Intrinsic Square Function Characterizations of Hardy Spaces with Variable Exponents

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Abstract   Let $p(\cdot): \mathbb{R}^n \to (0, \infty)$ be a measurable function satisfying some decay condition and some locally log-Hölder continuity. In this article, via first establishing characterizations of the variable exponent Hardy space $H^{p(\cdot)}(\mathbb{R}^n)$ in terms of the Littlewood-Paley g-function, the Lusin area function and the g_λ^*-function, the authors then obtain its intrinsic square function characterizations including the intrinsic Littlewood-Paley g-function, the intrinsic Lusin area function and the intrinsic g_λ^*-function. The $p(\cdot)$-Carleson measure characterization for the dual space of $H^{p(\cdot)}(\mathbb{R}^n)$, the variable exponent Campanato space $\mathcal{L}_{1,p(\cdot),s}(\mathbb{R}^n)$, in terms of the intrinsic function is also presented.

1   Introduction

Variable exponent Lebesgue spaces are a generalization of the classical $L^p(\mathbb{R}^n)$ spaces, in which the constant exponent p is replaced by an exponent function $p(\cdot): \mathbb{R}^n \to (0, \infty)$, namely, they consist of all functions f such that $\int_{\mathbb{R}^n} |f(x)|^{p(x)} dx < \infty$. These spaces were introduced by Birnbaum-Orlicz [3] and Orlicz [34], and widely used in the study of harmonic analysis as well as partial differential equations; see, for example, [1, 2, 6, 7, 8, 11, 12, 13, 14, 20, 30, 43, 48, 50]. For a systematic research about the variable exponent Lebesgue space, we refer the reader to [8, 13]

Recently, Nakai and Sawano [32] extended the theory of variable Lebesgue spaces via studying the Hardy spaces with variable exponents on $\mathbb{R}^n$, and Sawano in [35] further gave more applications of these variable exponent Hardy spaces. Independently, Cruz-Uribe and Wang in [9] also investigated the variable exponent Hardy space with some weaker conditions than those used in [32], which also extends the theory of variable exponent Lebesgue spaces. Recall that the classical Hardy spaces $H^p(\mathbb{R}^n)$ with $p \in (0, 1]$ on the Euclidean space $\mathbb{R}^n$ and their duals are well studied (see, for example, [10, 38]) and have been playing an important and fundamental role in various fields of analysis such as harmonic analysis and partial differential equations; see, for example, [4, 31].

On the other hand, the study of the intrinsic square function on function spaces, including Hardy spaces, has recently attracted many attentions. To be precise, Wilson [44] originally introduced intrinsic square functions, which can be thought of as "grand maximal" square functions of C. Fefferman and E. M. Stein from [10], to settle a conjecture proposed by R. Fefferman and E. M. Stein on the boundedness of the Lusin area function $S(f)$ from the weighted Lebesgue space $L^2_{\mathcal{M}(v)}(\mathbb{R}^n)$ to the weighted Lebesgue space $L^2_v(\mathbb{R}^n)$, where $0 \le v \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $\mathcal{M}$ denotes the Hardy-Littlewood maximal function. The boundedness of these intrinsic square functions on the weighted Lebesgue spaces $L^p_\omega(\mathbb{R}^n)$, when $p \in (1, \infty)$ and ω belongs to Muckenhoupt weights $A_p(\mathbb{R}^n)$, was proved by Wilson [45]. The intrinsic square functions dominate all square functions of the form $S(f)$ (and the classical ones as well), but are not essentially bigger than any one of them. Similar to the Fefferman-Stein and the Hardy-Littlewood maximal functions, their generic natures make them pointwise equivalent to each other and extremely easy to work with. Moreover, the intrinsic Lusin area function has the distinct advantage of being pointwise comparable at different cone openings, which is a property long known not to hold true for the classical Lusin area function; see Wilson [44, 45, 46, 47] and also Lerner [24, 25].

Later, Huang and Liu in [19] obtain the intrinsic square function characterizations of the weighted Hardy space $H^1_\omega(\mathbb{R}^n)$ under the additional assumption that $f \in L^1_\omega(\mathbb{R}^n)$, which was further generalized to the weighted Hardy space $H^p_\omega(\mathbb{R}^n)$ with $p \in (n/(n+\alpha), 1)$ and $\alpha \in (0, 1)$ by Wang and Liu in [42], under another additional assumption. Very recently, Liang and Yang in [28] established the s-order intrinsic square function characterizations of the Musielak-Orlicz Hardy space $H^\varphi(\mathbb{R}^n)$, which was introduced by Ky [23] and generalized both the Orlicz-Hardy space (see, for example, [21, 41]) and the weighted Hardy space (see, for example, [16, 36]), in terms of the intrinsic Lusin area function, the intrinsic g-function and the intrinsic g^*_λ-function with the best known range $\lambda \in (2 + 2(\alpha+s)/n, \infty)$. More applications of such intrinsic square functions were also given by Wilson [46, 47] and Lerner [24, 25].

Motivated by [28], in this article, we establish intrinsic square function characterizations of the variable exponent Hardy space $H^{p(\cdot)}(\mathbb{R}^n)$ introduced by Nakai and Sawano in [32], including the intrinsic Littlewood-Paley g-function, the intrinsic Lusin area function and the intrinsic g^*_λ-function by first obtaining characterizations of $H^{p(\cdot)}(\mathbb{R}^n)$ via the Littlewood-Paley g-function, the Lusin area function and the g^*_λ-function. We also establish the $p(\cdot)$-Carleson measure characterization for the dual space of $H^{p(\cdot)}(\mathbb{R}^n)$, the variable exponent Campanato space $\mathcal{L}_{1,p(\cdot),s}(\mathbb{R}^n)$ in [32], in terms of the intrinsic square function.

To state the results, we begin with some notation. In what follows, for a measurable function $p(\cdot) \colon \mathbb{R}^n \to (0, \infty)$ and a measurable set E of $\mathbb{R}^n$, let

$$p_-(E) := \operatorname*{ess\,inf}_{x \in E} p(x) \quad \text{and} \quad p_+(E) := \operatorname*{ess\,sup}_{x \in E} p(x).$$

For simplicity, we let $p_- := p_-(\mathbb{R}^n)$, $p_+ := p_+(\mathbb{R}^n)$ and $p^* := \min\{p_-, 1\}$. Denote by $\mathcal{P}(\mathbb{R}^n)$ the *collection of all measurable functions* $p(\cdot) \colon \mathbb{R}^n \to (0, \infty)$ *satisfying* $0 < p_- \le p_+ < \infty$.

For $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, the *space* $L^{p(\cdot)}(\mathbb{R}^n)$ is defined to be the set of all measurable functions such that

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} := \inf \left\{ \lambda \in (0, \infty) \colon \int_{\mathbb{R}^n} \left[\frac{|f(x)|}{\lambda} \right]^{p(x)} \, dx \le 1 \right\} < \infty.$$

Remark 1.1. It was pointed out in [32, p. 3671] (see also [8, Theorem 2.17]) that the follows hold true:

  (i) $\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \ge 0$, and $\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} = 0$ if and only if $f(x) = 0$ for almost every $x \in \mathbb{R}^n$;

  (ii) $\|\lambda f\|_{L^{p(\cdot)}(\mathbb{R}^n)} = |\lambda| \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}$ for any $\lambda \in \mathbb{C}$;

  (iii) $\|f + g\|^\ell_{L^{p(\cdot)}(\mathbb{R}^n)} \le \|f\|^\ell_{L^{p(\cdot)}(\mathbb{R}^n)} + \|g\|^\ell_{L^{p(\cdot)}(\mathbb{R}^n)}$ for all $\ell \in (0, p^*]$;

(iv) for all measurable functions f with $\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \neq 0$, $\int_{\mathbb{R}^n} [|f(x)|/\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}]^{p(x)}\, dx = 1$.

A function $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ is said to satisfy the *locally log-Hölder continuous condition* if there exists a positive constant C such that, for all $x,\ y \in \mathbb{R}^n$ and $|x - y| \leq 1/2$,

$$\tag{1.1} |p(x) - p(y)| \leq \frac{C}{\log(1/|x-y|)},$$

and $p(\cdot)$ is said to satisfy the *decay condition* if there exist positive constants C_∞ and p_∞ such that, for all $x \in \mathbb{R}^n$,

$$\tag{1.2} |p(x) - p_\infty| \leq \frac{C_\infty}{\log(e + |x|)}.$$

In the whole article, we denote by $\mathcal{S}(\mathbb{R}^n)$ the *space of all Schwartz functions* and by $\mathcal{S}'(\mathbb{R}^n)$ its *topological dual space*. Let $\mathcal{S}_\infty(\mathbb{R}^n)$ denote the *space of all Schwartz functions* φ satisfying $\int_{\mathbb{R}^n} \varphi(x) x^\beta\, dx = 0$ for all multi-indices $\beta \in \mathbb{Z}_+^n := (\{0,1,\ldots\})^n$ and $\mathcal{S}'_\infty(\mathbb{R}^n)$ its *topological dual space*. For $N \in \mathbb{N} := \{1,2,\ldots\}$, let

$$\tag{1.3} \mathcal{F}_N(\mathbb{R}^n) := \left\{ \psi \in \mathcal{S}(\mathbb{R}^n) : \sum_{\beta \in \mathbb{Z}_+^n,\, |\beta| \leq N} \sup_{x \in \mathbb{R}^n} (1 + |x|)^N |D^\beta \psi(x)| \leq 1 \right\},$$

where, for $\beta := (\beta_1, \ldots, \beta_n) \in \mathbb{Z}_+^n$, $|\beta| := \beta_1 + \cdots + \beta_n$ and $D^\beta := (\frac{\partial}{\partial x_1})^{\beta_1} \cdots (\frac{\partial}{\partial x_n})^{\beta_n}$. Then, for all $f \in \mathcal{S}'(\mathbb{R}^n)$, the *grand maximal function* $f_{N,+}^*$ of f is defined by setting, for all $x \in \mathbb{R}^n$,

$$f_{N,+}^*(x) := \sup \left\{ |f * \psi_t(x)| : t \in (0,\infty) \text{ and } \psi \in \mathcal{F}_N(\mathbb{R}^n) \right\},$$

where, for all $t \in (0,\infty)$ and $\xi \in \mathbb{R}^n$, $\psi_t(\xi) := t^{-n} \psi(\xi/t)$.

For any measurable set $E \subset \mathbb{R}^n$ and $r \in (0,\infty)$, let $L^r(E)$ be the *set of all measurable functions* f such that $\|f\|_{L^r(E)} := \{ \int_E |f(x)|^r\, dx \}^{1/r} < \infty$. For $r \in (0,\infty)$, denote by $L_{\mathrm{loc}}^r(\mathbb{R}^n)$ the *set of all r-locally integrable functions* on $\mathbb{R}^n$. Recall that the *Hardy-Littlewood maximal operator* $\mathcal{M}$ is defined by setting, for all $f \in L_{\mathrm{loc}}^1(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$\mathcal{M}(f)(x) := \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)|\, dy,$$

where the supremum is taken over all balls B of $\mathbb{R}^n$ containing x.

Now we recall the notion of the Hardy space with variable exponent, $H^{p(\cdot)}(\mathbb{R}^n)$, introduced by Nakai and Sawano in [32]. For simplicity, we also call $H^{p(\cdot)}(\mathbb{R}^n)$ the variable exponent Hardy space.

Definition 1.2. Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfy (1.1) and (1.2), and

$$\tag{1.4} N \in \left(\frac{n}{p_-} + n + 1, \infty \right) \cap \mathbb{N}.$$

The *Hardy space with variable exponent* $p(\cdot)$, denoted by $H^{p(\cdot)}(\mathbb{R}^n)$, is defined to be the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that $f_{N,+}^* \in L^{p(\cdot)}(\mathbb{R}^n)$ with the *quasi-norm* $\|f\|_{H^{p(\cdot)}(\mathbb{R}^n)} := \|f_{N,+}^*\|_{L^{p(\cdot)}(\mathbb{R}^n)}$.

Remark 1.3. (i) Independently, Cruz-Uribe and Wang in [9] introduced the variable exponent Hardy space, denoted by $\widetilde{H}^{p(\cdot)}$, in the following way: Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfy that there exist $p_0 \in (0, p_-)$ and a positive constant C, only depending on n, $p(\cdot)$ and p_0, such that

$$\tag{1.5} \|\mathcal{M}(f)\|_{L^{p(\cdot)/p_0}(\mathbb{R}^n)} \leq C \|f\|_{L^{p(\cdot)/p_0}(\mathbb{R}^n)}.$$

If $N \in (n/p_0 + n + 1, \infty)$, then the *variable exponent Hardy space* $\widetilde{H}^{p(\cdot)}$ is defined to be the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that $f_{N,+}^* \in L^{p(\cdot)}(\mathbb{R}^n)$. In [9, Theorem 3.1], it was shown that the space $\widetilde{H}^{p(\cdot)}$ is independent of the choice of $N \in (n/p_0 + n + 1, \infty)$.

(ii) We point out that, in [32, Theorem 3.3], it was proved that the space $H^{p(\cdot)}(\mathbb{R}^n)$ is independent of N as long as N is sufficiently large. Although the range of N is not presented explicitly in [32, Theorem 3.3], by the proof of [32, Theorem 3.3], we see that N as in (1.4) does the work.

Let $\phi \in \mathcal{S}(\mathbb{R}^n)$ be a radial real-valued function satisfying

$$(1.6) \qquad\qquad \operatorname{supp} \widehat{\phi} \subset \{\xi \in \mathbb{R}^n : 1/2 \le |\xi| \le 2\}$$

and

$$(1.7) \qquad\qquad |\widehat{\phi}(\xi)| \ge C \text{ if } 3/5 \le |\xi| \le 5/3,$$

where C denotes a positive constant independent of ξ and, for all $\phi \in \mathcal{S}(\mathbb{R}^n)$, $\widehat{\phi}$ denotes its *Fourier transform*. Obviously, $\phi \in \mathcal{S}_\infty(\mathbb{R}^n)$. Then, for all $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$, the *Littlewood-Paley g-function*, the *Lusin area function* and the g_λ^*-*function* with $\lambda \in (0, \infty)$ of f are, respectively, defined by setting, for all $x \in \mathbb{R}^n$,

$$g(f)(x) := \left\{ \int_0^\infty |f * \phi_t(x)|^2 \frac{dt}{t} \right\}^{1/2},$$

$$S(f)(x) := \left\{ \int_0^\infty \int_{\{y \in \mathbb{R}^n : |y-x| < t\}} |\phi_t * f(y)|^2 \frac{dy\, dt}{t^{n+1}} \right\}^{1/2}$$

and

$$(1.8) \qquad g_\lambda^*(f)(x) := \left\{ \int_0^\infty \int_{\mathbb{R}^n} \left(\frac{t}{t + |x - y|} \right)^{\lambda n} |\phi_t * f(y)|^2 \frac{dy\, dt}{t^{n+1}} \right\}^{\frac{1}{2}}.$$

For all $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$ and $\phi \in \mathcal{S}(\mathbb{R}^n)$ satisfying (1.6) and (1.7), we let, for all $t \in (0, \infty)$, $j \in \mathbb{Z}$, $a \in (0, \infty)$ and $x \in \mathbb{R}^n$,

$$(\phi_t^* f)_a(x) := \sup_{y \in \mathbb{R}^n} \frac{|\phi_t * f(x + y)|}{(1 + |y|/t)^a} \quad \text{and} \quad (\phi_j^* f)_a(x) := \sup_{y \in \mathbb{R}^n} \frac{|\phi_j * f(x + y)|}{(1 + 2^j |y|)^a}.$$

Then, for all $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$, $a \in (0, \infty)$ and $x \in \mathbb{R}^n$, define

$$g_{a,*}(f)(x) := \left\{ \int_0^\infty [(\phi_t^* f)_a(x)]^2 \frac{dt}{t} \right\}^{1/2} \quad \text{and} \quad \sigma_{a,*}(f)(x) := \left\{ \sum_{j \in \mathbb{Z}} \left[(\phi_j^* f)_a(x) \right]^2 \right\}^{1/2}.$$

The following conclusion is the first main result of this article.

Theorem 1.4. *Let* $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ *satisfy* (1.1) *and* (1.2). *Then* $f \in H^{p(\cdot)}(\mathbb{R}^n)$ *if and only if* $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$ *and* $S(f) \in L^{p(\cdot)}(\mathbb{R}^n)$; *moreover, there exists a positive constant* C, *independent of* f, *such that* $C^{-1} \|S(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le \|f\|_{H^{p(\cdot)}(\mathbb{R}^n)} \le C \|S(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)}$.

The same is true if $S(f)$ *is replaced, respectively, by* $g(f)$, $g_{a,*}(f)$ *and* $\sigma_{a,*}(f)$ *with* $a \in (n/\min\{p_-, 2\}, \infty)$.

Corollary 1.5. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $\lambda \in (1 + 2/\min\{2, p_-\}, \infty)$. Then $f \in H^{p(\cdot)}(\mathbb{R}^n)$ if and only if $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$ and $g^*_\lambda(f) \in L^{p(\cdot)}(\mathbb{R}^n)$; moreover, there exists a positive constant C, independent of f, such that $C^{-1}\|g^*_\lambda(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le \|f\|_{H^{p(\cdot)}(\mathbb{R}^n)} \le C\|g^*_\lambda(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)}$.*

Remark 1.6.   (i) We point out that the conclusion of Theorem 1.4 is understood in the following sense: if $f \in H^{p(\cdot)}(\mathbb{R}^n)$, then $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$ and there exists a positive constant C such that, for all $f \in H^{p(\cdot)}(\mathbb{R}^n)$, $\|S(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C\|f\|_{H^{p(\cdot)}(\mathbb{R}^n)}$; conversely, if $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$ and $S(f) \in L^{p(\cdot)}(\mathbb{R}^n)$, then there exists a unique extension $\widetilde{f} \in \mathcal{S}'(\mathbb{R}^n)$ such that, for all $h \in \mathcal{S}_\infty(\mathbb{R}^n)$, $\langle \widetilde{f}, h \rangle = \langle f, h \rangle$ and $\|\widetilde{f}\|_{H^{p(\cdot)}(\mathbb{R}^n)} \le C\|S(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)}$ with C being a positive constant independent of f. In this sense, we identify f with $\widetilde{f}$.

(ii) Recall that, Hou et al. [18] characterized the Musielak-Orlicz Hardy space $H^\varphi(\mathbb{R}^n)$, which was introduced by Ky [23], via the Lusin area function, and Liang et al. [26] established the Littlewood-Paley g-function and the g^*_λ-function characterizations of $H^\varphi(\mathbb{R}^n)$. Observe that, when

$$(1.9) \qquad \varphi(x, t) := t^{p(x)} \quad \text{for all} \quad x \in \mathbb{R}^n \quad \text{and} \quad t \in [0, \infty),$$

then $H^\varphi(\mathbb{R}^n) = H^{p(\cdot)}(\mathbb{R}^n)$. However, a general Musielak-Orlicz function φ satisfying all the assumptions in [23] (and hence [18, 26]) may not have the form as in (1.9). On the other hand, it was proved in [49, Remark 2.23(iii)] that there exists an exponent function $p(\cdot)$ satisfying (1.1) and (1.2), but $t^{p(\cdot)}$ is not a uniformly Muckenhoupt weight, which was required in [23] (and hence [18, 26]). Thus, the Musielak-Orlicz Hardy space $H^\varphi(\mathbb{R}^n)$ in [23] (and hence in [18, 26]) and the variable exponent Hardy space $H^{p(\cdot)}(\mathbb{R}^n)$ in [32] (and hence in the present article) can not cover each other.

Moreover, Liang et al. [26, Theorem 4.8] established the g^*_λ-function characterization of the Musielak-Orlicz Hardy space $H^\varphi(\mathbb{R}^n)$ with the best known range for λ. In particular, in the case of the classical Hardy space $H^p(\mathbb{R}^n)$, $\lambda \in (2/\min\{p, 2\}, \infty)$; see, for example, [15, p. 221, Corollary 7.4] and [37, p. 91, Theorem 2]. However, it is still unclear whether the g^*_λ-function, when $\lambda \in (2/\min\{p_-, 2\}, 1 + 2/\min\{p_-, 2\}]$, can characterize $H^{p(\cdot)}(\mathbb{R}^n)$ or not, since the method used in [26, Theorem 4.8] strongly depends on the properties of uniformly Muckenhoupt weights, which are not satisfied by $t^{p(\cdot)}$.

Indeed, a key fact that used in the proof of [26, Theorem 4.8], which may not hold in the present setting, is that, if φ is a Musielak-Orlicz function as in [26], then there exists a positive constant C such that, for all $\lambda \in (0, \infty)$, $\alpha \in (0, 1)$ and measurable set $E \subset \mathbb{R}^n$,

$$\int_{U(E; \alpha)} \varphi(x, \lambda) \, dx \le C\alpha^{nq} \int_E \varphi(x, \lambda) \, dx,$$

where $U(E; \alpha) := \{x \in \mathbb{R}^n : \mathcal{M}(\chi_E)(x) > \alpha\}$ and $q \in [1, \infty)$ is the uniformly Muckenhoupt weight index of φ. To see this, following [32, Example 1.3], for all $x \in \mathbb{R}$, let

$$p(x) := \max\left\{1 - e^{3 - |x|}, \min\left(6/5, \max\{1/2, 3/2 - x^2\}\right)\right\}.$$

Then $p(\cdot)$ satisfies (1.1) and (1.2). Now, let $E := (1, 2)$, then, for all $x \in \mathbb{R}$,

$$\mathcal{M}(\chi_E)(x) = \chi_E(x) + \frac{1}{1 + 2|x - 3/2|}\chi_{\mathbb{R} \setminus E}(x).$$

It is easy to see that, for all $\lambda \in (0, \infty)$, $\int_E \lambda^{p(x)} \, dx = \lambda^{1/2}$ and

$$\int_{U(E; 1/11)} \lambda^{p(x)} \, dx = \int_{-\frac{7}{2}}^{-\frac{13}{2}} \lambda^{p(x)} \, dx > \int_E \lambda^{p(x)} \, dx + \int_{-\frac{1}{2}}^{\frac{1}{2}} \lambda^{p(x)} \, dx = \lambda^{1/2} + \lambda^{6/5}.$$

Thus, we find that

$$\lim_{\lambda \to \infty} \frac{\int_{U(E;1/11)} \lambda^{p(x)} \, dx}{\int_E \lambda^{p(x)} \, dx} = \infty,$$

which implies that there does not exist a positive constant C, independent of λ, such that,

$$\int_{U(E;1/11)} \lambda^{p(x)} \, dx \leq C \int_E \lambda^{p(x)} \, dx.$$

Thus, the method used in the proof of [26, Theorem 4.8] is not suitable for the present setting.

For any $s \in \mathbb{Z}_+$, $C^s(\mathbb{R}^n)$ denotes the *set of all functions having continuous classical derivatives up to order not more than s*. For $\alpha \in (0, 1]$ and $s \in \mathbb{Z}_+$, let $\mathcal{C}_{\alpha,s}(\mathbb{R}^n)$ be the *family of functions* $\phi \in C^s(\mathbb{R}^n)$ such that supp $\phi \subset \{x \in \mathbb{R}^n : |x| \leq 1\}$, $\int_{\mathbb{R}^n} \phi(x) x^\gamma \, dx = 0$ for all $\gamma \in \mathbb{Z}_+^n$ and $|\gamma| \leq s$, and, for all x_1, $x_2 \in \mathbb{R}^n$ and $\nu \in \mathbb{Z}_+^n$ with $|\nu| = s$,

$$(1.10) \qquad\qquad\qquad |D^\nu \phi(x_1) - D^\nu \phi(x_2)| \leq |x_1 - x_2|^\alpha.$$

For all $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $(y, t) \in \mathbb{R}_+^{n+1} := \mathbb{R}^n \times (0, \infty)$, let

$$A_{\alpha,s}(f)(y, t) := \sup_{\phi \in \mathcal{C}_{\alpha,s}(\mathbb{R}^n)} |f * \phi_t(y)|.$$

Then, the *intrinsic g-function*, the *intrinsic Lusin area integral* and the *intrinsic g^*_λ-function* of f are, respectively, defined by setting, for all $x \in \mathbb{R}^n$ and $\lambda \in (0, \infty)$,

$$g_{\alpha,s}(f)(x) := \left\{ \int_0^\infty [A_{\alpha,s}(f)(x, t)]^2 \, \frac{dt}{t} \right\}^{1/2},$$

$$S_{\alpha,s}(f)(x) := \left\{ \int_0^\infty \int_{\{y \in \mathbb{R}^n : \, |y-x| < t\}} [A_{\alpha,s}(f)(y, t)]^2 \, \frac{dy \, dt}{t^{n+1}} \right\}^{1/2}$$

and

$$g^*_{\lambda,\alpha,s}(f)(x) := \left\{ \int_0^\infty \int_{\mathbb{R}^n} \left(\frac{t}{t + |x - y|} \right)^{\lambda n} [A_{\alpha,s}(f)(y, t)]^2 \, \frac{dy \, dt}{t^{n+1}} \right\}^{1/2}.$$

We also recall another kind of similar-looking square functions, defined via convolutions with kernels that have unbounded supports. For $\alpha \in (0, 1]$, $s \in \mathbb{Z}_+$ and $\epsilon \in (0, \infty)$, let $\mathcal{C}_{(\alpha,\epsilon),s}(\mathbb{R}^n)$ be the *family of functions* $\phi \in C^s(\mathbb{R}^n)$ such that, for all $x \in \mathbb{R}^n$, $\gamma \in \mathbb{Z}_+^n$ and $|\gamma| \leq s$, $|D^\gamma \phi(x)| \leq (1 + |x|)^{-n-\epsilon}$, $\int_{\mathbb{R}^n} \phi(x) x^\gamma \, dx = 0$ and, for all x_1, $x_2 \in \mathbb{R}^n$, $\nu \in \mathbb{Z}_+^n$ and $|\nu| = s$,

$$(1.11) \qquad |D^\nu \phi(x_1) - D^\nu \phi(x_2)| \leq |x_1 - x_2|^\alpha \left[(1 + |x_1|)^{-n-\epsilon} + (1 + |x_2|)^{-n-\epsilon} \right].$$

Remark that, in what follows, the parameter ϵ usually has to be chosen to be large enough. For all f satisfying

$$(1.12) \qquad\qquad\qquad\qquad |f(\cdot)| (1 + |\cdot|)^{-n-\epsilon} \in L^1(\mathbb{R}^n)$$

and $(y, t) \in \mathbb{R}_+^{n+1}$, let

$$(1.13) \qquad\qquad\qquad \widetilde{A}_{(\alpha,\epsilon),s}(f)(y, t) := \sup_{\phi \in \mathcal{C}_{(\alpha,\epsilon),s}(\mathbb{R}^n)} |f * \phi_t(y)|.$$

Then, for all $x \in \mathbb{R}^n$ and $\lambda \in (0, \infty)$, we let

$$\widetilde{g}_{(\alpha,\epsilon),s}(f)(x) := \left\{ \int_0^\infty \left[\widetilde{A}_{(\alpha,\epsilon),s}(f)(x,t) \right]^2 \frac{dt}{t} \right\}^{1/2},$$

$$\widetilde{S}_{(\alpha,\epsilon),s}(f)(x) := \left\{ \int_0^\infty \int_{\{y \in \mathbb{R}^n: \ |y-x|<t\}} \left[\widetilde{A}_{(\alpha,\epsilon),s}(f)(y,t) \right]^2 \frac{dy\,dt}{t^{n+1}} \right\}^{1/2}$$

and

$$\widetilde{g}^*_{\lambda,(\alpha,\epsilon),s}(f)(x) := \left\{ \int_0^\infty \int_{\mathbb{R}^n} \left(\frac{t}{t+|x-y|} \right)^{\lambda n} [\widetilde{A}_{(\alpha,\epsilon),s}(f)(y,t)]^2 \frac{dy\,dt}{t^{n+1}} \right\}^{1/2}.$$

These intrinsic square functions, when $s = 0$, were original introduced by Wilson [44], which were further generalized to $s \in \mathbb{Z}_+$ by Liang and Yang [28].

In what follows, for any $r \in \mathbb{Z}_+$, we use $\mathcal{P}_r(\mathbb{R}^n)$ to denote the *set of all polynomials on $\mathbb{R}^n$ with order not more than r.*

We now recall the notion of the Campanato space with variable exponent, which was introduced by Nakai and Sawano in [32].

Definition 1.7. Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, s be a nonnegative integer and $q \in [1, \infty)$. Then the *Campanato space* $\mathcal{L}_{q,p(\cdot),s}(\mathbb{R}^n)$ is defined to be the set of all $f \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$ such that

$$\|f\|_{\mathcal{L}_{q,p(\cdot),s}(\mathbb{R}^n)} := \sup_{Q \subset \mathbb{R}^n} \frac{|Q|}{\|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \left[\frac{1}{|Q|} \int_Q |f(x) - P^s_Q f(x)|^q \, dx \right]^{\frac{1}{q}} < \infty,$$

where the supremum is taken over all cubes Q of $\mathbb{R}^n$ and $P^s_Q g$ denotes the *unique polynomial $P \in \mathcal{P}_s(\mathbb{R}^n)$* such that, for all $h \in \mathcal{P}_s(\mathbb{R}^n)$, $\int_Q [f(x) - P(x)]h(x)\,dx = 0$.

Now we state the second main result of this article. Recall that $f \in \mathcal{S}'(\mathbb{R}^n)$ is said to *vanish weakly at infinity*, if, for every $\phi \in \mathcal{S}(\mathbb{R}^n)$, $f * \phi_t \to 0$ in $\mathcal{S}'(\mathbb{R}^n)$ as $t \to \infty$; see, for example, [15, p. 50].

Theorem 1.8. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfy (1.1), (1.2) and $p_+ \in (0,1]$. Assume that $\alpha \in (0,1]$, $s \in \mathbb{Z}_+$ and $p_- \in (n/n+\alpha+s, 1]$. Then $f \in H^{p(\cdot)}(\mathbb{R}^n)$ if and only if $f \in (\mathcal{L}_{1,p(\cdot),s}(\mathbb{R}^n))^*$, the dual space of $\mathcal{L}_{1,p(\cdot),s}(\mathbb{R}^n)$, f vanishes weakly at infinity and $g_{\alpha,s}(f) \in L^{p(\cdot)}(\mathbb{R}^n)$; moreover, it holds true that*

$$\frac{1}{C} \|g_{\alpha,s}(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le \|f\|_{H^{p(\cdot)}(\mathbb{R}^n)} \le C \|g_{\alpha,s}(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)}$$

with C being a positive constant independent of f.

The same is true if $g_{\alpha,s}(f)$ is replaced by $\widetilde{g}_{(\alpha,\epsilon),s}(f)$ with $\epsilon \in (\alpha+s, \infty)$.

Observe that, for all $x \in \mathbb{R}^n$, $S_{\alpha,s}(f)(x)$ and $g_{\alpha,s}(f)(x)$ as well as $\widetilde{S}_{(\alpha,\epsilon),s}(f)(x)$ and $\widetilde{g}_{(\alpha,\epsilon),s}(f)(x)$ are pointwise comparable (see [28, Proposition 2.4]), which, together with Theorem 1.8, immediately implies the following Corollary 1.9.

Corollary 1.9. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfy (1.1), (1.2) and $p_+ \in (0,1]$. Assume that $\alpha \in (0,1]$, $s \in \mathbb{Z}_+$ and $p_- \in (n/(n+\alpha+s), 1]$. Then $f \in H^{p(\cdot)}(\mathbb{R}^n)$ if and only if $f \in (\mathcal{L}_{1,p(\cdot),s}(\mathbb{R}^n))^*$, f vanishes weakly at infinity and $S_{\alpha,s}(f) \in L^{p(\cdot)}(\mathbb{R}^n)$; moreover, it holds true that*

$$\frac{1}{C} \|S_{\alpha,s}(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le \|f\|_{H^{p(\cdot)}(\mathbb{R}^n)} \le C \|S_{\alpha,s}(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)}$$

with C being a positive constant independent of f.

The same is true if $S_{\alpha,s}(f)$ is replaced by $\widetilde{S}_{(\alpha,\epsilon),s}(f)$ with $\epsilon \in (\alpha+s, \infty)$.

Theorem 1.10. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfy (1.1), (1.2) and $p_+ \in (0,1]$. Assume that $\alpha \in (0,1]$, $s \in \mathbb{Z}_+$, $p_- \in (n/(n+\alpha+s),1]$ and $\lambda \in (3+2(\alpha+s)/n, \infty)$. Then $f \in H^{p(\cdot)}(\mathbb{R}^n)$ if and only if $f \in (\mathcal{L}_{1,p(\cdot),s}(\mathbb{R}^n))^*$, f vanishes weakly at infinity and $g^*_{\lambda,\alpha,s}(f) \in L^{p(\cdot)}(\mathbb{R}^n)$; moreover, it holds true that*

$$\frac{1}{C}\|g^*_{\lambda,\alpha,s}(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq \|f\|_{H^{p(\cdot)}(\mathbb{R}^n)} \leq C\|g^*_{\lambda,\alpha,s}(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)}$$

with C being a positive constant independent of f.

The same is true if $g^*_{\lambda,\alpha,s}(f)$ is replaced by $\widetilde{g}^*_{\lambda,(\alpha,\epsilon),s}(f)$ with $\epsilon \in (\alpha+s, \infty)$.

Remark 1.11.   (i) We point out that there exists a positive constant C such that, for all $\phi \in \mathcal{C}_{\alpha,s}(\mathbb{R}^n)$, $C\phi \in \mathcal{C}_{(\alpha,\epsilon),s}(\mathbb{R}^n)$ and hence $\phi \in \mathcal{L}_{1,p(\cdot),s}(\mathbb{R}^n)$; see Lemma 2.8 below. Thus, the intrinsic square functions are well defined for functionals in $(\mathcal{L}_{1,p(\cdot),s}(\mathbb{R}^n))^*$. Observe that, if $\phi \in \mathcal{S}(\mathbb{R}^n)$, then $\phi \in \mathcal{L}_{1,p(\cdot),s}(\mathbb{R}^n)$; see also Lemma 2.8 below. Therefore, if $f \in (\mathcal{L}_{1,p(\cdot),s}(\mathbb{R}^n))^*$, then $f \in \mathcal{S}'(\mathbb{R}^n)$ and f vanishing weakly at infinity makes sense.

 (ii) Recall that Liang and Yang [28] characterized the Musielak-Orlicz Hardy space $H^\varphi(\mathbb{R}^n)$ in terms of the intrinsic square functions original introduced by Wilson [44]. Moreover, Liang and Yang [28] established the intrinsic g^*_λ-functions $g^*_{\lambda,\alpha,s}$ and $\widetilde{g}^*_{\lambda,(\alpha,\epsilon),s}$ with the best known range $\lambda \in (2+2(\alpha+s)/n, \infty)$ via some argument similar to that used in the proof of [26, Theorem 4.8]. However, it is still unclear whether the intrinsic g^*_λ-functions $g^*_{\lambda,\alpha,s}$ and $\widetilde{g}^*_{\lambda,(\alpha,\epsilon),s}$, when $\lambda \in (2+2(\alpha+s)/n, 3+2(\alpha+s)/n]$, can characterize $H^{p(\cdot)}(\mathbb{R}^n)$ or not. Based on the same reason as in Remark 1.6(ii), we see that the method used in the proof of [28, Theorem 1.8] is not available for the present setting.

 (iii) Let $p \in (0,1]$. When

(1.14) $p(x) := p \quad \text{for all} \quad x \in \mathbb{R}^n,$

then $H^{p(\cdot)}(\mathbb{R}^n) = H^p(\mathbb{R}^n)$. In this case, Theorem 1.8 and Corollary 1.9 coincide with the corresponding results of the classical Hardy space $H^p(\mathbb{R}^n)$; see [28, Theorem 1.6] and [28, Corollary 1.7].

 (iv) We also point out that the method used in this article does not work for the variable exponent Hardy space investigated by Cruz-Uribe and Wang in [9], since it strongly depends on the locally log-Hölder continuity condition (1.1) and the decay condition (1.2) of $p(\cdot)$. Thus, it is still unknown whether the variable exponent Hardy space in [9] has any intrinsic square function characterizations or not.

Definition 1.12. Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. A measure $d\mu$ on $\mathbb{R}^{n+1}_+$ is called a $p(\cdot)$-*Carleson measure* if

$$\|d\mu\|_{p(\cdot)} := \sup_{Q \subset \mathbb{R}^n} \frac{|Q|^{1/2}}{\|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \left\{\int_{\widehat{Q}} |d\mu(x,t)|\right\}^{1/2} < \infty,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ and $\widehat{Q}$ denotes the *tent* over Q, namely, $\widehat{Q} := \{(y,t) \in \mathbb{R}^{n+1}_+ : B(x,t) \subset Q\}$.

Theorem 1.13. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfy (1.1) and (1.2). Assume that $p_+ \in (0,1]$, $s \in \mathbb{Z}_+$, $p_- \in (n/(n+s+1),1]$ and $\phi \in \mathcal{S}(\mathbb{R}^n)$ is a radial function satisfying (1.6) and (1.7).*

 (i) *If $b \in \mathcal{L}_{1,p(\cdot),s}(\mathbb{R}^n)$, then $d\mu(x,t) := |\phi_t * b(x)|^2 \frac{dxdt}{t}$ for all $(x,t) \in \mathbb{R}^{n+1}_+$ is a $p(\cdot)$-Carleson measure on $\mathbb{R}^{n+1}_+$; moreover, there exists a positive constant C, independent of b, such that $\|d\mu\|_{p(\cdot)} \leq C\|b\|_{\mathcal{L}_{1,p(\cdot),s}(\mathbb{R}^n)}$.*

(ii) If $b \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$ and $d\mu(x,t) := |\phi_t * b(x)|^2 \frac{dxdt}{t}$ for all $(x,t) \in \mathbb{R}^{n+1}_+$ is a $p(\cdot)$-Carleson measure on $\mathbb{R}^{n+1}_+$, then $b \in \mathcal{L}_{1,p(\cdot),s}(\mathbb{R}^n)$ and, moreover, there exists a positive constant C, independent of b, such that $\|b\|_{\mathcal{L}_{1,p(\cdot),s}(\mathbb{R}^n)} \le C\|d\mu\|_{p(\cdot)}$.

In what follows, for $\alpha \in (0,1]$, $s \in \mathbb{Z}_+$, $\epsilon \in (0,\infty)$ and $b \in \mathcal{L}_{1,p(\cdot),s}(\mathbb{R}^n)$, the *measure* μ_b on $\mathbb{R}^{n+1}_+$ is defined by setting, for all $(x,t) \in \mathbb{R}^{n+1}_+$,

$$(1.15) \qquad d\mu_b(x,t) := [\widetilde{A}_{(\alpha,\epsilon),s}(b)(x,t)]^2 \frac{dxdt}{t},$$

where $\widetilde{A}_{(\alpha,\epsilon),s}(b)$ is as in (1.13) with f replaced by b.

Theorem 1.14. *Let $\alpha \in (0,1]$, $s \in \mathbb{Z}_+$, $\epsilon \in (\alpha+s,\infty)$, $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfy (1.1), (1.2), $p_+ \in (0,1]$ and $p_- \in (n/(n+\alpha+s),1]$.*

(i) *If $b \in \mathcal{L}_{1,p(\cdot),s}(\mathbb{R}^n)$, then $d\mu_b$ as in (1.15) is a $p(\cdot)$-Carleson measure on $\mathbb{R}^{n+1}_+$; moreover, there exists a positive constant C, independent of b, such that $\|d\mu_b\|_{p(\cdot)} \le C\|b\|_{\mathcal{L}_{1,p(\cdot),s}(\mathbb{R}^n)}$.*

(ii) *If $b \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$ and $d\mu_b$ as in (1.15) is a $p(\cdot)$-Carleson measure on $\mathbb{R}^{n+1}_+$, then it follows that $b \in \mathcal{L}_{1,p(\cdot),s}(\mathbb{R}^n)$; moreover, there exists a positive constant C, independent of b, such that*

$$\|b\|_{\mathcal{L}_{1,p(\cdot),s}(\mathbb{R}^n)} \le C\|d\mu_b\|_{p(\cdot)}.$$

Remark 1.15. (i) Fefferman and Stein [10] shed some light on the tight connection between BMO-functions and Carleson measures, which is the case of Theorem 1.13 when $s = 0$ and $p(x) := 1$ for all $x \in \mathbb{R}^n$.

(ii) When $p(\cdot)$ is as in (1.14) with $p \in (0,1]$, Theorem 1.13 is already known (see [29, Theorem 4.2]).

(iii) When $p(\cdot)$ is as in (1.14) with $p \in (0,1]$, Theorem 1.14 was obtained in [28, Theorem 1.11] with $p \in (n/(n+\alpha+s),1]$. Thus, the range of p_- in Theorem 1.14 is reasonable and the best known possible, even in the case that $p(\cdot)$ being as in (1.14) with $p \in (0,1]$.

This article is organized as follows.

Section 2 is devoted to the proofs of Theorems 1.4, 1.8, 1.10, 1.13 and 1.14. To prove Theorem 1.4, we establish an equivalent characterization of $H^{p(\cdot)}(\mathbb{R}^n)$ via the discrete Littlewood-Paley g-function (see Proposition 2.3 below) by using the nontangential maximal function characterization of $H^{p(\cdot)}(\mathbb{R}^n)$ obtained in this article and the Littlewood-Paley decomposition of $H^{p(\cdot)}(\mathbb{R}^n)$ which was proved in [32]. In the proof of Theorem 1.4, we also borrow some ideas from the proofs of [40, Theorem 2.8] (see also [27, Theorem 3.2]).

The key tools used to prove Theorem 1.8 are the Littlewood-Paley g-function characterization of $H^{p(\cdot)}(\mathbb{R}^n)$ in Theorem 1.4, the atomic decomposition of $H^{p(\cdot)}(\mathbb{R}^n)$ established in [32] (see also Lemma 2.11 below), the dual space of $H^{p(\cdot)}(\mathbb{R}^n)$, $\mathcal{L}_{1,p(\cdot),s}(\mathbb{R}^n)$, given in [32] and the fact that the intrinsic square functions are pointwise comparable proved in [28]. As an application of Theorems 1.4 and 1.8, we give the proof of Theorem 1.10 via showing that, for all $x \in \mathbb{R}^n$, the intrinsic square functions $\widetilde{S}_{(\alpha,\epsilon),s}(f)(x)$ and $\widetilde{g}^*_{\lambda,(\alpha,\epsilon),s}(f)(x)$ are pointwise comparable under the assumption $\lambda \in (3 + 2(\alpha+s)/n, \infty)$.

The proof of Theorem 1.13 is similar to that of [29, Theorem 4.2], which depends on atomic decomposition of the tent space with variable exponent, the fact that the dual space of $H^{p(\cdot)}(\mathbb{R}^n)$ is $\mathcal{L}_{1,p(\cdot),s}(\mathbb{R}^n)$ (see [32, Theorem 7.5]) and some properties of $\mathcal{L}_{1,p(\cdot),s}(\mathbb{R}^n)$. To complete the proof of Theorem 1.13, we first introduce the tent space with variable exponent and obtain its

atomic decomposition in Theorem 2.16 below. Then we give an equivalent norm of $\mathcal{L}_{1,p(\cdot),s}(\mathbb{R}^n)$ via establishing a John-Nirenberg inequality for functions in $\mathcal{L}_{1,p(\cdot),s}(\mathbb{R}^n)$. At the end of Section 2, we give the proof of Theorem 1.14 by using Theorem 1.13 and some ideas from the proof of [28, Theorem 1.11].

Finally, we make some conventions on notation. Throughout the paper, we denote by C a *positive constant* which is independent of the main parameters, but it may vary from line to line. The *symbol* $A \lesssim B$ means $A \le CB$. If $A \lesssim B$ and $B \lesssim A$, then we write $A \sim B$. If E is a subset of $\mathbb{R}^n$, we denote by χ_E its *characteristic function*. For any $x \in \mathbb{R}^n$ and $r \in (0,\infty)$, let $B(x,r) := \{y \in \mathbb{R}^n : |x-y| < r\}$ be the ball. For $\beta := (\beta_1, \ldots, \beta_n) \in \mathbb{Z}_+^n$, let $\beta! := \beta_1! \cdots \beta_n!$. For $\alpha \in \mathbb{R}$, we use $\lfloor \alpha \rfloor$ to denote the maximal integer not more than α. For a measurable function f, we use $\overline{f}$ to denote its *conjugate function*.

2   Proofs of main results

In what follows, for all $f \in \mathcal{S}'(\mathbb{R}^n)$ and $N \in \mathbb{N}$, the *nontangential maximal function* f_N^* of f is defined by setting, for all $x \in \mathbb{R}^n$,

$$(2.1) \qquad\qquad f_N^*(x) := \sup_{\psi \in \mathcal{F}_N(\mathbb{R}^n)} \sup_{\substack{t \in (0,\infty) \\ |y-x|<t}} |f * \psi_t(y)|,$$

where $\mathcal{F}_N(\mathbb{R}^n)$ is as in (1.3).

The following proposition is an equivalent characterization of $H^{p(\cdot)}(\mathbb{R}^n)$.

Proposition 2.1. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfy (1.1) and (1.2), and N be as in (1.4). Then $f \in H^{p(\cdot)}(\mathbb{R}^n)$ if and only if $f \in \mathcal{S}'(\mathbb{R}^n)$ and $f_N^* \in L^{p(\cdot)}(\mathbb{R}^n)$; moreover, there exists a positive constant C such that, for all $f \in H^{p(\cdot)}(\mathbb{R}^n)$,*

$$C^{-1}\|f\|_{H^{p(\cdot)}(\mathbb{R}^n)} \le \|f_N^*\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C\|f\|_{H^{p(\cdot)}(\mathbb{R}^n)}.$$

Proof. Let $f \in \mathcal{S}'(\mathbb{R}^n)$ and $f_N^* \in L^{p(\cdot)}(\mathbb{R}^n)$. Observing that, for all $x \in \mathbb{R}^n$, $f_{N,+}^*(x) \le f_N^*(x)$, we then conclude that $\|f\|_{H^{p(\cdot)}(\mathbb{R}^n)} = \|f_{N,+}^*\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le \|f_N^*\|_{L^{p(\cdot)}(\mathbb{R}^n)}$ and hence $f \in H^{p(\cdot)}(\mathbb{R}^n)$. This finishes the proof of the sufficiency of Proposition 2.1.

To prove the necessity, we need to show that, for all $f \in H^{p(\cdot)}(\mathbb{R}^n)$, $\|f_N^*\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim \|f\|_{H^{p(\cdot)}(\mathbb{R}^n)}$. To this end, for all $\Phi \in \mathcal{F}_N(\mathbb{R}^n)$, $x \in \mathbb{R}^n$, $t \in (0,\infty)$ and $y \in \mathbb{R}^n$ with $|y-x| < t$, let, for all $z \in \mathbb{R}^n$, $\psi(z) := \Phi(z + (y-x)/t)$. Then we see that

$$\sum_{\beta \in \mathbb{Z}_+^n, |\beta| \le N} \sup_{z \in \mathbb{R}^n} (1+|z|)^N |D^\beta \psi(z)| = \sum_{\beta \in \mathbb{Z}_+^n, |\beta| \le N} \sup_{z \in \mathbb{R}^n} \left(1 + \left|z - \frac{y-z}{t}\right|\right)^N |D^\beta \Phi(z)| \le 2^N,$$

which implies that $2^{-N}\psi \in \mathcal{F}_N(\mathbb{R}^n)$. From this, we deduce that

$$|f * \Phi_t(y)| = |f * \psi_t(x)| \le 2^N f_{N,+}^*(x),$$

and hence $f_N^*(x) \lesssim f_{N,+}^*(x)$ for all $x \in \mathbb{R}^n$, which further implies that

$$\|f_N^*\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim \|f_{N,+}^*\|_{L^{p(\cdot)}(\mathbb{R}^n)} \sim \|f\|_{H^{p(\cdot)}(\mathbb{R}^n)}.$$

This finishes the proof of the necessity part and hence Proposition 2.1. $\square$

Corollary 2.2. *Let $p(\cdot)$ be as in Proposition 2.1 and $f \in H^{p(\cdot)}(\mathbb{R}^n)$. Then f vanishes weakly at infinity.*

Proof. Observe that, for any $f \in H^{p(\cdot)}(\mathbb{R}^n)$ with $\|f\|_{H^{p(\cdot)}(\mathbb{R}^n)} \neq 0$, $\phi \in \mathcal{S}(\mathbb{R}^n)$, $x \in \mathbb{R}^n$, $t \in (0, \infty)$ and $y \in B(x,t)$, $|f * \phi_t(x)| \lesssim f_N^*(y)$, where f_N^* is as in (2.1) with N as in (1.4). By this and Remark 1.1(iv), we see that

$$
\begin{aligned}
\min\{|f * \phi_t(x)|^{p+}, |f * \phi_t(x)|^{p-}\} &\lesssim \inf_{y \in B(x,t)} \min\{[f_N^*(y)]^{p+}, [f_N^*(y)]^{p-}\} \\
&\lesssim \int_{B(x,t)} \min\{[f_N^*(y)]^{p+}, [f_N^*(y)]^{p-}\}\, dy |B(x,t)|^{-1} \\
&\lesssim \int_{B(x,t)} [f_N^*(y)]^{p(y)}\, dy |B(x,t)|^{-1} \\
&\lesssim \int_{\mathbb{R}^n} \left[\frac{f_N^*(y)}{\|f_N^*\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \right]^{p(y)} \|f_N^*\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{p(y)}\, dy |B(x,t)|^{-1} \\
&\lesssim |B(x,t)|^{-1} \max\{\|f_N^*\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{p-}, \|f_N^*\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{p+}\} \to 0,
\end{aligned}
$$

as $t \to \infty$, which implies that f vanishes weakly at infinity. This finishes the proof of Corollary 2.2. $\qquad\square$

In what follows, denote by $P_{\text{poly}}(\mathbb{R}^n)$ the *set of all polynomials* on $\mathbb{R}^n$. For $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$ and $\phi \in \mathcal{S}(\mathbb{R}^n)$ satisfying (1.6) and (1.7), let

$$
\sigma(f)(x) := \left[\sum_{j \in \mathbb{Z}} |\phi_j * f(x)|^2 \right]^{1/2}
$$

and

$$
H_\sigma^{p(\cdot)}(\mathbb{R}^n) := \left\{ f \in \mathcal{S}'_\infty(\mathbb{R}^n) : \|f\|_{H_\sigma^{p(\cdot)}(\mathbb{R}^n)} := \|\sigma(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)} < \infty \right\}.
$$

Proposition 2.3. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfy (1.1) and (1.2). Then $H^{p(\cdot)}(\mathbb{R}^n) = H_\sigma^{p(\cdot)}(\mathbb{R}^n)$ in the following sense: if $f \in H^{p(\cdot)}(\mathbb{R}^n)$, then $f \in H_\sigma^{p(\cdot)}(\mathbb{R}^n)$ and there exists a positive constant C such that, for all $f \in H^{p(\cdot)}(\mathbb{R}^n)$, $\|f\|_{H_\sigma^{p(\cdot)}(\mathbb{R}^n)} \leq C \|f\|_{H^{p(\cdot)}(\mathbb{R}^n)}$; conversely, if $f \in H_\sigma^{p(\cdot)}(\mathbb{R}^n)$, then there exists a unique extension $\widetilde{f} \in \mathcal{S}'(\mathbb{R}^n)$ such that, for all $h \in \mathcal{S}_\infty(\mathbb{R}^n)$, $\langle \widetilde{f}, h \rangle = \langle f, h \rangle$ and $\|\widetilde{f}\|_{H^{p(\cdot)}(\mathbb{R}^n)} \leq C \|f\|_{H_\sigma^{p(\cdot)}(\mathbb{R}^n)}$ with C being a positive constant independent of f.*

Proof. Let $f \in H^{p(\cdot)}(\mathbb{R}^n)$. Then $f \in \mathcal{S}'(\mathbb{R}^n) \subset \mathcal{S}'_\infty(\mathbb{R}^n)$ and, by [32, Theorem 5.7] (see also [35, Theorem 3.1]), we see that $\|f\|_{H_\sigma^{p(\cdot)}(\mathbb{R}^n)} \lesssim \|f\|_{H^{p(\cdot)}(\mathbb{R}^n)}$ and hence $f \in H_\sigma^{p(\cdot)}(\mathbb{R}^n)$.

Conversely, let $f \in H_\sigma^{p(\cdot)}(\mathbb{R}^n)$. Then $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$. From [17, Proposition 2.3.25], we deduce that there exists $\widetilde{f} \in \mathcal{S}'(\mathbb{R}^n)$ such that $f - \widetilde{f} \in P_{\text{poly}}(\mathbb{R}^n)$. By [32, Theorem 5.7] and the fact that $\phi_j * f = \phi_j * \widetilde{f}$ for all $j \in \mathbb{Z}$ and ϕ as in definition of $\sigma(f)$, we know that

$$
\|\widetilde{f}\|_{H^{p(\cdot)}(\mathbb{R}^n)} \lesssim \|\sigma(\widetilde{f})\|_{L^{p(\cdot)}(\mathbb{R}^n)} \sim \|\sigma(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \sim \|f\|_{H_\sigma^{p(\cdot)}(\mathbb{R}^n)},
$$

which implies that $\widetilde{f} \in H^{p(\cdot)}(\mathbb{R}^n)$.

Suppose that there exists another extension of f, for example, $\widetilde{g} \in H^{p(\cdot)}(\mathbb{R}^n)$. Then $\widetilde{g} \in \mathcal{S}'(\mathbb{R}^n)$ and $\widetilde{g} = f$ in $\mathcal{S}'_\infty(\mathbb{R}^n)$, which, together with [17, Proposition 2.3.25], implies $\widetilde{g} - \widetilde{f} \in P_{\text{poly}}(\mathbb{R}^n)$. From this, $\widetilde{g} - \widetilde{f} \in H^{p(\cdot)}(\mathbb{R}^n)$ and Corollary 2.2, we deduce that $\widetilde{g} = \widetilde{f}$ since nonzero polynomials fail to vanish weakly at infinity. Therefore, $\widetilde{f}$ is the unique extension of $f \in H_\sigma^{p(\cdot)}(\mathbb{R}^n)$, which completes the proof of Proposition 2.3. $\qquad\square$

The following estimate is a special case of [27, Lemma 3.5], which is further traced back to [40, (2.29)] and the argument used in the proof of [40, Theorem 2.6] (see also [27, Theorem 3.2]),

Lemma 2.4. *Let $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$, $N_0 \in \mathbb{N}$ and $\Phi \in \mathcal{S}(\mathbb{R}^n)$ satisfy (1.6) and (1.7). Then, for all $t \in [1,2]$, $a \in (0, N_0]$, $l \in \mathbb{Z}$ and $x \in \mathbb{R}^n$, it holds true that*

$$[(\Phi^*_{2^{-l}t}f)_a(x)]^r \leq C_{(r)} \sum_{k=0}^\infty 2^{-kN_0 r} 2^{(k+l)n} \int_{\mathbb{R}^n} \frac{|(\Phi_{k+l})_t * f(y)|^r}{(1 + 2^l|x-y|)^{ar}}\, dy,$$

where r is an arbitrary fixed positive number and $C_{(r)}$ a positive constant independent of Φ, f, l, t, but may depend on r.

We point out that Lemma 2.4 plays an important role in the proof of Theorem 1.4.

The following vector-valued inequality on the boundedness of the Hardy-Littlewood maximal operator $\mathcal{M}$ on the variable Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ was obtained in [6, Corollary 2.1].

Lemma 2.5. *Let $r \in (1, \infty)$. Assume that $p(\cdot) : \mathbb{R}^n \to [0, \infty)$ is a measurable function satisfying (1.1), (1.2) and $1 < p_- \leq p_+ < \infty$, then there exists a positive constant C_0 such that, for all sequences $\{f_j\}_{j=1}^\infty$ of measurable functions,*

$$\left\| \left[\sum_{j=1}^\infty (\mathcal{M}f_j)^r \right]^{1/r} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C_0 \left\| \left(\sum_{j=1}^\infty |f_j|^r \right)^{1/r} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

Proof of Theorem 1.4. We first prove that, for all $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$,

$$(2.2) \qquad \|g(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \sim \|S(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \sim \|g_{a,*}(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)}$$
$$\sim \|\sigma(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \sim \|\sigma_{a,*}(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

To prove (2.2), we first show that, for all $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$,

$$(2.3) \qquad \|g(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \sim \|g_{a,*}(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \text{ and } \|\sigma(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \sim \|\sigma_{a,*}(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

For similarity, we only give the proof for the first equivalence. By definitions, we easily see that $\|g(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq \|g_{a,*}(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)}$. Conversely, we show that $\|g_{a,*}(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim \|g(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)}$. Since $a \in (n/\min\{p_-, 2\}, \infty)$, it follows that there exists $r \in (0, \min\{p_-, 2\})$ such that $a \in (n/r, \infty)$. By Lemma 2.4 and the Minkowski integral inequality, we find that

$$g_{a,*}(f)(x) = \left\{ \sum_{j \in \mathbb{Z}} \int_1^2 [(\phi^*_{2^{-j}t}f)_a(x)]^2 \frac{dt}{t} \right\}^{1/2}$$

$$\lesssim \left\{ \sum_{j \in \mathbb{Z}} \int_1^2 \left[\sum_{k=0}^\infty 2^{-kN_0 r} 2^{(k+j)n} \int_{\mathbb{R}^n} \frac{|(\phi_{k+j})_t * f(y)|^r}{(1 + 2^j|x-y|)^{ar}}\, dy \right]^{\frac{2}{r}} \frac{dt}{t} \right\}^{1/2}$$

$$\lesssim \left\{ \sum_{j \in \mathbb{Z}} \left[\sum_{k=0}^\infty 2^{-kN_0 r} 2^{(k+j)n} \int_{\mathbb{R}^n} \frac{[\int_1^2 |(\phi_{k+j})_t * f(y)|^2 \frac{dt}{t}]^{\frac{r}{2}}}{(1 + 2^j|x-y|)^{ar}}\, dy \right]^{\frac{2}{r}} \right\}^{1/2},$$

which, together with the Minkowski series inequality and Remark 1.1(iii), implies that

$$(2.4) \qquad \|g_{a,*}(f)\|^r_{L^{p(\cdot)}(\mathbb{R}^n)}$$

$$\lesssim \left\| \sum_{k=0}^{\infty} 2^{-k(N_0 r - n)} \left(\sum_{j \in \mathbb{Z}} 2^{j \frac{2n}{r}} \left[\int_{\mathbb{R}^n} \frac{[\int_1^2 |(\phi_{k+j})_t * f(y)|^2 \frac{dt}{t}]^{\frac{r}{2}}}{(1 + 2^j |\cdot - y|)^{ar}} \, dy \right]^{\frac{2}{r}} \right)^{\frac{r}{2}} \right\|_{L^{\frac{p(\cdot)}{r}}(\mathbb{R}^n)}$$

$$\lesssim \sum_{k=0}^{\infty} 2^{-k(N_0 r - n)} \left\| \left\{ \sum_{j \in \mathbb{Z}} 2^{j \frac{2n}{r}} \left[\int_{\mathbb{R}^n} \frac{[\int_1^2 |(\phi_{k+j})_t * f(y)|^2 \frac{dt}{t}]^{\frac{r}{2}}}{(1 + 2^j |\cdot - y|)^{ar}} \, dy \right]^{\frac{2}{r}} \right\}^{\frac{1}{2}} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}^r$$

$$\lesssim \sum_{k=0}^{\infty} 2^{-k(N_0 r - n)} \left\| \left\{ \sum_{j \in \mathbb{Z}} 2^{j \frac{2n}{r}} \left(\sum_{i=0}^{\infty} 2^{-iar} \right. \right. \right.$$

$$\left. \left. \left. \times \int_{|\cdot - y| \sim 2^{i-j}} \left[\int_1^2 |(\phi_{k+j})_t * f(y)|^2 \frac{dt}{t} \right]^{\frac{r}{2}} \, dy \right)^{\frac{2}{r}} \right\}^{\frac{1}{2}} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}^r,$$

where $N_0 \in \mathbb{N}$ is sufficiently large and $|x - y| \sim 2^{i-j}$ means that $|x - y| < 2^{-j}$ if $i = 0$, or $2^{i-j-1} \le |x - y| < 2^{i-j}$ if $i \in \mathbb{N}$. Applying the Minkowski inequality and Lemma 2.5, we conclude that

$$\|g_{a,*}(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)}^r$$

$$\lesssim \sum_{k=0}^{\infty} 2^{-kN_0 r + kn} \sum_{i=0}^{\infty} 2^{-iar + in} \left\| \left\{ \sum_{j \in \mathbb{Z}} \left[\mathcal{M} \left(\left[\int_1^2 |(\phi_{k+j})_t * f|^2 \frac{dt}{t} \right]^{\frac{r}{2}} \right) \right]^{\frac{2}{r}} \right\}^{\frac{r}{2}} \right\|_{L^{\frac{p(\cdot)}{r}}(\mathbb{R}^n)}$$

$$\lesssim \sum_{k=0}^{\infty} 2^{-kN_0 r + kn} \sum_{i=0}^{\infty} 2^{-iar + in} \left\| \left\{ \sum_{j \in \mathbb{Z}} \left[\int_1^2 |(\phi_{k+j})_t * f|^2 \frac{dt}{t} \right]^2 \right\}^{\frac{1}{2}} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}^r \lesssim \|g(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)}^r,$$

which completes the proof of (2.3).

Next we prove that

(2.5)
$$\|S(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \sim \|g_{a,*}(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

It suffices to show that $\|g_{a,*}(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim \|S(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)}$, since the inverse inequality holds true trivially. From [27, (3.9)], we deduce that

$$\|g_{a,*}(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)}$$

$$= \left\| \left\{ \sum_{j \in \mathbb{Z}} \left[\sum_{k=0}^{\infty} 2^{-kN_0 r + 2(k+j)n} \right. \right. \right.$$

$$\left. \left. \left. \times \sum_{i=0}^{\infty} \int_{|\cdot - y| \sim 2^{i-j}} \left(\int_1^2 \int_{|z| < 2^{-(k+j)} t} |(\phi_{k+j})_t * f(y+z)|^2 \frac{dz \, dt}{t} \right)^{\frac{r}{2}} \, dy \right]^{\frac{2}{r}} \right\} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)},$$

where $N_0 \in \mathbb{N}$ is sufficiently large and $|\cdot - y| \sim 2^{i-j}$ is the same as in (2.4). Then, by an argument similar to that used in the proof of (2.3), we conclude that $\|g_{a,*}(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim \|S(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)}$, which completes the proof of (2.5).

By arguments similar to those used in the proofs of (2.3), (2.5) and [40, Theorem 2.8], we conclude that

$$\|\sigma(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim \|g_{a,*}(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim \|\sigma_{a,*}(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)}. \tag{2.6}$$

Now, from (2.3), (2.5) and (2.6), we deduce that (2.2) holds true, which, together with Proposition 2.3, implies that $f \in H^{p(\cdot)}(\mathbb{R}^n)$ if and only if $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$ and $S(f) \in L^{p(\cdot)}(\mathbb{R}^n)$; moreover, $\|f\|_{H^{p(\cdot)}(\mathbb{R}^n)} \sim \|S(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)}$. This finishes the proof of Theorem 1.4. $\qquad\square$

Proof of Corollary 1.5. Assume $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$ and $g^*_\lambda(f) \in L^{p(\cdot)}(\mathbb{R}^n)$. It is easy to see that, for all $\lambda \in (1, \infty)$ and $x \in \mathbb{R}^n$, $S(f)(x) \lesssim g^*_\lambda(f)(x)$, which, together with Theorem 1.4, implies that $f \in H^{p(\cdot)}(\mathbb{R}^n)$ and $\|f\|_{H^{p(\cdot)}(\mathbb{R}^n)} \sim \|S(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim \|g^*_\lambda(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)}$.

Conversely, let $f \in H^{p(\cdot)}(\mathbb{R}^n)$. Then $f \in \mathcal{S}'(\mathbb{R}^n) \subset \mathcal{S}'_\infty(\mathbb{R}^n)$. By the fact that $\lambda \in (1 + 2/\min\{2, p_-\}, \infty)$, we see that there exists $a \in (n/\min\{2, p_-\}, \infty)$ such that $\lambda \in (1 + 2a/n, \infty)$. Then, by this, we further find that, for all $x \in \mathbb{R}^n$,

$$
\begin{aligned}
g^*_\lambda(f)(x) &= \left\{ \int_0^\infty \int_{\mathbb{R}^n} \left(\frac{t}{t + |x - y|} \right)^{\lambda n} |\phi_t * f(y)|^2 \, dy \, \frac{dt}{t^{n+1}} \right\}^{1/2} \\
&\lesssim \left\{ \int_0^\infty [(\phi^*_t f)_a(x)]^2 \int_{\mathbb{R}^n} \left(1 + \frac{|x - y|}{t} \right)^{2a - \lambda n} \, dy \, \frac{dt}{t^{n+1}} \right\}^{1/2} \\
&\sim \left\{ \int_0^\infty [(\phi^*_t f)_a(x)]^2 \, \frac{dt}{t} \right\}^{1/2} \sim g_{a,*}(f)(x).
\end{aligned}
$$

From this and Theorem 1.4, we deduce that

$$\|g^*_\lambda(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim \|g_{a,*}(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \sim \|f\|_{H^{p(\cdot)}(\mathbb{R}^n)},$$

which completes the proof of Corollary 1.5. $\qquad\square$

To prove Theorem 1.8, we need more preparations. The following technical lemma is essentially contained in [32].

Lemma 2.6. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfy (1.1) and (1.2). Then there exists a positive constant C such that, for all cubes $Q_1 \subset Q_2$,*

$$\|\chi_{Q_1}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C \left(\frac{|Q_1|}{|Q_2|} \right)^{1/p_+} \|\chi_{Q_2}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \tag{2.7}$$

and

$$\|\chi_{Q_2}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C \left(\frac{|Q_2|}{|Q_1|} \right)^{1/p_-} \|\chi_{Q_1}\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

Proof. For similarity, we only show (2.7). Let $z_0 \in Q_1$. If $\ell(Q_2) \le 1$, then, by [32, Lemma 2.2(1)] and its proof, we see that

$$\frac{\|\chi_{Q_1}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_{Q_2}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \sim \left(\frac{|Q_1|}{|Q_2|} \right)^{\frac{1}{p(z_0)}} \lesssim \left(\frac{|Q_1|}{|Q_2|} \right)^{\frac{1}{p_+}}.$$

If $\ell(Q_1) \ge 1$, then by [32, Lemma 2.2(2)], we find that

$$\frac{\|\chi_{Q_1}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_{Q_2}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \sim \left(\frac{|Q_1|}{|Q_2|} \right)^{\frac{1}{p_\infty}} \lesssim \left(\frac{|Q_1|}{|Q_2|} \right)^{\frac{1}{p_+}},$$

where p_∞ is as in (1.2). If $\ell(Q_1) < 1 < \ell(Q_2)$, then by [32, Lemma 2.2], we know that

$$\frac{\|\chi_{Q_1}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_{Q_2}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \sim \frac{|Q_1|^{1/p(z_0)}}{|Q_2|^{1/p_\infty}} \lesssim \left(\frac{|Q_1|}{|Q_2|}\right)^{\frac{1}{p_+}},$$

which completes the proof of (2.7) and hence Lemma 2.6. $\square$

The following Lemma 2.7 comes from [39, p.38].

Lemma 2.7. *Let $g \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, $s \in \mathbb{Z}_+$ and Q be a cube in $\mathbb{R}^n$. Then there exists a positive constant C, independent of g and Q, such that*

$$\sup_{x \in Q} |P^s_Q g(x)| \leq \frac{C}{|Q|} \int_Q |g(x)| \, dx.$$

Lemma 2.8. *Let $\alpha \in (0,1]$, $s \in \mathbb{Z}_+$ and $\epsilon \in (\alpha + s, \infty)$. Assume that $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfies (1.1), (1.2) and $p_- \in (n/(n+\alpha+s), 1]$. If $f \in \mathcal{C}_{(\alpha,\epsilon),s}(\mathbb{R}^n)$ or $\mathcal{S}(\mathbb{R}^n)$, then $f \in \mathcal{L}_{1,p(\cdot),s}(\mathbb{R}^n)$.*

Proof. For similarity, we only give the proof for $\mathcal{C}_{(\alpha,\epsilon),s}(\mathbb{R}^n)$. For any $f \in \mathcal{C}_{(\alpha,\epsilon),s}(\mathbb{R}^n)$, $x \in \mathbb{R}^n$ and cube $Q := Q(x_0, r) \subset \mathbb{R}^n$ with $(x_0, r) \in \mathbb{R}^{n+1}_+$, let

$$p_Q(x) := \sum_{|\beta| \leq s} \frac{D^\beta f(x_0)}{\beta!} (x - x_0)^\beta \in \mathcal{P}_s(\mathbb{R}^n).$$

Then, from Lemma 2.7 and Taylor's remainder theorem, we deduce that, for any $x \in Q$, there exists $\xi(x) \in Q$ such that

$$
\begin{aligned}
(2.8) \qquad \int_Q |f(x) - P^s_Q f(x)| \, dx &\leq \int_Q |f(x) - p_Q(x)| \, dx + \int_Q |P^s_Q(p_Q - f)(x)| \, dx \\
&\lesssim \int_Q |f(x) - p_Q(x)| \, dx \\
&\sim \int_Q \left| \sum_{|\beta|=s} \frac{D^\beta f(\xi(x)) - D^\beta f(x_0)}{\beta!} (x - x_0)^\beta \right| \, dx.
\end{aligned}
$$

Now, if $|x_0| + r \leq 1$, namely, $Q \subset Q(0, \sqrt{n})$, then, by Lemma 2.6, (2.8), (1.11) and the fact that $p_- \in (n/(n+\alpha+s), 1]$, we see that

$$
\begin{aligned}
(2.9) \quad &\frac{1}{\|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \int_Q |f(x) - P^s_Q f(x)| \, dx \\
&\lesssim \left\{ \sup_{x,\, y \in \mathbb{R}^n, x \neq y} \sum_{|\beta|=s} \frac{|D^\beta f(x) - D^\beta f(y)|}{|x-y|^\alpha} \right\} \frac{1}{\|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \int_Q |\xi(x) - x_0|^\alpha |x - x_0|^s \, dx \\
&\lesssim |Q|^{1+(\alpha+s)/n - 1/p_-} \frac{|Q(0,\sqrt{n})|^{1/p_-}}{\|\chi_{Q(0,\sqrt{n})}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \lesssim 1.
\end{aligned}
$$

If $|x_0| + r > 1$ and $|x_0| \leq 2r$, then $r > 1/3$ and $|Q| \sim |Q(0, \sqrt{n}(|x_0| + r))|$. From Lemma 2.7 and $|f(x)| \leq (1 + |x|)^{-n-\epsilon}$ for all $x \in \mathbb{R}^n$, we deduce that

$$(2.10) \quad \frac{1}{\|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \int_Q |f(x) - P^s_Q f(x)| \, dx$$

$$\lesssim \frac{1}{\|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \int_Q |f(x)|\, dx \lesssim \sup_{y\in\mathbb{R}^n} \left[(1+|y|)^{n+\varepsilon}|f(y)|\right] \frac{1}{\|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \int_Q \frac{1}{(1+|x|)^{n+\varepsilon}}\, dx$$

$$\lesssim \left[\frac{|Q(0,\sqrt{n}(|x_0|+r))|}{|Q|}\right]^{1/p_-} \frac{1}{\|\chi_{Q(0,\sqrt{n}(|x_0|+r))}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \lesssim 1.$$

If $|x_0|+r>1$ and $|x_0|>2r$, then, for all $x\in Q$, it holds that $1\lesssim |x|\sim |x_0|$. By this, (2.8), Lemma 2.6 and (1.11), we find that

$$(2.11) \qquad \frac{1}{\|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \int_Q |f(x) - P_Q^s f(x)|\, dx$$

$$\lesssim \frac{1}{\|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \int_Q |\xi(x) - x_0|^\alpha (1+|x_0|)^{-n-\epsilon}|x-x_0|^s\, dx$$

$$\lesssim \frac{|Q|^{1+\frac{\alpha+s}{n}}}{\|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)}} (|x_0|+r)^{-n-\epsilon}$$

$$\lesssim \frac{|Q|^{1+(\alpha+s)/n}}{(|x_0|+r)^{n+\epsilon}} \left(\frac{|Q(0,\sqrt{n}(|x_0|+r))|}{|Q|}\right)^{\frac{1}{p_-}} \frac{1}{\|\chi_{Q(0,\sqrt{n}(|x_0|+r))}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \lesssim 1.$$

Combining (2.9), (2.10) and (2.11), we see that $f\in\mathcal{L}_{1,p(\cdot),s}(\mathbb{R}^n)$, which completes the proof of Lemma 2.8. $\qquad\square$

Remark 2.9. We point out that, from the proof of Lemma 2.8, we know that $\mathcal{C}_{(\alpha,\varepsilon),s}(\mathbb{R}^n)$ and $\mathcal{S}(\mathbb{R}^n)$ are continuously embedding into $\mathcal{L}_{1,p(\cdot),s}(\mathbb{R}^n)$, which, in the case of $s=0$ and $p(x):=1$ for all $x\in\mathbb{R}^n$, was proved in [33, Proposition 2.1]. Indeed, by the proof of Lemma 2.8, we see that, for all $f\in\mathcal{C}_{(\alpha,\varepsilon),s}(\mathbb{R}^n)$ or $\mathcal{S}(\mathbb{R}^n)$,

$$\|f\|_{\mathcal{L}_{1,p(\cdot),s}(\mathbb{R}^n)} \lesssim \sup_{x\in\mathbb{R}^n}(1+|x|)^{n+\varepsilon}|f(x)|$$

$$+ \sup_{x,\,y\in\mathbb{R}^n, x\neq y} \sum_{|\beta|=s}\left\{\left[\frac{1}{(1+|x|)^{n+\varepsilon}} + \frac{1}{(1+|y|)^{n+\varepsilon}}\right]^{-1} \frac{|D^\beta f(x) - D^\beta f(y)|}{|x-y|^\alpha}\right\};$$

moreover, if $f\in\mathcal{C}_{(\alpha,\varepsilon),s}(\mathbb{R}^n)$, then $\|f\|_{\mathcal{L}_{1,p(\cdot),s}(\mathbb{R}^n)}\lesssim 1$; if $f\in\mathcal{S}(\mathbb{R}^n)$, then

$$\|f\|_{\mathcal{L}_{1,p(\cdot),s}(\mathbb{R}^n)} \lesssim \sup_{x\in\mathbb{R}^n} \sum_{\beta\in\mathbb{Z}_+^n,\,|\beta|\leq s+1}(1+|x|)^{n+\varepsilon}|D^\beta f(x)|.$$

In this sense, $\mathcal{C}_{(\alpha,\varepsilon),s}(\mathbb{R}^n)$ and $\mathcal{S}(\mathbb{R}^n)$ are continuously embedding into $\mathcal{L}_{1,p(\cdot),s}(\mathbb{R}^n)$.

Now we recall the atomic Hardy space with variable exponent introduced by Nakai and Sawano [32]. Let $p(\cdot)\in\mathcal{P}(\mathbb{R}^n)$, $s\in(n/p_- - n - 1,\infty)\cap\mathbb{Z}_+$ and $q\in[1,\infty]$ satisfy that $q\in[p_+,\infty)$. Recall that a measurable function a on $\mathbb{R}^n$ is called a $(p(\cdot),q,s)$-*atom* if it satisfies the following three conditions:

(i) $\operatorname{supp} a\subset Q$ for some $Q\subset\mathbb{R}^n$;

(ii) $\|a\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq \dfrac{|Q|^{1/q}}{\|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)}}$;

(iii) $\int_{\mathbb{R}^n} a(x)x^\beta\, dx = 0$ for any $\beta\in\mathbb{Z}_+^n$ and $|\beta|\leq s$.

The *atomic Hardy space with variable* $p(\cdot)$, denoted by $H^{p(\cdot),q}_{\text{atom}}(\mathbb{R}^n)$, is defined to be the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ that can be represented as a sum of multiples of $(p(\cdot), q, s)$-atoms, namely, $f = \sum_j \lambda_j a_j$ in $\mathcal{S}'(\mathbb{R}^n)$, where, for each j, λ_j is a nonnegative number and a_j is a $(p(\cdot), q, s)$-atom supported in some cube Q_j with the property

$$\int_{\mathbb{R}^n} \left\{ \sum_j \left(\frac{\lambda_j \chi_{Q_j}(x)}{\|\chi_{Q_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \right)^{p^*} \right\}^{\frac{p(x)}{p^*}} dx < \infty$$

with $p^* := \min\{p_-, 1\}$. The *norm* of $f \in H^{p(\cdot),q}_{\text{atom}}(\mathbb{R}^n)$ is defined by

$$\|f\|_{H^{p(\cdot),q}_{\text{atom}}(\mathbb{R}^n)} := \inf \left\{ \mathcal{A}(\{\lambda_j\}_j, \{Q_j\}_j) : \ f = \sum_j \lambda_j a_j \text{ in } \mathcal{S}'(\mathbb{R}^n) \right\},$$

where the infimum is taken over all decompositions of f as above and

$$\mathcal{A}(\{\lambda_j\}_j, \{Q_j\}_j) := \inf \left\{ \lambda \in (0, \infty) : \ \int_{\mathbb{R}^n} \left(\sum_j \left[\frac{\lambda_j \chi_{Q_j}(x)}{\lambda \|\chi_{Q_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \right]^{p^*} \right)^{\frac{p(x)}{p^*}} dx \leq 1 \right\}.$$

The following conclusion is just [32, Lemma 4.11].

Lemma 2.10. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfy (1.1) and (1.2). Then there exist $\beta_0 \in (0, 1)$ and a positive constant C such that, if $q \in (0, \infty)$ satisfies $1/q \in (0, -\log_2 \beta_0/(n+1))$, then, for all sequences $\{\lambda_j\}_j$ of nonnegative numbers, measurable functions $\{b_j\}_j$ and cubes $\{Q_j\}_j$ satisfying $\operatorname{supp} b_j \subset Q_j$ and $\|b_j\|_{L^q(Q_j)} \neq 0$ for each j,*

$$\left\| \left\{ \sum_j \left(\frac{\lambda_j |b_j| |Q_j|^{1/q}}{\|b_j\|_{L^q(Q_j)} \|\chi_{Q_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \right)^{p^*} \right\}^{\frac{1}{p^*}} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C \mathcal{A}(\{\lambda_j\}_j, \{Q_j\}_j).$$

Let $q \in [1, \infty]$ and $s \in \mathbb{Z}_+$. Denote by $L^{q,s}_{\text{comp}}(\mathbb{R}^n)$ the *set of all functions* $f \in L^\infty(\mathbb{R}^n)$ *with compact* and

$$L^{q,s}_{\text{comp}}(\mathbb{R}^n) := \left\{ f \in L^q_{\text{comp}}(\mathbb{R}^n) : \ \int_{\mathbb{R}^n} f(x) x^\alpha \, dx = 0, \ |\alpha| \leq s \right\}.$$

As point out in [32, p. 3707], $L^{q,s}_{\text{comp}}(\mathbb{R}^n)$ is dense in $H^{p(\cdot),q}_{\text{atom}}(\mathbb{R}^n)$. The conclusions of the following Lemmas 2.11 and 2.12 were, respectively, just [32, Theorems 4.6] and [32, Theorem 7.5], which play key roles in the proof of Theorem 1.4.

Lemma 2.11. *Let $q \in [1, \infty]$ and $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfy (1.1), (1.2) and $p_+ \in (0, q)$. Assume that q is as in Lemma 2.10. Then $H^{p(\cdot)}(\mathbb{R}^n) = H^{p(\cdot),q}_{\text{atom}}(\mathbb{R}^n)$ with equivalent quasi-norms.*

Lemma 2.12. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfy (1.1), (1.2), $p_+ \in (0, 1]$, $q \in (p_+, \infty)$ and $s \in (n/p_- - n - 1, \infty) \cap \mathbb{Z}_+$. Then the dual space of $H^{p(\cdot),q}_{\text{atom}}(\mathbb{R}^n)$, denoted by $(H^{p(\cdot),q}_{\text{atom}}(\mathbb{R}^n))^*$, is $\mathcal{L}_{q',p(\cdot),s}(\mathbb{R}^n)$ in the following sense: for any $b \in \mathcal{L}_{q',p(\cdot),s}(\mathbb{R}^n)$, the linear functional*

$$(2.12) \qquad\qquad \ell_b(f) := \int_{\mathbb{R}^n} b(x) f(x) \, dx,$$

initial defined for all $f \in L^{q,s}_{\mathrm{comp}}(\mathbb{R}^n)$, *has a bounded extension to* $H^{p(\cdot),q}_{\mathrm{atom}}(\mathbb{R}^n)$; *conversely, if* ℓ *is a bounded linear functional on* $H^{p(\cdot),q}_{\mathrm{atom}}(\mathbb{R}^n)$, *then* ℓ *has the form as in* (2.12) *with a unique* $b \in \mathcal{L}_{q',p(\cdot),s}(\mathbb{R}^n)$.

Moreover,

$$\|b\|_{\mathcal{L}_{q',p(\cdot),s}(\mathbb{R}^n)} \sim \|\ell_b\|_{(H^{p(\cdot),q}_{\mathrm{atom}}(\mathbb{R}^n))^*},$$

where the implicit positive constants are independent of b.

The following Lemma 2.13 is just from [28, Theorem 2.6], which, in the case when $s = 0$, was first proved by Wilson [44, Theorem 2].

Lemma 2.13. *Let* $\alpha \in (0,1]$, $s \in \mathbb{Z}_+$ *and* $\epsilon \in (\max\{\alpha, s\}, \infty)$. *Then there exists a positive constant* C *such that, for all* f *satisfying* (1.12) *and* $x \in \mathbb{R}^n$,

$$\frac{1}{C} g_{\alpha,s}(f)(x) \le \widetilde{g}_{(\alpha,\epsilon),s}(f)(x) \le C g_{\alpha,s}(f)(x).$$

The following Lemma 2.14 is a special case of [28, Proposition 3.2].

Lemma 2.14. *Let* $\alpha \in (0,1]$, $s \in \mathbb{Z}_+$ *and* $q \in (1,\infty)$. *Then there exists a positive constant* C *such that, for all measurable functions* f,

$$\int_{\mathbb{R}^n} [g_{\alpha,s}(f)(x)]^q \, dx \le C \int_{\mathbb{R}^n} |f(x)|^q \, dx.$$

Now we come to give a proof of Theorem 1.8.

Proof of Theorem 1.8. For $\epsilon \in (\alpha + s, \infty)$, by Lemma 2.13, we see that $g_{\alpha,s}(f)$ and $\widetilde{g}_{(\alpha,\epsilon),s}(f)$ are pointwise comparable. Thus, to prove Theorem 1.8, it suffices to show that the conclusion of Theorem 1.8 holds true for the intrinsic square function $g_{\alpha,s}(f)$.

Let $f \in (\mathcal{L}_{1,p(\cdot),s}(\mathbb{R}^n))^*$ vanish weakly at infinity and $g_{\alpha,s}(f) \in L^{p(\cdot)}(\mathbb{R}^n)$. Then, by Lemma 2.8, we find that $f \in \mathcal{S}'(\mathbb{R}^n) \subset \mathcal{S}'_\infty(\mathbb{R}^n)$. Notice that, for all $x \in \mathbb{R}^n$, $g(f)(x) \lesssim \widetilde{g}_{(\alpha,\epsilon),s}(f)(x) \sim g_{\alpha,s}(f)(x)$ (see Lemma 2.13), it follows that $g(f) \in L^{p(\cdot)}(\mathbb{R}^n)$. From this and Theorem 1.4, we deduce that there exists a distribution $\widetilde{f} \in \mathcal{S}'(\mathbb{R}^n)$ such that $\widetilde{f} = f$ in $\mathcal{S}'_\infty(\mathbb{R}^n)$, $\widetilde{f} \in H^{p(\cdot)}(\mathbb{R}^n)$ and $\|\widetilde{f}\|_{H^{p(\cdot)}(\mathbb{R}^n)} \lesssim \|g(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)}$, which, together with Corollary 2.2 and the fact that f vanishes weakly at infinity, implies that $f = \widetilde{f}$ in $\mathcal{S}'(\mathbb{R}^n)$ and hence

$$\|f\|_{H^{p(\cdot)}(\mathbb{R}^n)} \sim \|\widetilde{f}\|_{H^{p(\cdot)}(\mathbb{R}^n)} \lesssim \|g(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim \|g_{\alpha,s}(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

This finishes the proof of the sufficiency of Theorem 1.4.

It remains to prove the necessity. Let $f \in H^{p(\cdot)}(\mathbb{R}^n)$. Then, by Corollary 2.2, we see that f vanishes weakly at infinity and, by Lemmas 2.11 and 2.12, we have $f \in (\mathcal{L}_{1,p(\cdot),s}(\mathbb{R}^n))^*$. If $q \in (1,\infty)$ is as in Lemma 2.10, then, by Lemma 2.11, we know that there exist a sequence $\{\lambda_j\}_j$ of nonnegative numbers and a sequence $\{a_j\}_j$ of $(p(\cdot),q,s)$-atoms, with supp $a_j \subset Q_j$ for all j, such that $f = \sum_j \lambda_j a_j$ in $\mathcal{S}'(\mathbb{R}^n)$ and also in $H^{p(\cdot)}(\mathbb{R}^n)$ and, moreover

$$(2.13) \qquad\qquad \mathcal{A}(\{\lambda_j\}_j, \{Q_j\}_j) \lesssim \|f\|_{H^{p(\cdot)}(\mathbb{R}^n)}.$$

Thus, by Lemma 2.8, we find that, for all $\phi \in \mathcal{C}_{(\alpha,\varepsilon),s}(\mathbb{R}^n)$, $f * \phi = \sum_j \lambda_j a_j * \phi$ pointwise and hence, for all $x \in \mathbb{R}^n$, $g_{\alpha,s}(f)(x) \le \sum_j \lambda_j g_{\alpha,s}(a_j)(x)$.

Now, for a $(p(\cdot),q,s)$-atom a with supp $a \subset Q := Q(x_0, r)$, we estimate $g_{\alpha,s}(a)$. By Lemma 2.14, we find that

$$(2.14) \qquad\qquad \|g_{\alpha,s}(a)\|_{L^q(2\sqrt{n}Q)} \lesssim \|a\|_{L^q(\mathbb{R}^n)} \lesssim \frac{|Q|^{1/q}}{\|Q\|_{L^{p(\cdot)}(\mathbb{R}^n)}},$$

here and hereafter, $2\sqrt{n}Q$ denotes the cube with the center same as Q but with the side length $2\sqrt{n}$ times Q.

On the other hand, for all $x \notin 2\sqrt{n}Q$, by the vanishing moment condition of a and (1.10), together with Taylor's remainder theorem, we see that

$$(2.15) \qquad |a * \phi_t(x)| = \frac{1}{t^n}\left|\int_{\mathbb{R}^n} a(y)\left[\phi\left(\frac{x-y}{t}\right) - \sum_{|\beta|\le s}\frac{D^\beta\phi(\frac{x-x_0}{t})}{\beta!}\left(\frac{x_0-y}{t}\right)^\beta\right] dy\right|$$

$$\lesssim \int_{\mathbb{R}^n}|a(y)|\frac{|y-x_0|^{\alpha+s}}{t^{n+\alpha+s}}\,dy \lesssim \frac{1}{\|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\left(\frac{r}{t}\right)^{n+\alpha+s}.$$

Notice that $\operatorname{supp}\phi \subset \{x \in \mathbb{R}^n : |x| \le 1\}$. If $x \notin 2\sqrt{n}Q$ and $\phi_t * a(x) \ne 0$, then, there exists a $y \in Q$ such that $|x-y|/t \le 1$ and hence $t \ge |x-y| \ge |x-x_0| - |x_0-y| > |x-x_0|/2$. From this and (2.15), we deduce that

$$g_{\alpha,s}(a)(x) = \left\{\int_0^\infty\left[\sup_{\phi\in\mathcal{C}_{\alpha,s}(\mathbb{R}^n)}|a*\phi_t(x)|\right]^2\frac{dt}{t}\right\}^{1/2}$$

$$\lesssim \frac{1}{\|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)}}r^{n+\alpha+s}\left\{\int_{\frac{|x-x_0|}{2}}^\infty t^{-2(n+\alpha+s)}\,dt\right\}^{1/2}$$

$$\lesssim \frac{1}{\|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\left(\frac{r}{|x-x_0|}\right)^{n+\alpha+s} \lesssim \frac{[\mathcal{M}(\chi_Q)(x)]^{\frac{n+\alpha+s}{n}}}{\|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)}},$$

which implies that

$$(2.16)\quad \|g_{\alpha,s}(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim \left\|\sum_j\lambda_j g_{\alpha,s}(a_j)\chi_{2\sqrt{n}Q_j}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} + \left\|\sum_j\lambda_j\frac{[\mathcal{M}(\chi_{Q_j})]^{\frac{n+\alpha+s}{n}}}{\|\chi_{Q_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}$$

$$=: \mathrm{I}_1 + \mathrm{I}_2.$$

For I_1, by taking $b_j := g_{\alpha,s}(a_j)\chi_{2\sqrt{n}Q_j}$ for each j in Lemma 2.10, (2.14) and Lemma 2.6, we conclude that

$$(2.17)\quad \mathrm{I}_1 \lesssim \left\|\sum_j\frac{\lambda_j b_j|Q_j|^{\frac{1}{q}}}{\|b_j\|_{L^q(2\sqrt{n}Q_j)}\|\chi_{2\sqrt{n}Q_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}$$

$$\lesssim \left\|\left\{\sum_j\left(\frac{\lambda_j b_j|Q_j|^{\frac{1}{q}}}{\|b_j\|_{L^q(2\sqrt{n}Q_j)}\|\chi_{2\sqrt{n}Q_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\right)^{p^*}\right\}^{1/p^*}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim \mathcal{A}(\{\lambda_j\}_j,\{Q_j\}_j).$$

For I_2, letting $\theta := (n+\alpha+s)/n$, by Lemma 2.5 and $p_- \in (n/(n+\alpha+s),\infty)$, we find that

$$\mathrm{I}_2 \lesssim \left\|\left\{\sum_j\frac{\lambda_j[\mathcal{M}(\chi_{Q_j})]^\theta}{\|\chi_{Q_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\right\}^{\frac{1}{\theta}}\right\|_{L^{\theta p(\cdot)}(\mathbb{R}^n)}^\theta \lesssim \left\|\sum_j\frac{\lambda_j\chi_{Q_j}}{\|\chi_{Q_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}$$

$$\lesssim \left\|\left\{\sum_j\left(\frac{\lambda_j\chi_{Q_j}}{\|\chi_{Q_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\right)^{p^*}\right\}^{\frac{1}{p^*}}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \sim \mathcal{A}(\{\lambda_j\}_j,\{Q_j\}_j).$$

From this, together with (2.13), (2.16) and (2.17), we deduce that

$$\|g_{\alpha,s}(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim \|f\|_{H^{p(\cdot)}(\mathbb{R}^n)},$$

which completes the proof of Theorem 1.8. $\square$

For $s \in \mathbb{Z}_+$, $\alpha \in (0,1]$ and $\epsilon \in (0,\infty)$, let $\mathcal{C}_{(\alpha,\epsilon),s}(y,t)$, with $y \in \mathbb{R}^n$ and $t \in (0,\infty)$, be the *family of functions* $\psi \in C^s(\mathbb{R}^n)$ *such that, for all* $\gamma \in \mathbb{Z}_+^n$, $|\gamma| \leq s$ and $x \in \mathbb{R}^n$, $|D^\gamma \psi(x)| \leq t^{-n-|\gamma|}(1 + |y - x|/t)^{-n-\epsilon}$, $\int_{\mathbb{R}^n} \psi(x) x^\gamma\, dx = 0$ and, for all $x_1,\ x_2 \in \mathbb{R}^n$, $\nu \in \mathbb{Z}_+^n$ and $|\nu| = s$,

$$|D^\nu \psi(x_1) - D^\nu \psi(x_2)| \leq \frac{|x_1 - x_2|^\alpha}{t^{n+\gamma+\alpha}} \left[\left(1 + \frac{|y - x_1|}{t}\right)^{-n-\epsilon} + \left(1 + \frac{|y - x_2|}{t}\right)^{-n-\epsilon} \right].$$

The proof of Theorem 1.10 needs the following Lemma 2.15, whose proof is trivial, the details being omitted.

Lemma 2.15. *Let* $s \in \mathbb{Z}_+$, $\alpha \in (0,1]$, $\epsilon \in (0,\infty)$ *and* f *be a measurable function satisfying* (1.12).

(i) *For any* $y \in \mathbb{R}^n$ *and* $t \in (0,\infty)$, *it holds true that*

$$\widetilde{A}_{(\alpha,\epsilon),s}(f)(y,t) = \sup_{\psi \in \mathcal{C}_{(\alpha,\epsilon),s}(y,t)} \left| \int_{\mathbb{R}^n} \psi(x) f(x)\, dx \right|.$$

(ii) *If* $t_1,\ t_2 \in (0,\infty)$, $t_1 < t_2$, $y \in \mathbb{R}^n$ *and* $\psi \in \mathcal{C}_{(\alpha,\epsilon),s}(y,t_1)$, *then* $(\frac{t_1}{t_2})^{n+s+\alpha}\psi \in \mathcal{C}_{(\alpha,\epsilon),s}(y,t_2)$.

Proof of Theorem 1.10. If $f \in (\mathcal{L}_{1,p(\cdot),s}(\mathbb{R}^n))^*$, $\widetilde{g}^*_{\lambda,(\alpha,\epsilon),s}(f) \in L^{p(\cdot)}(\mathbb{R}^n)$ and f vanishes weakly at infinity, then, by Lemma 2.8, we see that $f \in \mathcal{S}'(\mathbb{R}^n)$ and, by the fact that, for all $x \in \mathbb{R}^n$,

$$g^*_\lambda(f)(x) \lesssim g^*_{\lambda,\alpha,s}(f)(x) \lesssim \widetilde{g}^*_{\lambda,(\alpha,\epsilon),s}(f)(x)$$

and Theorem 1.4, we further know that $f \in H^{p(\cdot)}(\mathbb{R}^n)$ and

$$\|f\|_{H^{p(\cdot)}(\mathbb{R}^n)} \lesssim \|g^*_\lambda(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim \|g^*_{\lambda,\alpha,s}(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim \|\widetilde{g}^*_{\lambda,(\alpha,\epsilon),s}(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

This finishes the proof of the sufficiency of Theorem 1.10.

Next we prove the necessity of Theorem 1.10. Let $f \in H^{p(\cdot)}(\mathbb{R}^n)$. Then, as in the proof of Theorem 1.8, we see that $f \in (\mathcal{L}_{1,p(\cdot),s}(\mathbb{R}^n))^*$ and f vanishes weakly at infinity. For all $x \in \mathbb{R}^n$, we have

$$(2.18)\quad [\widetilde{g}^*_{\lambda,(\alpha,\epsilon),s}(f)(x)]^2$$

$$= \int_0^\infty \int_{|y-x|<t} \left(\frac{t}{t + |x - y|} \right)^{\lambda n} [\widetilde{A}_{(\alpha,\epsilon),s}(f)(y,t)]^2\, \frac{dy\, dt}{t^{n+1}}$$

$$+ \sum_{k=1}^\infty \int_0^\infty \int_{2^{k-1}t \leq |y-x| < 2^k t} \cdots \frac{dy\, dt}{t^{n+1}}$$

$$\lesssim [\widetilde{S}_{(\alpha,\epsilon),s}(f)(x)]^2 + \sum_{k=1}^\infty 2^{-k\lambda n} \int_0^\infty \int_{|y-x|<2^k t} [\widetilde{A}_{(\alpha,\epsilon),s}(f)(y,t)]^2\, \frac{dy\, dt}{t^{n+1}}$$

$$\sim [\widetilde{S}_{(\alpha,\epsilon),s}(f)(x)]^2 + \sum_{k=1}^\infty 2^{-k\lambda n} 2^{kn} \int_0^\infty \int_{|y-x|<t} [\widetilde{A}_{(\alpha,\epsilon),s}(f)(y,2^{-k}t)]^2\, \frac{dy\, dt}{t^{n+1}}.$$

By Lemma 2.15, we find that, for all $k \in \mathbb{N}$ and $(y, t) \in \mathbb{R}^{n+1}_+$,

$$\widetilde{A}_{(\alpha,\epsilon),s}(f)(y, 2^{-k}t) = \sup_{\psi \in \mathcal{C}_{(\alpha,\epsilon),s}(y,2^{-k}t)} \left| \int_{\mathbb{R}^n} \psi(x)f(x)\,dx \right|$$

$$\leq 2^{k(n+s+\alpha)} \sup_{\widetilde{\psi} \in \mathcal{C}_{(\alpha,\epsilon),s}(y,t)} \left| \int_{\mathbb{R}^n} \widetilde{\psi}(x)f(x)\,dx \right| = 2^{k(n+s+\alpha)} \widetilde{A}_{(\alpha,\epsilon),s}(f)(y,t),$$

which, together with (2.18) and $\lambda \in (3 + 2(s+\alpha)/n, \infty)$, implies that

$$[\widetilde{g}^*_{\lambda,(\alpha,\epsilon),s}(f)(x)]^2 \lesssim \sum_{k=0}^{\infty} 2^{-k\lambda n} 2^{k(3n+2s)} [\widetilde{S}_{(\alpha,\epsilon),s}(f)(x)]^2 \sim [\widetilde{S}_{(\alpha,\epsilon),s}(f)(x)]^2.$$

From this, together with Theorem 1.9, we deduce that $\widetilde{g}^*_{\lambda,(\alpha,\epsilon),s}(f) \in L^{p(\cdot)}(\mathbb{R}^n)$ and

$$\|g^*_{\lambda,\alpha,s}(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim \|\widetilde{g}^*_{\lambda,(\alpha,\epsilon),s}(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim \|\widetilde{S}_{(\alpha,\epsilon),s}(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim \|f\|_{H^{p(\cdot)}(\mathbb{R}^n)},$$

which completes the proof of Theorem 1.10. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

To prove Theorem 1.13, we first introduce the tent space with variable exponent. For all measurable functions g on $\mathbb{R}^{n+1}_+$ and $x \in \mathbb{R}^n$, define

$$\mathcal{A}(g)(x) := \left\{ \int_0^\infty \int_{\{y \in \mathbb{R}^n:\, |y-x| < t\}} |g(y,t)|^2 \frac{dy\,dt}{t^{n+1}} \right\}^{1/2}.$$

Recall that a measurable function g is said to belong to the *tent space* $T_2^p(\mathbb{R}^{n+1}_+)$ with $p \in (0, \infty)$, if $\|g\|_{T_2^p(\mathbb{R}^{n+1}_+)} := \|\mathcal{A}(g)\|_{L^p(\mathbb{R}^n)} < \infty$.

Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfy (1.1) and (1.2). In what follows, we denote by $T_2^{p(\cdot)}(\mathbb{R}^{n+1}_+)$ the *space* of all measurable functions g on $\mathbb{R}^{n+1}_+$ such that $\mathcal{A}(g) \in L^{p(\cdot)}(\mathbb{R}^n)$ and, for any $g \in T_2^{p(\cdot)}(\mathbb{R}^{n+1}_+)$, its *quasi-norm* is defined by

$$\|g\|_{T_2^{p(\cdot)}(\mathbb{R}^{n+1}_+)} := \|\mathcal{A}(g)\|_{L^{p(\cdot)}(\mathbb{R}^n)} := \inf \left\{ \lambda \in (0, \infty) : \int_{\mathbb{R}^n} \left(\frac{\mathcal{A}(g)(x)}{\lambda} \right)^{p(x)} dx \leq 1 \right\}.$$

Let $p \in (1, \infty)$. A function a on $\mathbb{R}^{n+1}_+$ is called a $(p(\cdot), p)$-*atom* if there exists a cube $Q \subset \mathbb{R}^n$ such that supp $a \subset \widehat{Q}$ and $\|a\|_{T_2^p(\mathbb{R}^{n+1}_+)} \leq |Q|^{1/p} \|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{-1}$. Furthermore, if a is a $(p(\cdot), p)$-atom for all $p \in (1, \infty)$, we then call a a $(p(\cdot), \infty)$-*atom*.

For functions in the space $T_2^{p(\cdot)}(\mathbb{R}^{n+1}_+)$, we have the following atomic decomposition.

Theorem 2.16. *Let* $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ *satisfy* (1.1) *and* (1.2). *Then, for any* $f \in T_2^{p(\cdot)}(\mathbb{R}^{n+1}_+)$, *there exist* $\{\lambda_j\}_j \subset \mathbb{C}$ *and a sequence* $\{a_j\}_j$ *of* $(p(\cdot), \infty)$-*atoms such that, for almost every* $(x, t) \in \mathbb{R}^{n+1}_+$, $f(x,t) = \sum_j \lambda_j a_j(x,t)$. *Moreover, there exists a positive constant* C *such that, for all* $f \in T_2^{p(\cdot)}(\mathbb{R}^{n+1}_+)$, $\mathcal{A}^*(\{\lambda_j\}_j, \{Q_j\}_j) \leq C\|f\|_{T_2^{p(\cdot)}(\mathbb{R}^{n+1}_+)}$, *where*

$$(2.19) \quad \mathcal{A}^*(\{\lambda_j\}_j, \{Q_j\}_j) := \inf \left\{ \lambda \in (0, \infty) : \sum_j \int_{Q_j} \left[\frac{\lambda_j}{\lambda \|\chi_{Q_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \right]^{p(x)} dx \leq 1 \right\}$$

and, for each j, Q_j *appears in the support of* a_j.

Remark 2.17. Assume that $p_+ \in (0,1]$. Then, by [32, Remark 4.4], we know that, for any sequences $\{\lambda_j\}_j$ of nonnegative numbers and cubes $\{Q_j\}_j$, $\sum_j \lambda_j \leq \mathcal{A}^*(\{\lambda_j\}_j, \{Q_j\}_j)$.

The proof of Theorem 2.16 is similar to that of [18, Theorem 3.2] (see also [22, Theorem 3.1]). To this end, we need some known facts as follows (see, for example, [22, Theorem 3.1]).

Let F be a closed subset of $\mathbb{R}^n$ and $O := \mathbb{R}^n \backslash F =: F^{\complement}$. Assume that $|O| < \infty$. For any fixed $\gamma \in (0,1)$, $x \in \mathbb{R}^n$ is said to have the *global γ-density* with respect to F if, for all $t \in (0,\infty)$, $|B(x,t) \cap F|/|B(x,t)| \geq \gamma$. Denote by F_γ^* the *set of all such x* and let $O_\gamma^* := (F_\gamma^*)^{\complement}$. Then

$$O_\gamma^* = \{x \in \mathbb{R}^n : \; \mathcal{M}(\chi_O)(x) > 1 - \gamma\}$$

is open, $O \subset O_\gamma^*$ and there exists a positive constant $C_{(\gamma)}$, depending on γ, such that $|O_\gamma^*| \leq C_{(\gamma)}|O|$. For any $\nu \in (0,\infty)$ and $x \in \mathbb{R}^n$, let $\Gamma_\nu(x) := \{(y,t) \in \mathbb{R}_+^{n+1} : \; |x - y| < \nu t\}$ be the *cone of aperture ν with vertex $x \in \mathbb{R}^n$* and $\Gamma(x) := \Gamma_1(x)$. Denote by $\mathcal{R}_\nu F$ the *union of all cones with vertices in F*, namely, $\mathcal{R}_\nu F := \cup_{x \in F} \Gamma_\nu(x)$.

The following Lemma 2.18 is just [22, Lemma 3.1].

Lemma 2.18. *Let ν, $\eta \in (0,\infty)$. Then there exist positive constants $\gamma \in (0,1)$ and C such that, for any closed subset F of $\mathbb{R}^n$ whose complement has finite measure, and any nonnegative measurable function H on $\mathbb{R}_+^{n+1}$,*

$$\int_{\mathcal{R}_\nu(F_\gamma^*)} H(y,t)t^n \, dy \, dt \leq C \int_F \left\{ \int_{\Gamma_\eta} H(y,t) \, dy \, dt \right\} dx,$$

where F_γ^* denotes the set of points in $\mathbb{R}^n$ with the global γ-density with respect to F.

Proof of Theorem 2.16. Assume that $f \in T_2^{p(\cdot)}(\mathbb{R}_+^{n+1})$. For any $k \in \mathbb{Z}$, we let

$$O_k := \left\{ x \in \mathbb{R}^n : \; \mathcal{A}(f)(x) > 2^k \right\}$$

and $F_k := O_k^{\complement}$. Since $f \in T_2^{p(\cdot)}(\mathbb{R}_+^{n+1})$, for each k, O_k is an open set of $\mathbb{R}^n$ and $|O_k| < \infty$. Let $\gamma \in (0,1)$ be as in Lemma 2.18 with $\eta = 1 = \nu$. In what follows, we denote $(F_k)_\gamma^*$ and $(O_k)_\gamma^*$ simply by F_k^* and O_k^*. By the proof of [18, Theorem 3.2], we know that supp $f \subset (\cup_{k \in \mathbb{Z}} \widehat{O_k^*} \cup E)$, where $E \subset \mathbb{R}_+^{n+1}$ satisfies that $\int_E \frac{dy \, dt}{t} = 0$.

For each $k \in \mathbb{Z}$, considering the Whitney decomposition of the open set of O_k^*, we obtain a set I_k of indices and a family $\{Q_{k,j}\}_{j \in I_k}$ of closed cubes with disjoint interiors such that

(i) $\cup_{j \in I_k} Q_{k,j} = O_k^*$ and, if $i \neq j$, then $\mathring{Q}_{k,j} \cap \mathring{Q}_{k,i} = \emptyset$, where $\mathring{E}$ denotes the *interior* of the set E;

(ii) $\sqrt{n}\ell(Q_{k,j}) \leq \text{dist}(Q_{k,j}, (O_k^*)^{\complement}) \leq 4\sqrt{n}\ell(Q_{k,j})$, where $\ell(Q_{k,j})$ denotes the *side-length* of $Q_{k,j}$ and $\text{dist}(Q_{k,j}, (O_k^*)^{\complement}) := \inf\{|z - w| : \; z \in Q_{k,j}, \; w \in (O_k^*)^{\complement}\}$.

Now, for each $j \in I_k$, let $R_{k,j}$ be the *cube with the same center as $Q_{k,j}$ and with the radius $11\sqrt{n}/2$-times $\ell(Q_{k,j})$*. Set

$$A_{k,j} := \widehat{R_{k,j}} \cap (Q_{k,j} \times (0,\infty)) \cap (\widehat{O_k^*} \backslash \widehat{O_{k+1}^*}),$$

$$a_{k,j} := 2^{-k} \|\chi_{R_{k,j}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{-1} f \chi_{A_{k,j}}$$

and $\lambda_{k,j} := 2^k \|\chi_{R_{k,j}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}$. Notice that $(Q_{k,j} \times (0,\infty)) \cap (\widehat{O_k^*} \backslash \widehat{O_{k+1}^*}) \subset \widehat{R_{k,j}}$. From this and supp $f \subset (\cup_{k \in \mathbb{Z}} \widehat{O_k^*} \cup E)$, we deduce that $f = \sum_{k \in \mathbb{Z}} \sum_{j \in I_k} \lambda_{k,j} a_{k,j}$ almost everywhere on $\mathbb{R}_+^{n+1}$.

Next we first show that, for each $k \in \mathbb{Z}$ and $j \in I_k$, $a_{k,j}$ is a $(p(\cdot), \infty)$-atom support in $\widehat{R_{k,j}}$. Let $p \in (1, \infty)$ and $h \in T_2^{p'}(\mathbb{R}_+^{n+1})$ with $\|h\|_{T_2^{p'}(\mathbb{R}_+^{n+1})} \leq 1$. Since $A_{k,j} \subset (\widehat{O_{k+1}^*})^{\complement} = \mathcal{R}_1(F_{k+1}^*)$, by Lemma 2.18 and the Hölder inequality, we have

$$
\begin{aligned}
|\langle a_{k,j}, h \rangle| &:= \left| \int_{\mathbb{R}_+^{n+1}} a_{k,j}(y,t) \chi_{A_{k,j}}(y,t) h(y,t) \frac{dy\,dt}{t} \right| \\
&\lesssim \int_{F_{k+1}} \int_{\Gamma(x)} |a_{k,j}(y,t) h(y,t)| \frac{dy\,dt}{t^{n+1}} dx \lesssim \int_{F_{k+1}} \mathcal{A}(a_{k,j})(x) \mathcal{A}(h)(x)\, dx \\
&\lesssim 2^{-k} \|\chi_{R_{k,j}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{-1} \left\{ \int_{(3R_{k,j}) \cap F_{k+1}} [\mathcal{A}(f)(x)]^p \, dx \right\}^{1/p} \|h\|_{T_2^{p'}(\mathbb{R}_+^{n+1})} \\
&\lesssim |R_{k,j}|^{1/p} \|\chi_{R_{k,j}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{-1},
\end{aligned}
$$

which, together with $(T_2^p(\mathbb{R}_+^{n+1}))^* = T_2^{p'}(\mathbb{R}_+^{n+1})$ (see [5]), where $(T_2^p(\mathbb{R}_+^{n+1}))^*$ denotes the *dual space* of $T_2^p(\mathbb{R}_+^{n+1})$, implies that $\|a_{k,j}\|_{T_2^p(\mathbb{R}_+^{n+1})} \lesssim |R_{k,j}|^{1/p} \|\chi_{R_{k,j}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{-1}$. Thus, $a_{k,j}$ is a $(p(\cdot), p)$-atom support in $\widehat{R_{k,j}}$ up to a harmless constant for all $p \in (1, \infty)$ and hence a $(p(\cdot), \infty)$-atom up to a harmless constant.

Finally, we prove that $\mathcal{A}^*(\{\lambda_j\}_j, \{Q_j\}_j) \lesssim \|f\|_{T_2^{p(\cdot)}(\mathbb{R}_+^{n+1})}$. By the fact that $\chi_{R_{k,j}} \lesssim \mathcal{M}(\chi_{Q_{k,j}}^r)$ for any $r \in (0, p_-)$, we know that

$$
\begin{aligned}
&\mathcal{A}^*(\{\lambda_{k,j}\}, \{R_{k,j}\}) \\
&\leq \left\| \left\{ \sum_{k \in \mathbb{Z}} \sum_{j \in I_k} \frac{|\lambda_{k,j}|^{p_-} \chi_{R_{k,j}}}{\|\chi_{R_{k,j}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{p_-}} \right\}^{\frac{1}{p_-}} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} = \left\| \left\{ \sum_{k \in \mathbb{Z}} \sum_{j \in I_k} \left(2^k \chi_{R_{k,j}} \right)^{p_-} \right\}^{\frac{1}{p_-}} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&\lesssim \left\| \left\{ \sum_{k \in \mathbb{Z}} \sum_{j \in I_k} \left[\mathcal{M}(2^{kr} \chi_{Q_{k,j}}^r)(x) \right]^{\frac{p_-}{r}} \right\}^{1/p_-} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)},
\end{aligned}
$$

which, together with Lemma 2.5 and the Whitney decomposition of O_k^*, implies that

$$
\mathcal{A}^*(\{\lambda_{k,j}\}, \{R_{k,j}\}) \lesssim \left\| \left\{ \sum_{k \in \mathbb{Z}} \sum_{j \in I_k} \left(2^k \chi_{Q_{k,j}} \right)^{p_-} \right\}^{1/p_-} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \sim \left\| \left\{ \sum_{k \in \mathbb{Z}} \left(2^k \chi_{O_k^*} \right)^{p_-} \right\}^{1/p_-} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}.
$$

From the fact that $\chi_{O_k^*} \lesssim \mathcal{M}(\chi_{O_k}^r)$ with $r \in (0, p_-)$ and Lemma 2.5 again, we further deduce that

$$
\begin{aligned}
&\mathcal{A}^*(\{\lambda_{k,j}\}, \{R_{k,j}\}) \\
&\lesssim \left\| \left\{ \sum_{k \in \mathbb{Z}} \left(\mathcal{M}(2^{kr} \chi_{O_k}^r) \right)^{p_-/r} \right\}^{1/p_-} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim \left\| \left\{ \sum_{k \in \mathbb{Z}} \left(2^k \chi_{O_k} \right)^{p_-} \right\}^{1/p_-} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&\sim \left\| \left\{ \sum_{k \in \mathbb{Z}} \left(2^k \chi_{O_k \setminus O_{k+1}} \right)^{p_-} \right\}^{1/p_-} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \sim \|\mathcal{A}(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \sim \|f\|_{T_2^{p(\cdot)}(\mathbb{R}_+^{n+1})},
\end{aligned}
$$

which completes the proof of Theorem 2.16. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

To prove Theorem 1.13, we also need following technical lemmas.

Lemma 2.19. *Let* $Q := Q(x_0, \delta) \subset \mathbb{R}^n$, $\varepsilon \in (n(1/p_- - 1), \infty)$, $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ *satisfy* (1.1) *and* (1.2), *and* $s \in (n/p_- - n - 1, \infty) \cap \mathbb{Z}_+$. *Then there exists a positive constant* C *such that, for all* $f \in \mathcal{L}_{1,p(\cdot),s}(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} \frac{\delta^\varepsilon |f(x) - P_Q^s f(x)|}{\delta^{n+\varepsilon} + |x - x_0|^{n+\delta}} \, dx \leq C \frac{\|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{|Q|} \|f\|_{\mathcal{L}_{1,p(\cdot),s}(\mathbb{R}^n)}.$$

To prove Lemma 2.19, we need the following Lemma 2.20 which was proved in [32, Lemma 6.5].

Lemma 2.20. *Let* $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ *and* $q \in [1, \infty]$. *Assume that* $p(\cdot)$ *satisfies* (1.1) *and* (1.2), *and* $s \in (n/p_- - n - 1, \infty) \cap \mathbb{Z}_+$. *Then there exists a positive constant* C *such that, for all* $Q \in \mathcal{Q}$, $j \in \mathbb{Z}$ *and* $f \in \mathcal{L}_{q,p(\cdot),s}(\mathbb{R}^n)$,

$$\left\{ \frac{1}{|2^j Q|} \int_{2^j Q} |f(x) - P_Q^s f(x)|^q \, dx \right\}^{1/q} \leq C 2^{jn(\frac{1}{p_-} - 1)} \frac{\|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{|Q|} \|f\|_{\mathcal{L}_{q,p(\cdot),s}(\mathbb{R}^n)},$$

where $2^j Q$ *denotes the cube with the same center as* Q *but* 2^j *times side-length of* Q.

Proof of Lemma 2.19. For any $k \in \mathbb{Z}$, let $Q_k := 2^k Q$, namely, Q_k has the same center with Q but with 2^k times side-length of Q. Then we have

$$I := \int_{\mathbb{R}^n} \frac{\delta^\varepsilon |f(x) - P_Q^s f(x)|}{\delta^{n+\varepsilon} + |x - x_0|^{n+\delta}} \, dx$$

$$= \left(\int_Q + \sum_{k=0}^\infty \int_{Q_{k+1} \setminus Q_k} \right) \frac{\delta^\varepsilon |f(x) - P_Q^s f(x)|}{\delta^{n+\varepsilon} + |x - x_0|^{n+\delta}} \, dx$$

$$\lesssim \frac{1}{|Q|} \int_Q |f(x) - P_Q^s f(x)| \, dx + \sum_{k=0}^\infty (2^k \delta)^{-n-\varepsilon} \delta^\varepsilon \int_{Q_{k+1}} |f(x) - P_Q^s f(x)| \, dx$$

$$\lesssim \frac{\|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{|Q|} \|f\|_{\mathcal{L}_{1,p(\cdot),s}(\mathbb{R}^n)} + \sum_{k=1}^\infty \frac{2^{-k(n+\varepsilon)}}{|Q|} \int_{Q_k} \left[|f(x) - P_{Q_k}^s f(x)| + |P_{Q_k}^s f(x) - P_Q^s f(x)| \right] \, dx.$$

By Lemmas 2.7 and 2.20, we find that, for all $x \in Q_k$,

$$|P_{Q_k}^s f(x) - P_Q^s f(x)| = |P_{Q_k}^s (f - P_Q^s f)(x)| \lesssim \frac{1}{|Q_k|} \int_{Q_k} |f(x) - P_Q^s f(x)| \, dx$$

$$\lesssim 2^{k(\frac{n}{p_-} - n)} \frac{\|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{|Q|} \|f\|_{\mathcal{L}_{1,p(\cdot),s}(\mathbb{R}^n)},$$

which, together with $\varepsilon \in (n(1/p_- - 1), \infty)$, implies that

$$I \lesssim \frac{\|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{|Q|} \|f\|_{\mathcal{L}_{1,p(\cdot),s}(\mathbb{R}^n)} + \sum_{k=1}^\infty 2^{-k(\varepsilon + n - \frac{n}{p_-})} \frac{\|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{|Q|} \|f\|_{\mathcal{L}_{1,p(\cdot),s}(\mathbb{R}^n)}$$

$$\sim \frac{\|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{|Q|} \|f\|_{\mathcal{L}_{1,p(\cdot),s}(\mathbb{R}^n)}.$$

This finishes the proof of Lemma 2.19. $\square$

Next we establish a John-Nirenberg inequality for functions in $\mathcal{L}_{1,p(\cdot),s}(\mathbb{R}^n)$.

Lemma 2.21. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfy (1.1) and (1.2), $f \in \mathcal{L}_{1,p(\cdot),s}(\mathbb{R}^n)$ with $s \in (n/p_- - n - 1, \infty) \cap \mathbb{Z}_+$. Assume that $p_+ \in (0,1]$. Then there exist positive constants c_1 and c_2, independent of f, such that, for all cubes $Q \subset \mathbb{R}^n$ and $\lambda \in (0,\infty)$,*

$$\left| \{ x \in Q : |f(x) - P_Q^s f(x)| > \lambda \} \right| \le c_1 \exp \left\{ -\frac{c_2 |Q| \lambda}{\|f\|_{\mathcal{L}_{1,p(\cdot),s}(\mathbb{R}^n)} \|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \right\} |Q|.$$

Proof. Let $f \in \mathcal{L}_{1,p(\cdot),s}(\mathbb{R}^n)$ and a cube $Q \subset \mathbb{R}^n$. Without loss of generality, we may assume that $\|f\|_{\mathcal{L}_{1,p(\cdot),s}(\mathbb{R}^n)} \|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)} = |Q|$. Otherwise, we replace f by $f|Q|/[\|f\|_{\mathcal{L}_{1,p(\cdot),s}(\mathbb{R}^n)} \|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)}]$. Thus, to show the conclusion of Lemma 2.21, it suffices to show that

$$(2.20) \qquad \left| \{ x \in Q : |f(x) - P_Q^s f(x)| > \lambda \} \right| \le c_1 \exp \left\{ -c_2 \lambda \right\} |Q|.$$

For any $\lambda \in (0,\infty)$ and cube $R \subset Q$, let $\mathrm{I}(\lambda, R) := |\{ x \in R : |f(x) - P_R^s f(x)| > \lambda \}|$ and

$$(2.21) \qquad \mathcal{F}(\lambda, Q) := \sup_{R \subset Q} \frac{\mathrm{I}(\lambda, R)}{|R|}.$$

Then it is easy to see that $\mathcal{F}(\lambda, Q) \le 1$. From Lemma 2.6, $\|f\|_{\mathcal{L}_{1,p(\cdot),s}(\mathbb{R}^n)} \|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)} = |Q|$ and $p_+ \in (0,1]$, we deduce that there exist a positive constant c_0 such that, for any cube $R \subset Q$,

$$\frac{1}{R} \int_R |f(x) - P_R^s f(x)| \, dx \le \frac{\|\chi_R\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{|R|} \|f\|_{\mathcal{L}_{1,p(\cdot),s}(\mathbb{R}^n)} \le c_0.$$

Applying the Calderón-Zygmund decomposition of $|f - P_R^s f|$ at height $\sigma \in (c_0, \infty)$ on the cube R, there exists a family $\{R_k\}_k$ of cubes of R such that $|f(x) - P_R^s f(x)| \le \sigma$ for almost every $x \in R \backslash (\cup_k R_k)$, $R_k \cap R_j = \emptyset$ if $k \ne j$ and, for all k, $\sigma < \int_{R_k} |f(x) - P_R^s f(x)| \, dx / |R_k| \le 2^n \sigma$. From this, we deduce that

$$(2.22) \qquad \sum_k |R_k| \le \frac{1}{\sigma} \sum_k \int_{R_k} |f(x) - P_R^s f(x)| \, dx \le \frac{1}{\sigma} \int_R |f(x) - P_R^s f(x)| \, dx \le \frac{c_0}{\sigma} |R|.$$

If $\lambda \in (\sigma, \infty)$, then, for almost every $x \in R \backslash (\cup_k R_k)$, $|f(x) - P_R^s f(x)| \le \sigma < \lambda$ and hence

$$(2.23) \quad \mathrm{I}(\lambda, R) \le \sum_k |\{ x \in R_k : |f(x) - P_R^s f(x)| > \lambda \}|$$

$$\le \sum_k \mathrm{I}(\lambda - \eta, R_k) + \sum_k |\{ x \in R_k : |P_{R_k}^s f(x) - P_R^s f(x)| > \eta \}| =: \mathrm{I}_1 + \mathrm{I}_2,$$

where $\eta \in (0, \lambda)$ is determined later. For I_1, by (2.21) and (2.22), we have

$$(2.24) \qquad \mathrm{I}_1 \le \sum_k \mathcal{F}(\lambda - \eta, Q) |R_k| \le \frac{C_0}{\sigma} \mathcal{F}(\lambda - \eta, Q) |R|.$$

For I_2, by Lemma 2.7, we find that there exists a positive constant C_1 such that, for any $x \in R_k$,

$$|P_{R_k}^s f(x) - P_R^s f(x)| = |P_{R_k}^s (f - P_R^s f)(x)| \le \frac{C_1}{|R_k|} \int_{R_k} |f(x) - P_R^s f(x)| \, dx \le 2^n C_1 \sigma.$$

Now, let $\sigma := 2c_0$ and $\eta = 2^n C_1 \sigma$. Then, when $\lambda \in (\eta, \infty)$, $\mathrm{I}_2 = 0$, which, together with (2.23) and (2.24), implies that $\mathrm{I}(\lambda, R) \le \mathcal{F}(\lambda - \eta, Q) |R|/2$ for all $R \subset Q$. Thus, it follows that $\mathcal{F}(\lambda, Q) \le \mathcal{F}(\lambda - \eta, Q)/2$. If $m \in \mathbb{N}$ satisfies $m\eta < \lambda \le (m+1)\eta$, then

$$\mathcal{F}(\lambda, Q) \le \frac{1}{2} \mathcal{F}(\lambda - \eta, Q) \le \cdots \le \frac{1}{2^m} \mathcal{F}(\lambda - m\eta, Q).$$

From $\mathcal{F}(\lambda - m\eta, Q) \leq 1$ and $m \geq \lambda/\eta - 1$, we deduce that

$$\mathcal{F}(\lambda, Q) \leq 2^{-m} \leq 2^{1-\lambda/\eta} = 2e^{(-\frac{1}{\eta}\log 2)\lambda}.$$

Therefore, when $\lambda \in (\eta, \infty)$, we conclude that (2.20) holds true with $c_1 := 2$ and $c_2 := (\log 2)/\eta$. On the other hand, when $\lambda \in (0, \eta)$, (2.20) holds true trivially. This finishes the proof of Lemma 2.21. $\qquad\square$

By the Hölder inequality and Lemma 2.21, we immediately obtain the following Corollary 2.22, the details being omitted.

Corollary 2.22. *Let $p(\cdot)$, s be as in Lemma 2.21 and $r \in (1, \infty)$. Then $f \in \mathcal{L}_{1,p(\cdot),s}(\mathbb{R}^n)$ if and only if $f \in \mathcal{L}_{r,p(\cdot),s}(\mathbb{R}^n)$.*

Now we prove Theorem 1.13.

Proof of Theorem 1.13. We first prove (i). Let $b \in \mathcal{L}_{1,p(\cdot),s}(\mathbb{R}^n)$. For any $Q_0 := Q(x_0, r)$, write

$$(2.25) \qquad b = P_{2Q_0}^s b + (b - P_{2Q_0}^s b)\chi_{2Q_0} + (b - P_{2Q_0}^s b)\chi_{\mathbb{R}^n \setminus (2Q_0)} =: b_1 + b_2 + b_3.$$

For b_1, since $\int_{\mathbb{R}^n} \phi(x)x^\gamma \, dx = 0$ for all $\gamma \in \mathbb{Z}_+^n$, we see that, for all $t \in (0, \infty)$, $\phi_t * b_1 \equiv 0$ and hence

$$(2.26) \qquad \int_{\widehat{Q_0}} |\phi_t * b_1(x)|^2 \frac{dxdt}{t} = 0.$$

For b_2, by the fact that the boundedness of the square function $g(f)$ on $L^2(\mathbb{R}^n)$ (see, for example, [17, p. 356, Exercise 5.1.4]), we find that

$$\int_{\widehat{Q_0}} |\phi_t * b_2(x)|^2 \frac{dxdt}{t} \leq \int_{\mathbb{R}_+^{n+1}} |\phi_t * b_2(x)|^2 \frac{dxdt}{t} \lesssim \|b_2\|_{L^2(\mathbb{R}^n)}^2 \sim \int_{2Q_0} |b(x) - P_{2Q_0}^s b(x)|^2 \, dx,$$

which, together with Lemma 2.6 and Corollary 2.22, implies that

$$(2.27) \qquad \frac{|Q_0|^{1/2}}{\|\chi_{Q_0}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \left\{ \int_{\widehat{Q_0}} |\phi_t * b_2(x)|^2 \frac{dxdt}{t} \right\}^{1/2} \lesssim \|b\|_{\mathcal{L}_{1,p(\cdot),s}(\mathbb{R}^n)}.$$

For b_3, let ε be as in Lemma 2.19. Then, for all $(x, t) \in \widehat{Q_0}$, we have

$$\begin{aligned}
|\phi_t * b_3(x)| &\lesssim \int_{\mathbb{R}^n \setminus (2Q_0)} \frac{t^\varepsilon}{(t + |x - y|)^{n+\varepsilon}} |b(y) - P_{2Q_0}^s(y)| \, dy \\
&\lesssim \int_{\mathbb{R}^n \setminus (2Q_0)} \frac{t^\varepsilon}{(t + |x_0 - y|)^{n+\varepsilon}} |b(y) - P_{2Q_0}^s(y)| \, dy \\
&\lesssim \frac{t^\varepsilon}{r^\varepsilon} \frac{\|\chi_{Q_0}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{|Q_0|} \|b\|_{\mathcal{L}_{1,p(\cdot),s}(\mathbb{R}^n)},
\end{aligned}$$

which implies that

$$(2.28) \qquad \frac{|Q_0|}{\|\chi_{Q_0}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \left\{ \frac{1}{|Q_0|} \int_{\widehat{Q_0}} |\phi_t * b_3(x)|^2 \frac{dxdt}{t} \right\}^{1/2} \lesssim \|b\|_{\mathcal{L}_{1,p(\cdot),s}(\mathbb{R}^n)}.$$

From this, (2.25), (2.26) and (2.27), we deduce that

$$\frac{|Q_0|}{\|\chi_{Q_0}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \left\{ \frac{1}{|Q_0|} \int_{\widehat{Q_0}} |\phi_t * b(x)|^2 \frac{dxdt}{t} \right\}^{1/2} \lesssim \|b\|_{\mathcal{L}_{1,p(\cdot),s}(\mathbb{R}^n)},$$

which, together with the arbitrariness of $Q_0 \subset \mathbb{R}^n$, implies that $d\mu$ is a $p(\cdot)$-Carleson measure on $\mathbb{R}_+^{n+1}$ and $\|d\mu\|_{p(\cdot)} \lesssim \|b\|_{\mathcal{L}_{1,p(\cdot),s}(\mathbb{R}^n)}$.

Next, we prove (ii). To this end, let $f \in L_{\mathrm{comp}}^{\infty,s}(\mathbb{R}^n)$. Then, by $f \in L^\infty(\mathbb{R}^n)$ with compact support, $b \in L_{\mathrm{loc}}^2(\mathbb{R}^n)$ and the Plancherel formula, we conclude that

$$(2.29) \qquad \left| \int_{\mathbb{R}^n} f(x)\overline{b(x)} \, dx \right| \sim \left| \int_{\mathbb{R}_+^{n+1}} \phi_t * f(x)\overline{\phi_t * b(x)} \, \frac{dxdt}{t} \right|.$$

Moreover, from $f \in H^{p(\cdot)}(\mathbb{R}^n)$ and Theorem 1.4, we deduce that $\phi_t * f \in T_2^{p(\cdot)}(\mathbb{R}_+^{n+1})$, which, combined with Theorem 2.16, implies that there exist $\{\lambda_j\}_j \subset \mathbb{C}$ and a sequence $\{a_j\}_j$ of $(p(\cdot), \infty)$-atoms with $\mathrm{supp}\,a_j \subset \widehat{Q_j}$ such that $\phi_t * f(x) = \sum_j \lambda_j a_j(x,t)$ almost everywhere. By this, (2.29), the Hölder inequality and Remark 2.17, we find that

$$\left| \int_{\mathbb{R}^n} f(x)\overline{b(x)} \, dx \right| \leq \sum_j |\lambda_j| \int_{\mathbb{R}_+^{n+1}} |a_j(x,t)||\phi_t * b(x)| \, \frac{dxdt}{t}$$

$$\leq \sum_j |\lambda_j| \left\{ \int_{\widehat{Q_j}} |a_j(x,t)|^2 \right\}^{1/2} \left\{ \int_{\widehat{Q_j}} |\phi_t * b(x)|^2 \, \frac{dxdt}{t} \right\}^{1/2}$$

$$\lesssim \sum_j |\lambda_j| \frac{|Q_j|^{1/2}}{\|\chi_{Q_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \left\{ \int_{\widehat{Q_j}} |\phi_t * b(x)|^2 \, \frac{dxdt}{t} \right\}^{1/2}$$

$$\lesssim \|f\|_{H^{p(\cdot)}(\mathbb{R}^n)} \|d\mu\|_{p(\cdot)} \lesssim \|d\mu\|_{p(\cdot)},$$

which, together with [32, Theorem 7.5] and the fact that $L_{\mathrm{comp}}^{\infty,s}(\mathbb{R}^n)$ is dense in $H^{p(\cdot)}(\mathbb{R}^n)$, implies that $\|b\|_{\mathcal{L}_{1,p(\cdot),s}(\mathbb{R}^n)} \lesssim \|d\mu\|_{p(\cdot)}$ and hence completes the proof of Theorem 1.13. $\square$

We conclude this section by giving the proof of Theorem 1.14.

Proof of Theorem 1.14. From Theorem 1.13(ii) and the fact that, for all $(x,t) \in \mathbb{R}_+^{n+1}$, $|\phi_t * b(x)| \lesssim \widetilde{A}_{(\alpha,\epsilon),s}(b)(x,t)$ with ϕ as in Theorem 1.13, we deduce that the conclusion of Theorem 1.14(ii) holds true.

It therefore remains to prove (i). Let $b \in \mathcal{L}_{1,p(\cdot),s}(\mathbb{R}^n)$. Then, for any cube $Q_0 \subset \mathbb{R}^n$, write

$$b = P_{2Q_0}^s b + (b - P_{2Q_0}^s b)\chi_{2Q_0} + (b - P_{2Q_0}^s b)\chi_{\mathbb{R}^n \setminus (2Q_0)} =: b_1 + b_2 + b_3.$$

For b_1, since $\int_{\mathbb{R}^n} \phi(x)x^\gamma \, dx = 0$ for $\phi \in \mathcal{C}_{(\alpha,\epsilon),s}(\mathbb{R}^n)$ and $\gamma \in \mathbb{Z}_+^n$ with $|\gamma| \leq s$, we see that, for all $t \in (0, \infty)$, it holds true that $\phi_t * b_1 \equiv 0$ and hence

$$(2.30) \qquad \int_{\widehat{Q_0}} [\widetilde{A}_{(\alpha,\epsilon),s}(b_1)(x,t)]^2 \, \frac{dxdt}{t} = 0.$$

For b_2, from Lemmas 2.14 and 2.13, we deduce that

$$\int_{\widehat{Q_0}} [\widetilde{A}_{(\alpha,\epsilon),s}(b_2)(x,t)]^2 \, \frac{dxdt}{t} \lesssim \|b_2\|_{L^2(\mathbb{R}^n)}^2 \sim \int_{2Q_0} |b(x) - P_{Q_0}^s b(x)|^2 \, dx,$$

which, together with Corollary 2.22, implies that

$$(2.31) \qquad \frac{|Q_0|^{1/2}}{\|\chi_{Q_0}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \left\{ \int_{\widehat{Q_0}} |\widetilde{A}_{(\alpha,\epsilon),s}(b_2)(x,t)|^2 \frac{dxdt}{t} \right\} \lesssim \|b\|_{\mathcal{L}_{1,p(\cdot),s}(\mathbb{R}^n)}.$$

By an argument similar to that used in the proof of (2.28), we find that

$$\frac{|Q_0|^{1/2}}{\|\chi_{Q_0}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \left\{ \int_{\widehat{Q_0}} |\widetilde{A}_{(\alpha,\epsilon),s}(b_3)(x,t)|^2 \frac{dxdt}{t} \right\} \lesssim \|b\|_{\mathcal{L}_{1,p(\cdot),s}(\mathbb{R}^n)}.$$

From this, combining (2.30) and (2.31), we conclude that $d\mu_b$ is a $p(\cdot)$-Carleson measure on $\mathbb{R}_+^{n+1}$ and $\|d\mu_b\|_{p(\cdot)} \lesssim \|b\|_{\mathcal{L}_{1,p(\cdot),s}(\mathbb{R}^n)}$, which completes the proof of Theorem 1.14. □

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