_k(t)\|^2 \leq E(t) \leq E(0)$, $\forall t \in (T^*, \hat{T})$. This contradicts with (28). The contradiction implies that T^* meets with T_{max} and (26) holds.

Let $I_k(\varphi_k) < 0$, then by Lemma 6.1, we get that $\frac{p-2}{2p}\|\nabla \varphi_k\|^2 > d_k,\ k = 1, 2$. Define

$$T_* = \sup\{t \in [0, T_{max}] : \frac{p-2}{2p}\|\nabla u_k(s)\|^2 > d_k,\ k = 1, 2,\ 0 \leq s < t\}$$

If $T_* < T_{max}$, then we have

$$\frac{p-2}{2p}\|\nabla u_1(T_*)\|^2 = d_1, \quad \frac{p-2}{2p}\|\nabla u_2(T_*)\|^2 \le d_2$$

or

$$\frac{p-2}{2p}\|\nabla u_1(T_*)\|^2 \le d_1, \quad \frac{p-2}{2p}\|\nabla u_2(T_*)\|^2 = d_2$$

Therefore, $I_k(u_k(T_*)) > 0,\ k = 1, 2$. Then by use of Lemma 6.1, we obtain

$$\sum \frac{p-2}{2p}\|\nabla u_k(T_*)\|^2 \le E(T_*) \tag{29}$$

Combining (*), (29) and noting that $E(0) < \min\{d_1, d_2\}$, we get that $\frac{p-2}{2p}\|\nabla u_k(T_*)\|^2 < d_k$. A contradiction. Thus (27) is true.

We give the estimates in the following. For any $T > 0$ we may consider $L : [0, T] \to R^+$ defined by

$$L(t) = \sum_{k=1}^{2} \|u_k(t)\|_2^2 + \int_0^t \|u_k(\tau)\|_*^2 d\tau + (T-t)\|\varphi_k\|_*^2 + \beta(t+s_0)^2 \tag{30}$$

where β and $s_0 > 0$ are constants to be determined.

Lemma 6.3. Let the assumptions of Theorem 3 be satisfied. then

$$L''(t) \ge (2+p)\sum_{k=1}^{2} \|u_{kt}\|^2 + 2p\sum_{k=1}^{2}\int_0^t \|u_{kt}\|_*^2 + (2+p)\beta$$

Proof:

$$\begin{aligned} L'(t) &= \sum_{k=1}^{2} 2\int_\Omega u_k u_{kt} + \|u_k\|_*^2 - \|\varphi_k\|_*^2 + 2\beta(t+s_0) \\ &= \sum_{k=1}^{2} 2\int_\Omega u_k u_{kt} + 2\int_0^t (u_k(\tau), u_{kt}(\tau))_* d\tau + 2\beta(t+s_0) \end{aligned} \tag{31}$$

$$L''(t) = \sum_{k=1}^{2} 2\langle u_{ktt}, u_k\rangle + 2\|u_{kt}\|_*^2 + 2(u_{kt}, u_k)_* + 2\beta \tag{32}$$

By (32), we get that

$$L''(t) = 2\Big[\sum_{k=1}^{2}(\|u_{kt}\|^2 - \|\nabla u_k\|^2 - \int_\Omega u_k f_k(u)dx + a_k\|u_k\|_p^p)\Big] + 2\beta$$

Using the energy equality and A6 and Lemma 6.2 , we have

$$\begin{aligned}
L''(t) \ &\geq (2+p)\sum_{k=1}^{2}\|u_{kt}\|^2 + (p-2)\sum_{k=1}^{2}\|\nabla u_k\|^2 + 2p\sum_{k=1}^{2}\int_0^t \|u_{kt}(\tau)\|_*^2 d\tau \\
&-2pE(0) + 2p\int_\Omega F(u)dx - \int_\Omega u_k f_k(u)dx + 2\beta \\
&\geq (2+p)\sum_{k=1}^{2}\|u_{kt}\|^2 + 2p\sum_{k=1}^{2}\int_0^t \|u_{kt}(\tau)\|_*^2 d\tau + (p-2)\sum_{k=1}^{2}\frac{2p}{p-2}d_k \\
&-2pE(0) + 2\Big[\frac{1}{n_0}\int_\Omega F(u)dx - \sum_{k=1}^{2}\int_\Omega u_k f_k(u)dx\Big] + 2\beta \\
&\geq (2+p)\sum_{k=1}^{2}\|u_{kt}\|^2 + 2p\sum_{k=1}^{2}\int_0^t \|u_{kt}(\tau)\|_*^2 d\tau + 2p\sum_{k=1}^{2}d_k - 2pE(0) + 2\beta
\end{aligned}$$

By the assumptions $2p\sum_{k=1}^{2}d_k - 2pE(0) > 0$. Now take $\beta = 2(\sum_{k=1}^{2}d_k - E(0))$ we get that

$$L''(t) \geq (2+p)\sum_{k=1}^{2}\|u_{kt}\|^2 + 2p\sum_{k=1}^{2}\int_0^t \|u_{kt}\|_*^2 + (2+p)\beta$$

Proof of Theorem 3

Assume by contradiction that the solution u is global. Define :

$$P = \sum_{k=1}^{2}(\|u_k\|^2 + \int_0^t \|u_k(\tau)\|_*^2 d\tau) + \beta(t+s_0)^2$$

$$Q = \sum_{k=1}^{2}\Big((u_k, u_{kt}) + \int_0^t (u_k, u_{kt})_* d\tau\Big) + \beta(t+s_0)$$

$$U = \sum_{k=1}^{2}(\|u_{kt}\|^2 + \int_0^t \|u_{kt}(\tau)\|_*^2 d\tau) + \beta$$

It is easy to check $0 < P \leq L(t)$, $Q = \frac{1}{2}L'(t)$, $0 < U \leq \dfrac{L''(t)}{2+p}$. For any real pair (λ, η) and for all $t \in [0, T]$, we have

$$P\lambda^2 + 2Q\lambda\eta + U\eta^2 = \sum_{k=1}^{2} \|\lambda u_k + \eta u_{kt}\|^2 + \sum_{k=1}^{2} \int_0^t \|\lambda u_k + \eta u_{kt}\|_*^2 d\tau + \beta[(t+s_0)\lambda + \eta]^2$$

Therefore, $PU - Q^2 \geq 0$. we infer from the above inequality that

$$L(t)L''(t) - \frac{2+p}{4}(L'(t))^2 \geq 0, \ t \in [0, T] \tag{33}$$

(33) implies that $[L^{-\theta}(t)]'' \leq 0, \ t \in [0, T]$ where $\theta = \frac{p-2}{4}$. Now taking $s_0 > -\dfrac{1}{\beta}\sum(\varphi_k, \psi_k)$, we get that $L'(0) > 0$ and $(L^{-\theta})'(0) < 0$. Choosing $T \geq -\dfrac{(L^{-\theta})(0)}{(L^{-\theta})'(0)}$, then by the concavity Lemma, there exists T_1 satisfying

$$0 < T_1 \leq -\frac{(L^{-\theta})(0)}{(L^{-\theta})'(0)}, \quad L^{-\theta}(T_1) = 0 \tag{34}$$

From (34), we see that $\lim\limits_{t \to T_1^-} L(t) = \infty$, which implies that $\lim\limits_{t \to T_1^-} \sum_{k=1}^{2}(\|u_k(t)\|_2^2 + \|\nabla u_k\|_2^2) = \infty$. This leads to a contraction with $T_{max} = \infty$. Now we give the estimate of T. If (34) holds, it suffices

$$\frac{\sum \|\varphi_k\|^2 + \beta s_0^2}{2\theta\Big[\sum(\varphi_k, \psi_k) + \beta s_0\Big] - \omega \sum \|\nabla\varphi_k\|^2 - \mu \sum \|\varphi_k\|^2} \leq T$$

where if it is necessary we may take s_0 sufficiently large such that

$$2\theta\Big[\sum(\varphi_k, \psi_k) + \beta s_0\Big] - \omega \sum \|\nabla\varphi_k\|^2 - \mu \sum \|\varphi_k\|^2 > 0$$

Therefore, we only need to take

$$\begin{aligned} T &= \inf\left\{ \frac{\sum \|\varphi_k\|^2 + \beta s_0^2}{2\theta\Big[\sum(\varphi_k, \psi_k) + \beta s_0\Big] - \omega \sum \|\nabla\varphi_k\|^2 - \mu \sum \|\varphi_k\|^2} \ \Big| \ s_0 > -\frac{1}{\beta}\sum(\varphi_k, \psi_k) \right\} \\ &= \frac{1}{2\theta^2\beta}\Bigg\{ \Big[\Big(\sum \Big(2\theta(\varphi_k, \psi_k) - \omega \sum \|\nabla\varphi_k\|^2 - \mu \sum \|\varphi_k\|^2\Big)\Big)^2 + 4\theta^2\beta \sum \|\varphi_k\|\Big]^{\frac{1}{2}} \\ &+ \Big(\omega \sum \|\nabla\varphi_k\|^2 + \mu \sum \|\varphi_k\|^2 - 2\theta(\varphi_k, \psi_k)\Big)\Bigg\} \end{aligned}$$

we have completed the proof of Theorem 3.

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