A geometric approach for solving Troesch's problem

M.S. Hashemi [a,*], S. Abbasbandy [b]

[a] *Department of Mathematics, Basic Science Faculty, University of Bonab, Bonab 55517, Iran*

[b] *Department of Mathematics, Imam Khomeini International University, Ghazvin 34149, Iran*

Abstract

In this paper, the Lie-group shooting method (LGSM) which is a numerical geometric integrator, is applied to the intrinsically unstable Troesch's problem. The calculated results illuminate the efficiency and precision of Lie-group shooting method (LGSM) for this problem.

Key words: Lie-group shooting method; Group preserving scheme; Troesch's problem.

1 Introduction

In this paper, we consider a nonlinear two-point boundary value problem, Troesch's problem, arises in the theory of gas porous electrodes [28,8] and investigation of the confinement of a plasma column by radiation pressure [37]. Troesch's problem, is defined by:

$$u'' = \mu \sinh(\mu u), \tag{1.1}$$
$$u(0) = 0, \quad u(1) = 1, \tag{1.2}$$

where μ is a positive constant.
Roberts, et al. [31] have been shown that exact solution of Eqs. (1.1)-(1.2) in terms of the elliptic function $sc(n|m)$ is as follows:

* Corresponding author: hashemi_math396@yahoo.com, hashemi@bonabu.ac.ir

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$$u(x) = \frac{2}{\mu} \sinh^{-1}\Big[\frac{u'(0)}{2} sc(\mu x|m)\Big], \tag{1.3}$$

where $m = 1 - \frac{1}{4}(u'(0))^2$ and satisfies

$$sc(\mu|m) = \frac{\sinh\left(\frac{\mu}{2}\right)}{\sqrt{1-m}}.$$

Obviously

$$x_s = \frac{1}{\mu} \ln\Big(\frac{8}{y'(0)}\Big),$$

is the singularity of Troesch's problem which located at a pole of $sc(\mu x|m)$, and singularity lies within the integration range, if $y'(0) > 8e^{-\mu}$. This results in the problem being very difficult to solve.
Some of the methods, such as the variational iteration method [30], the homotopy perturbation method [29], the modified homotopy perturbation method [7] and the Adomian decomposition method [13,6], fail to solve the Troesch's problem for $\mu > 1$. However, some other methods such as finite element method [14], the method of transformation group [5], inverse shooting method [34], the simple shooting method [11,12,35,1], the invariant imbedding method [33], Monte Carlo method [36] and a combination of the multi-point shooting method with the continuation and perturbation technique [32] have been successfully applied to this problem.
Difficulties to solve the Eqs. (1.1)-(1.2) take place, when $\mu > 1$. Hence, we split the topics of this paper in two major cases, $\mu \leq 1$ and $\mu > 1$. Thus, to avoid overflow or excessive error growth during numerical integration and also compatibility with LGSM, we introduce two different transformations.
The present paper, provides a LGSM for Troesch's problem, which is based on the group preserving scheme (GPS), developed by Liu [16] for the integration of initial value problems. Also a combination of GPS and Lie symmetries are introduced by Hashemi et.al in [9]. The LSGM able our to find the initial slope condition through a minimum solution of r in interval $(0, 1)$, and determined by matching the right-end boundary condition. The factor r is used in a generalized mid-point rule for the Lie-group of one-step GPS. LGSM is an effective and powerful method which is easy to implement and time saving method.
Recently, LGSM successfully applied to various problems [17–19,2,4,20,21,24,38,25,3,26,27]. Also, Reproducing kernel Hilbert space method is applied to solve the Bratu's problem which is an two-point boundary problem, by Inc et. al. in [10].

2 Coordinate transformations

In this paper, firstly we convert the boundary value problem (1.1)-(1.2) to an initial value problem by using the LGSM and then we solve it by GPS. Since solving the Troesch's problem for $\mu > 1$ is so difficult, we introduce two cases of transformations, namely $(a): \ \mu \leq 1$ and $(b): \ \mu > 1$.

2.1 Case (a): $\mu \leq 1$

In this case we use the following transformation:

$$y(x) = u(x) - x + c, \quad c > 0. \tag{2.4}$$

Hence, Troesch's problem transforms to the following problem:

$$y''(x) = \mu \sinh\Big(\mu\big[y(x) + x - c\big]\Big), \tag{2.5}$$

$$y(0) = y(1) = c. \tag{2.6}$$

Then, we convert (2.5)-(2.6) to the following equivalent system

$$y_1' = y_2, \tag{2.7}$$

$$y_2' = \mu \sinh\Big(\mu\big[y_1 + x - c\big]\Big), \tag{2.8}$$

$$y_1(0) = c, \quad y_2(0) = A, \tag{2.9}$$

which A is unknown and LGSM has applied to find it.

2.2 Case (b): $\mu > 1$

For this case which is difficult than the first one, we introduce the following transformation:

$$y(x) = \tanh\Big(\mu u(x)/4\Big) - \tanh\Big(\mu/4\Big)x + c, \quad c > 0. \tag{2.10}$$

Hence, Troesch's problem transforms to the following problem:

$$
\begin{aligned}
y''(x) =& 2\frac{y(x)+\tanh\left(\mu/4\right)x-c}{\left(y(x)+\tanh\left(\mu/4\right)x-c\right)^2-1}\left(y'(x)+\tanh\left(\mu/4\right)\right)^2 && (2.11)\\
&+\mu^2\frac{\left(y(x)+\tanh\left(\mu/4\right)x-c\right)\left(1+\left(y(x)+\tanh\left(\mu/4\right)x-c\right)^2\right)}{1-\left(y(x)+\tanh\left(\mu/4\right)x-c\right)^2},\\
y(0) =& y(1)=c, && (2.12)
\end{aligned}
$$

which equivalent one-order ODEs system is as follows

$$
\begin{aligned}
y_1' =& y_2, && (2.13)\\
y_2' =& 2\frac{y_1+\tanh\left(\mu/4\right)x-c}{\left(y_1+\tanh\left(\mu/4\right)x-c\right)^2-1}.\left(y_2+\tanh\left(\mu/4\right)\right)^2\\
&+\mu^2\frac{\left(y_1+\tanh\left(\mu/4\right)x-c\right)\left(1+\left(y_1+\tanh\left(\mu/4\right)x-c\right)^2\right)}{1-\left(y_1+\tanh\left(\mu/4\right)x-c\right)^2}, && (2.14)\\
y_1(0) =& c, \quad y_2(0)=A, && (2.15)
\end{aligned}
$$

where A is unknown and we apply the LGSM to find it.

3 One-step group-preserving scheme

3.1 The group-preserving scheme

Let us write Eqs. (2.7)-(2.8) and Eqs. (2.13)-(2.14) in a vector form:

$$\mathbf{Y}' = \mathbf{f}(x, \mathbf{Y}), \tag{3.16}$$

where for case $a)$:

$$\mathbf{Y} := \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \quad \mathbf{f} := \begin{bmatrix} y_2 \\ \mu \sinh\left(\mu\left[y_1 + x - c\right]\right) \end{bmatrix}, \tag{3.17}$$

and for case $b)$:

$$\mathbf{Y} := \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \quad \mathbf{f} := \begin{bmatrix} y_2 \\ f_2 \end{bmatrix}, \tag{3.18}$$

where

$$
\begin{aligned}
f_2 = & 2\frac{y_1+\tanh\left(\mu/4\right)x-c}{\left(y_1+\tanh\left(\mu/4\right)x-c\right)^2-1}\left(y_2+\tanh\left(\mu/4\right)\right)^2 \\
& +\mu^2\frac{\left(y_1+\tanh\left(\mu/4\right)x-c\right)\left(1+\left(y_1+\tanh\left(\mu/4\right)x-c\right)^2\right)}{1-\left(y_1+\tanh\left(\mu/4\right)x-c\right)^2}.
\end{aligned}
$$

Liu [16] has embedded Eq. (3.16) in general form, into an augmented system:

$$
\mathbf{X}' := \frac{d}{dx}\begin{bmatrix} \mathbf{Y} \\ \|\mathbf{Y}\| \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{2\times 2} & \frac{\mathbf{f}(x,\mathbf{Y})}{\|\mathbf{Y}\|} \\ \frac{\mathbf{f}^T(x,\mathbf{Y})}{\|\mathbf{Y}\|} & 0 \end{bmatrix}\begin{bmatrix} \mathbf{Y} \\ \|\mathbf{Y}\| \end{bmatrix} := A\mathbf{X}, \tag{3.19}
$$

where A is an element of the Lie algebra $so(2,1)$ satisfying

$$
A^T g + gA = 0, \tag{3.20}
$$

with

$$
g = \begin{bmatrix} I_2 & \mathbf{0}_{2\times 1} \\ \mathbf{0}_{1\times 2} & -1 \end{bmatrix}, \tag{3.21}
$$

a Minkowski metric. The augmented variable $\mathbf{X}$ satisfies the cone condition:

$$
\mathbf{X}^T g\mathbf{X} = \mathbf{Y}\cdot\mathbf{Y} = \|\mathbf{Y}\|^2 = 0. \tag{3.22}
$$

Accordingly, Liu [16] has developed a group-preserving scheme (GPS) as follows:

$$
\mathbf{X}_{n+1} = G(n)\mathbf{X}_n, \tag{3.23}
$$

where $\mathbf{X}_n$ denotes the numerical value of $\mathbf{X}$ at the discrete x_n, and $G(n) \in SO_o(2,1)$ satisfies

$$
G^T g G = g, \tag{3.24}
$$

$$
det(G) = 1, \tag{3.25}
$$

$$
G_0^0 > 0, \tag{3.26}
$$

where G_0^0 is the $00'$th component of G and $SO_o(2,1)$ is the 3-dimensional Lorentz group.

Applying scheme (3.23) to Eq. (3.19) with a specified initial condition $\mathbf{X}(0) = \mathbf{X}_0$, we can compute the solution $\mathbf{X}(x)$ by GPS. Assuming that the step size used in GPS is $\Delta x = 1/K$ and starting from an initial augmented condition $\mathbf{X}_0 = \mathbf{X}(0) = \Big((\mathbf{Y}^0)^T, \|\mathbf{Y}^0\|\Big)^T$, we attempt to calculate the value $\mathbf{X}(1) = \Big(\mathbf{Y}^T(1), \|\mathbf{Y}(1)\|\Big)^T$ at $x = 1$.
By applying Eq. (3.23) step-by-step we can obtain

$$\mathbf{X}(1) \cong \mathbf{X}_f = G_K(\Delta x) \cdots G_1(\Delta x)\mathbf{X}_0. \tag{3.27}$$

However, let us recollect that each $G_i,\ i = 1, \ldots, K$, is an element of the Lie group $SO_o(n, 1)$, and by the closure property of the Lie groups, $G_K(\Delta x) \cdots G_1(\Delta x)$ is also a Lie group denoted by G. Hence we have

$$\mathbf{X}_f = G\mathbf{X}_0, \tag{3.28}$$

which is a one-step transformation from $\mathbf{X}_0$ to $\mathbf{X}_f$.
We can calculate G by a generalized mid-point rule, which is obtained from an exponential mapping of A by taking the values of the argument variables of A at a generalized mid-point. The Lie group generated from $A \in so(2, 1)$, is known as a proper orthochronous Lorentz group, which admits a closed-form representation:

$$G = \begin{bmatrix} I_2 + \frac{(\alpha-1)}{\|\tilde{\mathbf{f}}\|^2}\tilde{\mathbf{f}}\tilde{\mathbf{f}}^T & \frac{\beta\tilde{\mathbf{f}}}{\|\tilde{\mathbf{f}}\|} \\ \frac{\beta\tilde{\mathbf{f}}^T}{\|\tilde{\mathbf{f}}\|} & \alpha \end{bmatrix}, \tag{3.29}$$

where

$$\tilde{\mathbf{Y}} = r\mathbf{Y}^0 + (1-r)\mathbf{Y}^f, \tag{3.30}$$

$$\tilde{\mathbf{f}} = \mathbf{f}(\tilde{x}, \tilde{\mathbf{Y}}), \tag{3.31}$$

$$\alpha = \cosh\Big(\frac{\|\tilde{\mathbf{f}}\|}{\|\tilde{\mathbf{Y}}\|}\Big), \tag{3.32}$$

$$\beta = \sinh\Big(\frac{\|\tilde{\mathbf{f}}\|}{\|\tilde{\mathbf{Y}}\|}\Big). \tag{3.33}$$

Here, we employ the initial $\mathbf{Y}^0 = (y_1(0), y_2(0))^T$ and the final $\mathbf{Y}^f = (y_1(1), y_2(1))^T$ through a suitable weighting factor r to calculate G, where $r \in (0, 1)$ is a parameter and $\tilde{x} = r$. The above method is applied by a generalized mid-point rule on the calculation of G, and the result is a single-parameter Lie group element denoted by $G(r)$. A suitable r can be determined by matching the right-end boundary condition.

3.3 A Lie group mapping between two points on the cone

Let us define a new vector

$$F := \frac{\tilde{\mathbf{f}}}{\|\tilde{\mathbf{Y}}\|}, \tag{3.34}$$

such that Eqs. (3.29), (3.32) and (3.33) can also be expressed as

$$G = \begin{bmatrix} I_2 + \frac{(\alpha-1)}{\|F\|^2} F F^T & \frac{\beta F}{\|F\|} \\ \frac{\beta F^T}{\|F\|} & \alpha \end{bmatrix}, \tag{3.35}$$

$$\alpha = \cosh(\|F\|), \tag{3.36}$$

$$\beta = \sinh(\|F\|). \tag{3.37}$$

From Eqs. (3.28) and (3.35) it follows that

$$\mathbf{Y}^f = \mathbf{Y}^0 + \eta F, \tag{3.38}$$

$$\|\mathbf{Y}^f\| = \alpha \|\mathbf{Y}^0\| + \beta \frac{F \cdot \mathbf{Y}^0}{\|F\|}, \tag{3.39}$$

where

$$\eta := \frac{(\alpha - 1) F \cdot \mathbf{Y}^0 + \beta \|\mathbf{Y}^0\| \|F\|}{\|F\|^2}. \tag{3.40}$$

Substituting

$$F = \frac{1}{\eta}(\mathbf{Y}^f - \mathbf{Y}^0) \tag{3.41}$$

into Eq. (3.39) we obtain

$$\frac{\|\mathbf{Y}^f\|}{\|\mathbf{Y}^0\|} = \alpha + \beta \frac{(\mathbf{Y}^f - \mathbf{Y}^0) \cdot \mathbf{Y}^0}{\|\mathbf{Y}^f - \mathbf{Y}^0\| \|\mathbf{Y}^0\|}, \tag{3.42}$$

where

$$\alpha = \cosh\left(\frac{\|\mathbf{Y}^f - \mathbf{Y}^0\|}{\eta}\right), \tag{3.43}$$

$$\beta = \sinh\left(\frac{\|\mathbf{Y}^f - \mathbf{Y}^0\|}{\eta}\right), \tag{3.44}$$

are obtained by inserting Eq. (3.41) for F into Eqs. (3.36) and (3.37).
Let

$$\cos\theta := \frac{(\mathbf{Y}^f - \mathbf{Y}^0)\cdot\mathbf{Y}^0}{\|\mathbf{Y}^f - \mathbf{Y}^0\|\|\mathbf{Y}^0\|}, \tag{3.45}$$

$$\zeta := \|\mathbf{Y}^f - \mathbf{Y}^0\|, \tag{3.46}$$

where $\theta \in [0,\pi]$ is the intersection angle between vectors $\mathbf{Y}^f - \mathbf{Y}^0$ and $\mathbf{Y}^0$ and thus from Eqs. (3.42)-(3.44) we have

$$\frac{\|\mathbf{Y}^f\|}{\|\mathbf{Y}^0\|} = \cosh\Big(\frac{\zeta}{\eta}\Big) + \cos\theta\sinh\Big(\frac{\zeta}{\eta}\Big). \tag{3.47}$$

By defining

$$Z := exp\Big(\frac{\zeta}{\eta}\Big), \tag{3.48}$$

we obtain a quadratic equation for Z from Eq. (3.47):

$$(1+\cos\theta)Z^2 - \frac{2\|\mathbf{Y}^f\|}{\|\mathbf{Y}^0\|}Z + 1 - \cos\theta = 0. \tag{3.49}$$

On the other hand, inserting Eq. (3.41) for F into Eq. (3.40) follows that

$$\|\mathbf{Y}^f - \mathbf{Y}^0\|^2 = (\alpha-1)(\mathbf{Y}^f - \mathbf{Y}^0)\cdot\mathbf{Y}^0 + \beta\|\mathbf{Y}^f - \mathbf{Y}^0\|\|\mathbf{Y}^0\|. \tag{3.50}$$

Dividing both sides by $\|\mathbf{Y}^f - \mathbf{Y}^0\|\|\mathbf{Y}^0\|$ and using Eqs. (3.43)-(3.46) and (3.48) we obtain another quadratic equation for Z:

$$(1+\cos\theta)Z^2 - 2\left(\cos\theta + \frac{\|\mathbf{Y}^f - \mathbf{Y}^0\|}{\|\mathbf{Y}^0\|}\right)Z + \cos\theta - 1 = 0. \tag{3.51}$$

By cancelling the quadratic term Z^2 from Eqs. (3.49) and (3.51) we obtain

$$Z = \frac{(\cos\theta - 1)\|\mathbf{Y}^0\|}{\cos\theta\|\mathbf{Y}^0\| + \|\mathbf{Y}^f - \mathbf{Y}^0\| - \|\mathbf{Y}^f\|}. \tag{3.52}$$

The above Z is indeed the solution of Eqs. (3.49) and (3.51).
When Z is available from Eq. (3.52), then from Eqs. (3.46) and (3.48) we obtain

$$\eta = \frac{\|\mathbf{Y}^f - \mathbf{Y}^0\|}{\ln Z}. \tag{3.53}$$

Therefore, between any two points $(\mathbf{Y}^0, \|\mathbf{Y}^0\|)$ and $(\mathbf{Y}^f, \|\mathbf{Y}^f\|)$ on the cone, there exists a Lie-group element $G \in SO_o(2,1)$ mapping $(\mathbf{Y}^0, \|\mathbf{Y}^0\|)$ onto $(\mathbf{Y}^f, \|\mathbf{Y}^f\|)$, which is given by

$$\begin{bmatrix} \mathbf{Y}^f \\ \|\mathbf{Y}^f\| \end{bmatrix} = G \begin{bmatrix} \mathbf{Y}^0 \\ \|\mathbf{Y}^0\| \end{bmatrix}, \tag{3.54}$$

where G is uniquely determined by $\mathbf{Y}^0$ and $\mathbf{Y}^f$ through the Eqs. (3.35), (3.36), (3.37), (3.41) and (3.53).

4 The Lie-group shooting method

In this section, we apply the LGSM to different cases: $(a): \ \mu \leq 1$ and $(b): \ \mu > 1$.

4.1 Case (a):

From Eqs. (2.7)-(2.9) it follows that

$$y_1' = y_2, \tag{4.55}$$

$$y_2' = \mu \sinh\Big(\mu\big[y_1 + x - c\big]\Big), \tag{4.56}$$

$$y_1(0) = c, \quad y_1(1) = c, \tag{4.57}$$

$$y_2(0) = A, \quad y_2(1) = B, \tag{4.58}$$

where A and B are two unknown constants, and c is a given constant.
From Eqs. (3.38), (4.57) and (4.58) it follows that

$$F := \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \frac{1}{\eta} \begin{bmatrix} 0 \\ B - A \end{bmatrix}. \tag{4.59}$$

From Eqs. (3.45), (3.52) and (3.53) by inserting Eq. (3.17) for $\mathbf{Y}$ and noting that

$$\mathbf{Y}^0 = \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} c \\ A \end{bmatrix}, \quad \mathbf{Y}^f = \begin{bmatrix} y_1(1) \\ y_2(1) \end{bmatrix} = \begin{bmatrix} c \\ B \end{bmatrix}, \tag{4.60}$$

we obtain

$$\eta = \frac{\sqrt{(A-B)^2}}{\ln Z}, \tag{4.61}$$

$$Z = \frac{\frac{\sqrt{c^2+B^2}}{\sqrt{c^2+A^2}} + \sqrt{\frac{c^2+B^2}{c^2+A^2} - 1 + \cos^2\theta}}{1+\cos\theta}, \tag{4.62}$$

$$\cos\theta = \frac{A(B-A)}{\sqrt{(A-B)^2}\sqrt{c^2+A^2}}. \tag{4.63}$$

When comparing Eq. (4.59) with Eq. (3.34), with the aid of Eqs. (3.30), (3.31) and (4.55)-(4.58) we obtain

$$rA + (1-r)B = 0, \tag{4.64}$$

$$A - B + \frac{\mu\eta}{\xi}\sinh(\mu r) = 0, \tag{4.65}$$

where

$$\xi := \sqrt{c^2 + \Big[rA + (1-r)B\Big]^2} = c. \tag{4.66}$$

From (4.64) we obtain $B = rA/(r-1)$ which has a different sign with A because $0 < r < 1$. Therefore, we obtain an algebraic equation for A:

$$A + \frac{\mu\sqrt{A^2}}{c\ln Z}\sinh(\mu r) = 0, \tag{4.67}$$

where

$$Z = \frac{\sqrt{c^2+B^2} + \sqrt{B^2}}{\sqrt{c^2+A^2} - \sqrt{A^2}}. \tag{4.68}$$

4.2 Case (b):

From Eqs. (2.13)-(2.15) it follows that

$$
\begin{aligned}
&y_1' = y_2, && (4.69)\\
&y_2' = 2\frac{y_1 + \tanh\big(\mu/4\big)x - c}{\Big(y_1 + \tanh\big(\mu/4\big)x - c\Big)^2 - 1}\Big(y_2 + \tanh\big(\mu/4\big)\Big)^2\\
&+\mu^2\frac{\Big(y_1 + \tanh\big(\mu/4\big)x - c\Big)\Big(1 + \Big(y_1 + \tanh\big(\mu/4\big)x - c\Big)^2\Big)}{1 - \Big(y_1 + \tanh\big(\mu/4\big)x - c\Big)^2}, && (4.70)\\
&y_1(0) = c, \quad y_1(1) = c, && (4.71)\\
&y_2(0) = A, \quad y_2(1) = B, && (4.72)
\end{aligned}
$$

where A and B are two unknown constants, and c is a given constant.
From Eqs. (3.38), (4.71) and (4.72) it follows that

$$
F := \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \frac{1}{\eta}\begin{bmatrix} 0 \\ B - A \end{bmatrix}. \tag{4.73}
$$

From Eqs. (3.45), (3.52) and (3.53) by inserting Eq. (3.18) for $\mathbf{Y}$ and noting that

$$
\mathbf{Y}^0 = \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} c \\ A \end{bmatrix}, \quad \mathbf{Y}^f = \begin{bmatrix} y_1(1) \\ y_2(1) \end{bmatrix} = \begin{bmatrix} c \\ B \end{bmatrix}, \tag{4.74}
$$

we obtain

$$
\begin{aligned}
&\eta = \frac{\sqrt{(A-B)^2}}{\ln Z}, && (4.75)\\
&Z = \frac{\frac{\sqrt{c^2+B^2}}{\sqrt{c^2+A^2}} + \sqrt{\frac{c^2+B^2}{c^2+A^2} - 1 + \cos^2\theta}}{1+\cos\theta}, && (4.76)\\
&\cos\theta = \frac{A(B-A)}{\sqrt{(A-B)^2}\sqrt{c^2+A^2}}. && (4.77)
\end{aligned}
$$

When comparing Eq. (4.59) with Eq. (3.34), with the aid of Eqs. (3.30), (3.31) and (4.69)-(4.72) we obtain

$$
\begin{aligned}
&rA + (1-r)B = 0, && (4.78)\\
&A - B + \frac{2\eta}{\xi}\Big(\frac{\tanh^3(\frac{\mu}{4})r}{\tanh^2(\frac{\mu}{4})r^2 - 1}\Big) + \frac{\eta\mu^2}{\xi}\tanh(\frac{\mu}{4})r\Big(\frac{1+\tanh^2(\frac{\mu}{4})r^2}{1-\tanh^2(\frac{\mu}{4})r^2}\Big) = 0, && (4.79)
\end{aligned}
$$

where

$$\xi := \sqrt{c^2 + \Big[rA + (1-r)B\Big]^2} = c. \tag{4.80}$$

From (4.64) we obtain $B = rA/(r-1)$ which because of $0 < r < 1$, it has a different sign with A. Therefore, we obtain an algebraic equation for A:

$$A + \frac{\sqrt{A^2}}{c.\ln Z} T(r) = 0, \tag{4.81}$$

where

$$T(r) = \frac{2\tanh^3(\frac{\mu}{4})r}{\tanh^2(\frac{\mu}{4})r^2 - 1} + \mu^2 \tanh(\frac{\mu}{4})r\Big(\frac{1+\tanh^2(\frac{\mu}{4})r^2}{1-\tanh^2(\frac{\mu}{4})r^2}\Big), \tag{4.82}$$

$$Z = \frac{\sqrt{c^2+B^2} + \sqrt{B^2}}{\sqrt{c^2+A^2} - \sqrt{A^2}}. \tag{4.83}$$

5 The solution of A

Firstly, Liu [19] analytically solved A for general second-order BVPs. Here we consider only the case $A < 0$ for Troesch's problem. From (4.67) and (4.81) we obtain:
case (a):

$$\ln Z = \frac{\mu}{c} \sinh(\mu r), \tag{5.84}$$

case (b):

$$\ln Z = \frac{-T(r)}{c}, \tag{5.85}$$

where $T(r)$ defined by (4.82). Defining

$$\Theta(r) := exp\Big(\frac{\mu}{c}\sinh(\mu r)\Big), \tag{5.86}$$

for case (a) and

$$\Theta(r) := exp\Big(\frac{-T(r)}{c}\Big), \tag{5.87}$$

for case b) and substituting Eq. (4.68) for Z into Eq. (5.84) and substituting Eq. (4.83) for Z into Eq. (5.85) we obtain

$$\Theta = \frac{\sqrt{c^2+B^2}+\sqrt{B^2}}{\sqrt{c^2+A^2}-\sqrt{A^2}}. \tag{5.88}$$

By using $A < 0$ and $B > 0$, Eq. (5.88) can be written as

$$\Theta A + B = \sqrt{c^2+B^2} - \Theta\sqrt{c^2+A^2}. \tag{5.89}$$

Squaring the above equation and cancelling the common terms we can rearrange it to

$$2\Theta\sqrt{c^2+B^2}\sqrt{c^2+A^2} = (1+\Theta^2)c^2 - 2\Theta AB. \tag{5.90}$$

Squaring again and cancelling the common term and factor we get

$$4\Theta^2(A^2+B^2) + 4\Theta(1+\Theta^2)AB = (1-\Theta^2)^2c^2. \tag{5.91}$$

Inserting $B = rA/(r-1)$ and through some algebraic manipulations we eventually obtain

$$\frac{4\Theta}{(r-1)^2}\Big[(1+\Theta)^2r^2 - (1+\Theta)^2r + \Theta\Big]A^2 = (1-\Theta^2)^2c^2. \tag{5.92}$$

If the following condition holds

$$\Psi(r) := (1+\Theta)^2r^2 - (1+\Theta)^2r + \Theta > 0, \tag{5.93}$$

then A has a negative solution:

$$A = -\sqrt{\frac{(r-1)^2(1-\Theta^2)^2c^2}{4\Psi\Theta}}. \tag{5.94}$$

The discriminant function $\Psi(r)$ is an open-up distorted parabola of r since Θ is also a function of r. By inspection, $\Psi(r)$ has the following properties:

$$\Psi(0) = \Psi(1) = \Theta, \tag{5.95}$$

and there exist two roots of r for $\Psi(r) = 0$:

$$r_1 = \frac{1}{2} - \frac{\Theta-1}{2(\Theta+1)} = \frac{1}{\Theta+1},$$
$$r_2 = \frac{1}{2} + \frac{\Theta-1}{2(\Theta+1)} = \frac{\Theta}{\Theta+1},$$

where $0 < r_1 < 0.5 < r_2 < 1$. There exist solutions of A given by Eq. (5.94) in the following ranges of r:

$$0 < r < r_1, \quad r_2 < r < 1. \tag{5.96}$$

6 Adjusting the slope A

We have derived a closed-form solution to calculate the slope A for each r in its admissible range. If A is available, then we can apply the GPS method given below to integrate the $(\mathbf{Y}, x) - IVP$ in Eqs. (4.55)-(4.58) and (4.69)-(4.72). Up to this point we should note that the LGSM is an exactly solving technique for the second-order nonlinear BVPs without making any assumption or the approximation in derivations of all required formulas. However, how to determine a correct r and thus A requires a numerical integration of the nonlinear ODEs.

6.1 The GPS

We have derived the closed-form solutions to calculate the slope A for each r in its admissible range, and thus we can integrate the $(\mathbf{Y}, x) - IVP$ in Eqs. (4.55)-(4.58) and (4.69)-(4.72) by the following GPS method:

$$\mathbf{Y}_{n+1} = \mathbf{Y}_n + \frac{2\tau\|\mathbf{Y}_n\|^2 + 2\tau^2 \mathbf{f}_n \cdot \mathbf{Y}_n}{\|\mathbf{Y}_n\|^2 - \tau^2\|\mathbf{f}_n\|^2}\mathbf{f}_n, \tag{6.97}$$

where

$$\mathbf{f}_n = \mathbf{f}(x_n, \mathbf{Y}_n), \quad \tau = h/2. \tag{6.98}$$

The GPS, was first derived by Liu [16], which has used the Cayley transformation and the Padé approximations in the augmented space, namely Minkowski. The major difference between GPS and the traditional numerical methods is that those schemes are all formulated directly in the usual Euclidean space $\mathbb{R}^n$; none of them are considered in the Minkowski space $\mathbb{M}^{n+1}$. One of the benefits of GPS in the augmented Minkowski space is that the resulting schemes can avoid the spurious solutions and ghost fixed points.

6.2 *Adjusting A for the Troesch's problem*

For a trial r in admissible range, which can be identify by (5.93), we can calculate A and then numerically integrate Eqs. (4.55)-(4.58) or (4.69)-(4.72) from $x = 0$ to $x = 1$, and compare the end value of $y_1^r(1)$ with the exact one $y_1(1) = c$. If $|y_1^r(1) - c|$ is smaller than a given error tolerance ϵ, then the process of finding A is finished. Otherwise, we need to calculate the end values of $y_1(1)$ corresponding to different $r_1 < r$ and $r_2 > r$, which are denoted by $y_1^{r_1}(1)$ and $y_1^{r_2}(1)$, respectively. If $\Big(y_1^{r_1}(1) - c\Big)\Big(y_1^r(1) - c\Big) < 0$, then there exists one root between r_1 and r; otherwise, the root is located between r and r_2. Then, we apply the half-interval method to find a suitable r, that requires us to to calculate Eqs. (4.55)-(4.58) or (4.69)-(4.72) at each of the calculation of $y_1^r(1) - c$, until $|y_1^r(1) - c|$ is small enough to satisfy the criterion of $|y_1^r(1) - c| < \epsilon$.

7 Numerical results

7.1 *Troesch's problem for $\mu = 0.5$*

We first consider the Troesch's problem for $\mu = 0.5$, in Eq. (1.1). By taking $c = 1$ in transformation (2.4), system of (2.7)-(2.9) will be as follows:

$$y_1' = y_2, \tag{7.99}$$

$$y_2' = 0.5 \sinh\left(0.5\Big[y_1 + x - 1\Big]\right), \tag{7.100}$$

$$y_1(0) = 1, \quad y_2(0) = A, \tag{7.101}$$

which firstly we calculate A by the LGSM. Fig. 1, shows $A(r)$ for $r \in (0, 1)$, which can be obtain from Eqs. (5.88), (5.93) and (5.94).
By explained procedure in previous section we obtain $r_1 = 0.13751069258$ and $r_2 = 0.86735612699$, that for both of them we have $A(r_1) = A(r_2) = -0.040944051019$. Positions of these roots displayed at Fig. 2.
By fixing the step-size $h = 0.0001$ in this calculation, we plot the exact and obtained solutions for transformed and original equations in Fig. 3. Also, Table 1. shows the comparison of LGSM absolute errors with other methods.

7.2 *Troesch's problem for $\mu = 1$*

For the second test problem, we consider the Troesch's problem for $\mu = 1$, in Eq. (1.1). By letting $c = 1$ in transformation (2.4), system of (2.7)-(2.9) will be as follows:

$$y_1' = y_2, \tag{7.102}$$

$$y_2' = \sinh\Big(y_1 + x - 1\Big), \tag{7.103}$$

$$y_1(0) = 1, \quad y_2(0) = A, \tag{7.104}$$

which firstly by the Lie-group shooting method we calculate A. From Eqs. (5.88), (5.93) and (5.94), we display the $A(r)$ for $r \in (0,1)$ in Fig. 4.
Roots of $r_1 = 0.130539194$ and $r_2 = 0.9059082$, obtained from explained procedure, yield $A(r_1) = A(r_2) = -0.154746$ that plotted in Fig. 5.
By taking the step-size $h = 0.0001$ in this calculation, we plot the exact and obtained solutions for transformed and original equations in Fig. 6. For this case, efficiency of LGSM with respect to traditional applied methods has been demonstrated in Table 2.

7.3 Troesch's problem for $\mu = 5$

Now we consider the Troesch's problem for $\mu = 5$. By letting $c = 1$ in transformation (2.10), system of (2.13)-(2.15) will be as follows:

$$y_1' = y_2, \tag{7.105}$$

$$y_2' = 2\frac{y_1 + \tanh\left(\frac{5}{4}\right)x - 1}{\left(y_1 + \tanh\left(\frac{5}{4}\right)x - 1\right)^2 - 1}.\left(y_2 + \tanh\left(\frac{5}{4}\right)\right)^2$$
$$+25\frac{\left(y_1 + \tanh\left(\frac{5}{4}\right)x - 1\right)\left(1 + \left(y_1 + \tanh\left(\frac{5}{4}\right)x - 1\right)^2\right)}{1 - \left(y_1 + \tanh\left(\frac{5}{4}\right)x - 1\right)^2}, \tag{7.106}$$

$$y_1(0) = 1, \quad y_2(0) = A. \tag{7.107}$$

Similar to previous examples, we apply the LGSM to obtain A. Since, for this case we have more than two roots, we plot the $A(r)$ for $r \in (0, 0.14)$ in Fig. 7, which can be obtain from Eqs. (5.88), (5.93) and (5.94).
By explained procedure in previous section we obtain one of the roots, i.e. $r_1 = 0.0348183155869$, and hence $A(r) = -0.791062$. Fig. 8, clarify the position of this root.
Fig. 9 shows the approximate and exact solutions for transformed and original equations, by taking $h = 0.0001$.
Finally, comparison of LGSM absolute errors with finite element method based on B-Spline technique have been shown in Table 3. at given points.

8 Conclusion

In this work, the Lie-group shooting method has been used for the approximate solution of the Troesch's problem. Because of unstably of Troesch's problem, two different transformations has been introduced to overcome this difficulty. From the test examples, we may conclude the low computational complexity and storage requirement with high accuracy of the proposed procedure. Also, comparison with other applied techniques for Troesch problem, clarify that LGSM is better than the previous ones.

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Table 1: Comparison of absolute errors for $\mu = 0.5$

x_i	$Laplace$[13]	HPM[7]	$Finite\ element$[14]	$LGSM$
0.1	7.7×10^{-4}	8.2×10^{-4}	7.7×10^{-4}	1.1×10^{-7}
0.2	1.5×10^{-3}	1.6×10^{-3}	1.5×10^{-3}	4.6×10^{-7}
0.3	2.1×10^{-3}	2.3×10^{-3}	2.1×10^{-3}	1.1×10^{-6}
0.4	2.7×10^{-3}	2.9×10^{-3}	2.7×10^{-3}	1.9×10^{-6}
0.5	3.0×10^{-3}	3.2×10^{-3}	3.0×10^{-3}	3.0×10^{-6}
0.6	3.1×10^{-3}	3.4×10^{-3}	3.1×10^{-3}	4.4×10^{-6}
0.7	3.0×10^{-3}	3.2×10^{-3}	3.0×10^{-3}	6.0×10^{-6}
0.8	2.4×10^{-3}	2.7×10^{-3}	2.4×10^{-3}	7.9×10^{-6}
0.9	1.5×10^{-3}	1.6×10^{-3}	1.5×10^{-3}	1.0×10^{-5}

Table 2: Comparison of absolute errors for $\mu = 1$

x_i	$Laplace$[13]	HPM[7]	$Finite\ element$[14]	$LGSM$
0.1	2.9×10^{-3}	3.6×10^{-3}	2.8×10^{-3}	4.8×10^{-6}
0.2	5.9×10^{-3}	7.1×10^{-2}	5.6×10^{-3}	8.9×10^{-6}
0.3	8.2×10^{-3}	1.0×10^{-2}	8.2×10^{-3}	1.2×10^{-5}
0.4	1.0×10^{-2}	1.3×10^{-2}	1.0×10^{-2}	1.4×10^{-5}
0.5	1.2×10^{-2}	1.6×10^{-2}	1.2×10^{-2}	1.6×10^{-5}
0.6	1.3×10^{-2}	1.7×10^{-2}	1.3×10^{-2}	1.7×10^{-5}
0.7	1.3×10^{-2}	1.7×10^{-2}	1.3×10^{-2}	1.6×10^{-5}
0.8	1.1×10^{-2}	1.5×10^{-2}	1.1×10^{-2}	1.4×10^{-5}
0.9	7.4×10^{-3}	9.7×10^{-3}	7.4×10^{-3}	1.2×10^{-5}

Table 3: Comparison of absolute errors for $\mu = 5$

x_i	$Finite\ element$[14]	$LGSM$
0.1	$--$	4.9×10^{-6}
0.2	7.3×10^{-3}	9.7×10^{-6}
0.3	$--$	1.5×10^{-5}
0.4	2.2×10^{-3}	2.1×10^{-5}
0.5	$--$	2.7×10^{-5}
0.6	$--$	2.9×10^{-5}
0.7	$--$	1.8×10^{-5}
0.8	1.4×10^{-2}	1.3×10^{-6}
0.9	3.0×10^{-2}	7.8×10^{-6}

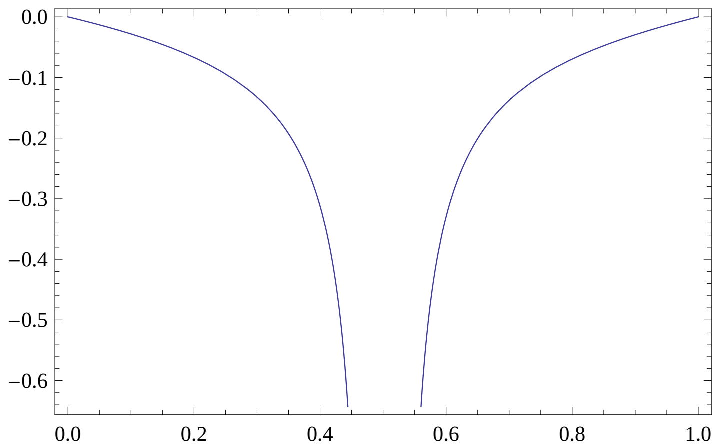

Fig. 1. Plot of A with respect to r for $\mu = 0.5$.

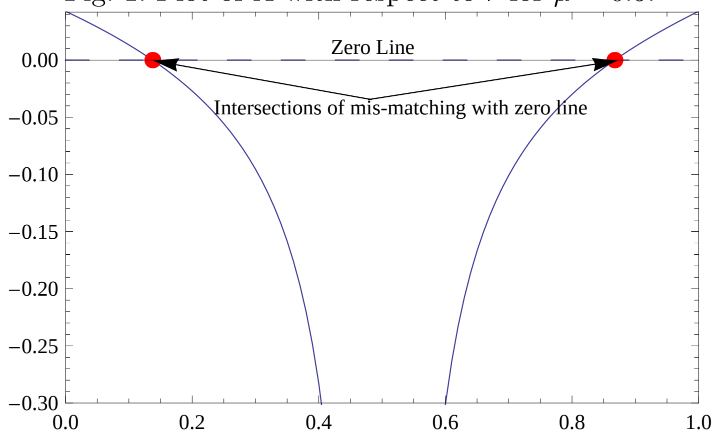

Fig. 2. Plot of $y_1(1) - 1$ with respect to r for $\mu = 0.5$.

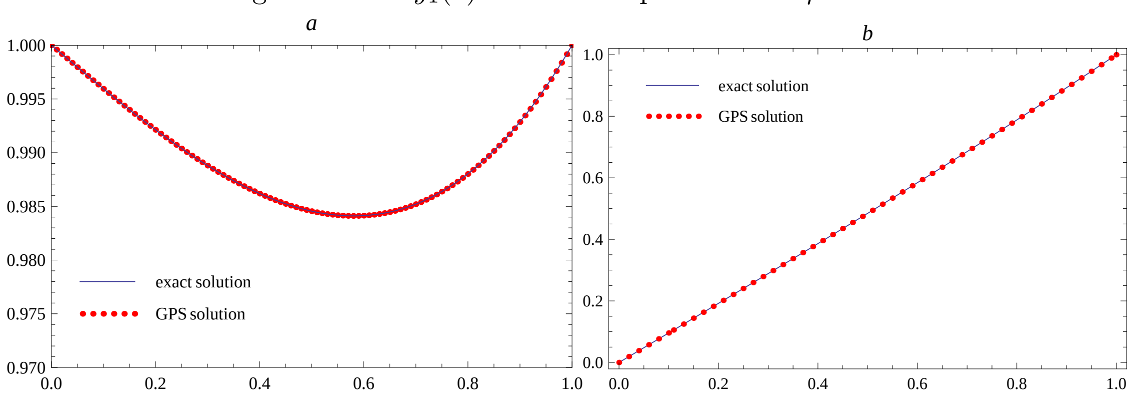

Fig. 3. Plot of exact and approximated solutions by GPS for (a) transformed Troesch's problem and (b) original Troesch's problem for $\mu = 0.5$.

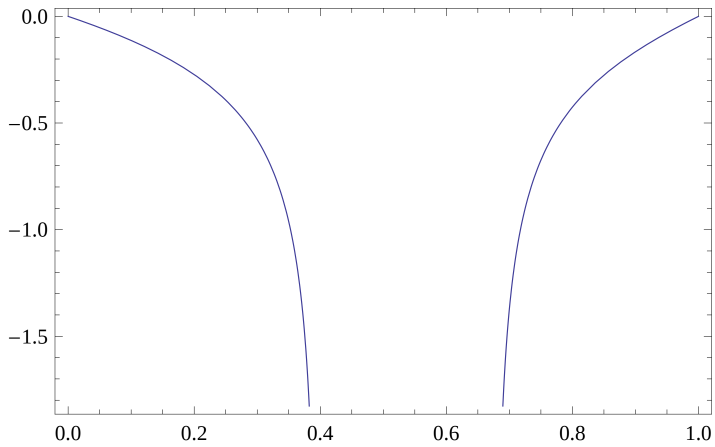

Fig. 4. Plot of A with respect to r for $\mu = 1$.

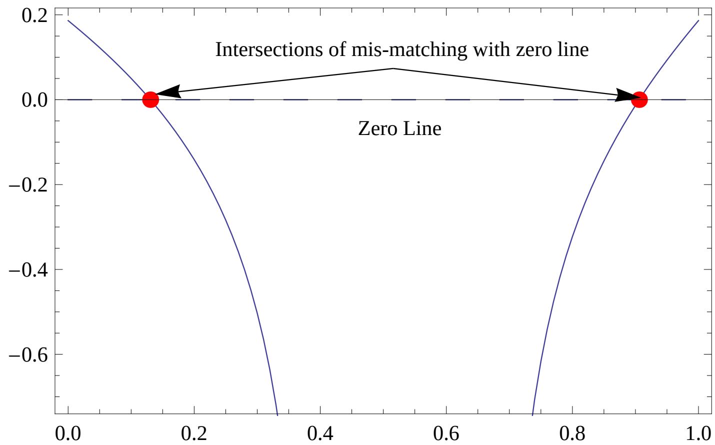

Fig. 5. Plot of $y_1(1) - 1$ with respect to r for $\mu = 1$.

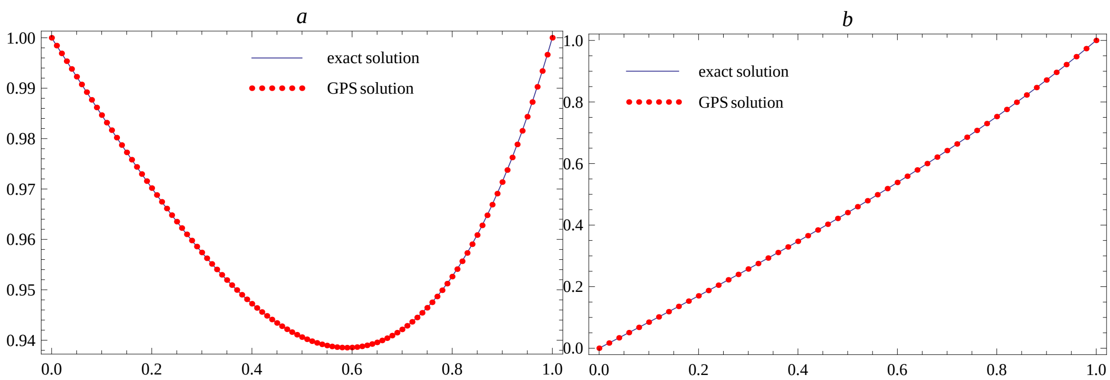

Fig. 6. Plot of exact and approximated solutions by GPS for (a) transformed Troesch's problem and (b) original Troesch's problem for $\mu = 1$.

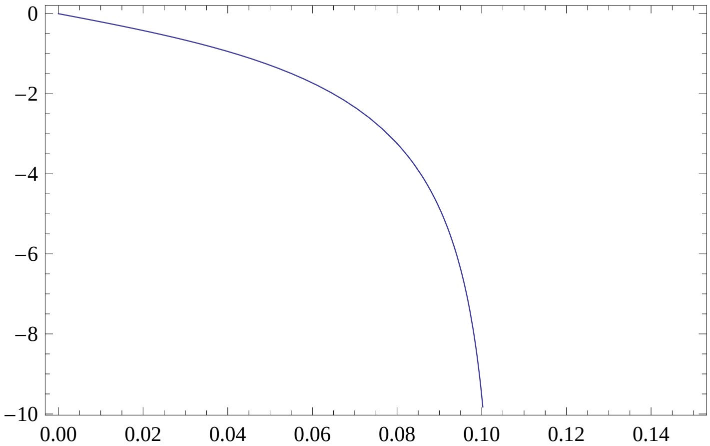

Fig. 7. Plot of A with respect to r for $\mu = 5$.

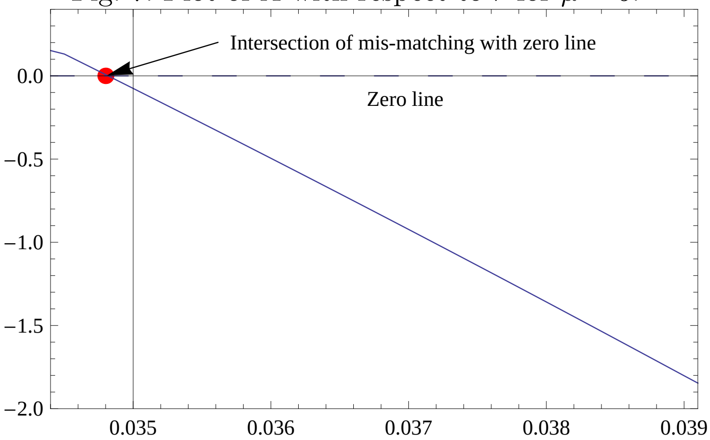

Fig. 8. Plot of $y_1(1) - 1$ with respect to r for $\mu = 5$.

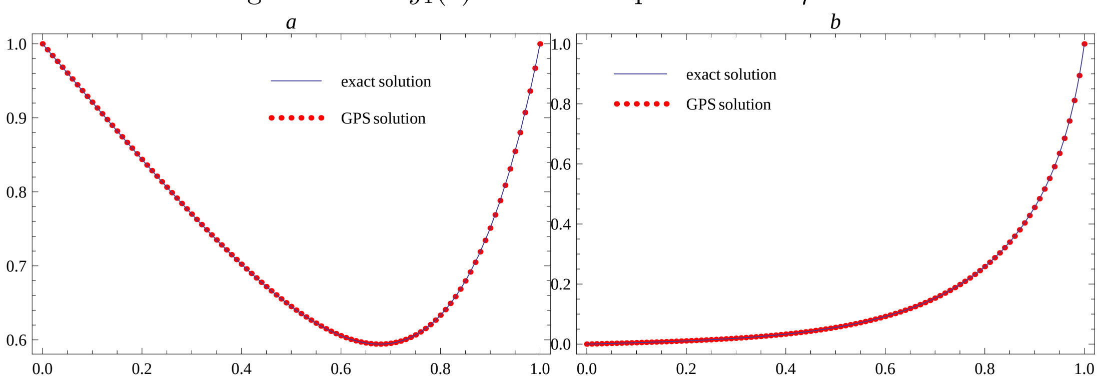

Fig. 9. Plot of exact and approximated solutions by GPS for (a) transformed Troesch's problem and (b) original Troesch's problem for $\mu = 5$.