Characterization of Signed Graphs whose Iterated Signed Line Graphs are Balanced or $S-$Consistent

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Abstract

A *signed graph* is a graph in which every edge is designated to be either *positive* or *negative*; it is *balanced* if every cycle contains an even number of negative edges. A *marked signed graph* is a signed graph each vertex of which is designated to be positive or negative, and it is *consistent* if every cycle in the signed graph possesses an even number of negative vertices. *Signed line graph* $L(S)$ of a given signed graph $S = (G, \sigma)$, as given by Behzad and Chartrand [9], is the signed graph with the standard *line graph* $L(G)$ of G as its underlying graph and whose edges are assigned the signs according to the rule: for any $e_ie_j \in E(L(S))$, $e_ie_j \in E^-L(S) \Leftrightarrow$ the edges e_i and e_j of S are both negative in S. Iterated signed line graphs $L^k(S)=L(L^{k-1}(S))$, $k \in N$, $S := L^0(S)$ is defined similarly. Further, $L(S)$ is $S-$*consistent* if to each vertex e of $L(S)$, which is an edge of S, one assigns the sign $\sigma(e)$ then the resulting marked signed graph $(L(S))_\mu$ is consistent. In this paper, we give a characterization of signed graphs S whose iterated signed line graphs $L^k(S)$ are balanced or $S-$consistent.

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1 Introduction

Unless mentioned or defined otherwise, for all terminology and notation in graph theory the reader is referred to [27]. We consider only finite simple graphs without self-loops.

Cartwright and Harary [12] considered graphs in which vertices represent persons in a social group and edges represent *dyadic* (symmetric) relations amongst persons each of which is given a positive or a negative sign according to whether the nature of the corresponding relationship is positive (friendly, like, etc.) or negative (hostile, dislike, etc.). Such a network (*i.e., weighted* graph) S is called a *signed graph* (Chartrand [13]; Harary *et al.* [16]). Thus, graphs may be regarded as signed graphs in which all the edges are positive, or the so-called *all-positive signed graphs* (*all-negative signed graphs* are defined similarly). A signed graph is said to be *homogeneous* if it is either all-positive or all-negative and *heterogeneous* otherwise.

Signed graphs are much studied in literature because of their extensive use in modelling a variety of cognitive and/or socio-psychological processes (*e.g. see* Acharya [4]; Katai and Iwai [20, 21]; Roberts [24]; Roberts and Xu [25]) and also because of their interesting connections with many classical mathematical systems (Zaslavsky [28]).

The number of positive (negative) edges incident at vertex v, denoted by $d^+(v)$ ($d^-(v)$) is called *positive* (*negative*) degree of the vertex v in S. The *total degree* $d(v)$ of the vertex v in S is the sum $d(v) = d^+(v) + d^-(v)$.

A cycle in a signed graph S is said to be *positive* if the product of the signs of its edges is positive or, equivalently, if the number of negative edges in it is even. A cycle which is not positive is said to be negative. A signed graph is said to be *balanced* if every cycle in it is positive (Harary [15]; Cartwright and Harary [12]; Acharya and Acharya [5]). The following characterization of balanced signed graphs is well known.

Theorem 1. (**Harary [15]**). *A signed graph is balanced if and only if there exists a partition of its vertex set into two subsets, one of them possibly empty, such that every positive edge joins two vertices in the same subset and every negative edge joins two vertices from different subsets.*

A spectral characterization of balanced signed graphs was given by Acharya [1]. Harary and Kabell [17, 18] developed a simple algorithm to detect balance in signed graphs as also enumerated them.

By a *negative section* (Gill and Patwardhan [14]) of a subgraph S' of a signed graph S we mean a maximal edge-induced subgraph in S' consisting of only the negative edges of S; in particular, a negative section in a cycle of S is essentially a maximal all-negative path in the cycle or the whole cycle itself. Since a cycle is positive if and only if it has an even number of negative sections of odd length, the following is another simple characterization of balanced signed graphs.

Theorem 2. *A signed graph is balanced if and only if every cycle in it has an even number of negative sections of odd length.*

Lemma 3. (**Zaslavsky [29]**). *A signed graph in which every chordless cycle is positive, is balanced.*

A *marked signed graph* is an ordered pair $S_\mu = (S, \mu)$ where $S = (G, \sigma)$ is a signed graph on its *underlying* graph $G = (V, E)$ and $\sigma : E(G) \to \{+1, -1\}$ is a function from the edge set $E(G)$ of G, called signing of G into the set $\{+1, -1\}$, whose elements are called *signs* and $\mu : V(G) \to \{+1, -1\}$ is a function from the vertex set $V(G)$ of G into the set $\{+1, -1\}$ whose elements are called *marks*. The mark $\mu(G)$ (sign $\sigma(G)$) $s(G')$ of a nonempty subgraph S' of S_μ is then defined as the product of the marks (signs) of vertices (edges) in S'. A cycle Z in S_μ is said to be *consistent* if $\mu(Z) = +1$; otherwise, it is said to be *inconsistent*. Further, S is said to be *consistent* if every cycle in it is consistent (Beineke and Harary [10]). Beineke and Harary [10, 11] were the first to pose the problem of characterizing consistent marked graphs, which was subsequently settled by Acharya [2, 3], Rao [23] and Hoede [19]. Recently, new characterizations of consistent marked graphs have been obtained by Roberts and Xu [25].

Theorem 4. (**Hoede [19]**). *A marked graph G_μ is consistent if and only if for any spanning tree T of G all fundamental cycles are consistent and all common paths of pairs of fundamental cycles have end vertices with the same marking.*

The following definition of *signed line graph* $L(S)$ of a given signed graph S was given by Behzad and Chartrand [9]: the vertices of $L(S)$ correspond one-to-one with the edges of S, $e_ie_j \in E(L(S)) \Leftrightarrow$ the edges of S corresponding to the vertices e_i and e_j of $L(S)$ have a vertex in common in S, and for any $e_ie_j \in E(L(S))$ one has $e_ie_j \in E^-L(S) \Leftrightarrow$ the edges of S corresponding to e_i and e_j are both negative in S.

A signed graph $\Gamma = (H, \xi)$ is $(S, \mathcal{R})-$*marked* if there exists a signed graph $S = (G, \sigma)$, a bijection $\varphi : E(S) \to V(H)$, a binary relation $\mathcal{R}$ on $E(S)$ and marking $\mu : V(H) \to \{+1, -1\}$ of H satisfying the following *compatibility conditions*,

(CC1): for every $u, v \in V(\Gamma),\ uv \in E(H) \Leftrightarrow \{\varphi^{-1}(u),\ \varphi^{-1}(v)\} \in \mathcal{R}$

(CC2): $\{\mu(u), \mu(v)\} = \{\sigma(\varphi^{-1}(u)), \sigma(\varphi^{-1}(v))\}$

Further, Γ is $(S, \mathcal{R})-$*consistent* if the following additional condition is satisfied:

(CC3): $\prod_{v \in V(Z)} \mu(v) = 1 \ \ \forall \ \ Z \in \mathcal{C}_\Gamma$ where $\mathcal{C}_\Gamma$ denotes the set of cycles in Γ;

The case when $\mathcal{R}$ is defined by the condition that $\varphi^{-1}(u) \cap \ \varphi^{-1}(v) \neq \phi$ is treated in Sinha [26] in respect of *signed graph equations* involving signed line graphs; in this particular case, the term '$(S, \mathcal{R})-$marked' and '$(S, \mathcal{R})-$consistent' will be reduced to '$S-$marked' and '$S-$consistent' respectively and similarly reduced terminology and notation will be adopted without specific mention in other notions using '$(S, \mathcal{R})-$format' in the above sense.

We begin with the following formal definition: Given any signed graph $S = (G, \sigma)$, $L(S)$ is $S-$consistent if $L(S)$ is consistent with respect to the marking $\mu_\sigma : V(L(S)) \to \{+1, -1\}$ which assigns to each vertex e in $L(S)$ the sign $\sigma(e)$ of the edge e in S, *i.e.*, $\mu_\sigma(e) = \sigma(e)$ for every $e \in V(L(S)) = E(S)$.

Clearly, if Γ_μ is $S-$consistent then S must be balanced.

The following characterization of signed graphs S whose signed line graphs $L(S)$ are S-consistent has been obtained recently.

Theorem 5. (**Acharya *et al.*** [**8**]). *For any isolate-free signed graph S of order p, $L(S)$ is S-consistent if and only if the following conditions hold in S:*

(1) *S is balanced; and*

(2) *$d(v_i) \geq 3$ in S for every v_i, $1 \leq i \leq p$, and*

  (*a*) *if $d(v_i) > 3$ then $d^-(v_i) = 0$; or*

  (*b*) *if $d(v_i) = 3$ then either $d^-(v_i) = 0$ or $d^-(v_i) = 2$; and*

(c) *if $d^-(v_i) = 2$ and v_i lies on a cycle of S then the negative degree of v_i is due to the negative edges of the cycle.*

In this note, we give a characterization of signed graphs S whose iterated signed line graphs $L^k(S)$=$L(L^{k-1}(S))$, $k \in N$, S:= $L^0(S)$, are $S-$consistent.

Length of a path P in S is the number of edges in it and is denoted by $\ell(P)$. Towards solving the above problem, we first need to characterize signed graphs S for which $L^k(S)$ is balanced. Hence, we need to have an extension of the following characterization of signed graphs whose signed line graphs are balanced, which is our objective in the next section.

Theorem 6. *(***Acharya and Sinha**, [**6**]*). For any signed graph S, $L(S)$ is balanced if and only if the following conditions hold in S:*

(1) *for any cycle Z in S;*

  (a) *if Z is all negative then Z is of even length;*

  (b) *if Z is heterogeneous then there is an even number of negative sections of non-zero even length in Z;*

(2) *for any vertex v in S, if the degree exceeds two then there is at most one negative edge incident at v.*

2 Balanced iterated signed line graphs

In this section, we extend Theorem 6 to any iterated signed line graph $L^k(S)$, $k \in N$.

Theorem 7. *For any signed graph S, and for any positive integer k, $L^k(S)$ is balanced if and only if the following conditions are satisfied by S:*

(1) *for any cycle Z in S;*

  (a) *if Z is all-negative then Z is of even length; and*

  (b) *if Z is heterogeneous then the number of negative sections of odd (even) length greater than k is even if k is even (odd); and*

(2) *for any vertex v in S, if $d(v) > 2$ then, $d^-(v) < 3$ and if $d^-(v) = 2$ then the length of any negative section through v is at most k.*

Proof. ***Necessity***: Suppose $L^k(S)$, $k \in N$ is balanced for a given signed graph S. Then, by the definition of balanced signed graphs, every cycle Z' in $L^k(S)$ must have an even number of negative edges.

Now, every vertex of $L^k(S), k \in N$, corresponds to an edge of $L^{k-1}(S), k \geq 1$, and hence the edges of every cycle in $L^{k-1}(S), k \geq 1$, create a cycle Z' in $L^k(S)$. Since the length of every negative section in S is reduced by one in its image in $L(S)$ as per definition of the latter and the same continues to hold in each subsequent iterated signed line graph $L^t(S)$, $t = 2, 3, ..., k$, we see that at the k^{th} iteration the length of a negative section greater than k in a cycle Z of S is reduced by k and all the other negative sections of lengths $\leq k$ in Z vanish when we see its image Z' in $L^k(S)$. Now, if there exists an all-negative cycle Z in S then it must be of even length due to our hypothesis that $L^k(S)$ is balanced and hence (1)(a) follows.

Next, let Z be heterogeneous and $N_1, N_2, N_3, ..., N_r$ be the negative sections of even lengths $> k$ and $P_1, P_2, ..., P_m$ be the negative-sections of odd lengths $> k$. Now, if $\ell(N_i), 1 \leq i \leq r$, and $\ell(P_j), 1 \leq j \leq m$, denote the lengths of negative sections in Z then the length of each negative section in Z' will be $\ell(N_i) - k$ and $\ell(P_j) - k$.

Therefore, $L^k(S)$ is balanced

$$\Leftrightarrow \sum_{i=1}^{r}\{\ell(N_i) - k\} + \sum_{j=1}^{m}\{\ell(P_j) - k\} \equiv 0 \pmod 2;$$

$$\Leftrightarrow \sum_{i=1}^{r}\ell(N_i) - kr + \sum_{j=1}^{m}\ell(P_j) - km \equiv 0 \pmod 2;$$

$$\Leftrightarrow \sum_{i=1}^{r}\ell(N_i) + \sum_{j=1}^{m}\ell(P_j) - k(r+m) \equiv 0 \pmod 2.$$

Now, we consider the following cases:

Case1: k is even.

Then we have,

$$\sum_{j=1}^{m}\ell(P_j) \equiv 0 \pmod 2$$

whence,

$$m \equiv 0 \pmod 2. \tag{A}$$

Case2: k is odd.

Then, we see that if m is odd whence,

$$\begin{aligned} kr &\equiv 0 \pmod 2 \\ \Longrightarrow r &\equiv 0 \pmod 2 \end{aligned} \tag{B}$$

If m is even then,

$$\begin{aligned} \sum_{j=1}^{m}\{\ell(P_j)\} - kr &\equiv 0 \pmod 2 \\ \Longrightarrow r &\equiv 0 \pmod 2 \end{aligned} \tag{C}$$

Thus, (1)(b) follows from (A), (B) and (C). On the other hand, suppose that not all the vertices of the cycle Z' in $L^k(S)$ correspond to the edges of a cycle in $L^{k-1}(S)$ for some $k \geq 1$. Such a cycle Z' must arise in $L^k(S)$ due to a vertex v of $d(v) \geq 3$ in S. Suppose $d^-(v) \geq 3$. Then any of the three negative edges incident to v will form an all-negative triangle in $L(S)$ and hence at every iteration k, $k \geq 1$, $L^k(S)$ would have an all-negative triangle, contradicting the hypothesis that $L^k(S)$ is balanced. Therefore, $d^-(v) < 3$. Now, the negative section through v is a path in S. If $d^-(v) = 0$ or 1, then the clique formed in $L(S)$ due to the edges incident at v would comprise of all-positive triangles and hence will not affect the balance of $L^k(S)$ within the clique in $L^k(S)$ constituted by the vertices which have evolved through the iterative process of obtaining $L^k(S)$ from the edges incident at v in S. Next, let $d^-(v) = 2$ and the length of a negative section through v be greater than k; specifically, let $P : v_0, e_1, v_1, e_2, v_2, ..., e_j, v_j = v, e_{j+1}, v_{j+1}..., e_t, v_t$ be the negative section in S containing v, where $t \geq k+1$. Then, there arise the following two cases:

Case(a): None of the edges in P lies on a cycle.

In this case, P would reduce to a path having only $t-k$ negative edges and at least one of these must create a negative triangle Z' in $L^k(S)$ since it is contained in a clique in $L^k(S)$ constituted by the vertices which have evolved through the iterative process of obtaining $L^k(S)$ from the edges incident at

$v \in P$ in S. Clearly, this contradics the balance of $L^k(S)$.

Case(b): Some edge in P lies on a (heterogeneous) cycle.

Without loss of generality, let v lie on a cycle. In this case, P will again reduce to a path having only $t-k$ negative edges. At least one of these must tend to create a negative triangle Z' in $L^k(S)$ by the same argument as above since P is an edge induced subgraph of S containing v, contradicting our hypothesis that $L^k(S)$ is balanced.

Hence (2) follows.

Sufficiency: Suppose conditions (1) and (2) hold for a given signed graph S. We shall show that $L^k(S)$ is balanced. If S is all-positive, $L^k(S)$ is trivially balanced. So, let S be heterogeneous and suppose $L^k(S)$ is unbalanced. Then there exists a cycle Z' in $L^k(S)$ which has an odd number of negative edges. Without loss of generality, let Z' have the least possible length. First, suppose that the vertices of Z' correspond to the edges of a cycle in $L^{k-1}(S)$. If Z' is all-negative, it must then be of odd length, contradicting (1)(a). Therefore, Z' must be heterogeneous. If all its vertices correspond to the edges of a cycle in $L^{k-1}(S)$, $k \geq 1$, then by definition, the length of each negative section of length greater than k in the cycle Z of S resulting in Z' is reduced by k in Z'. Let $\ell(N_i)$, $1 \leq i \leq r$, and $\ell(P_j)$, $1 \leq j \leq m$, denote the lengths of negative sections N_i and P_j of even and odd lengths $> k$ respectively in Z. Since Z' contains an odd number of negative edges and

$$\sum_{i=1}^{r} \ell(N_i) \equiv 0 \pmod 2$$

it follows that

$$\sum_{j=1}^{m} \ell(P_j) - k(r+m) \not\equiv 0 \pmod 2. \tag{$*$}$$

Hence, if k is even, then

$$\sum_{j=1}^{m} \ell(P_j) \not\equiv 0 \pmod 2$$

whence $m \not\equiv 0 \pmod 2$, contradicting (1)(b). So, k must be odd. Then, m cannot be even, for otherwise from $(*)$ we see that r must be odd, contradicting (1)(b). But, then, $(*)$ implies that r should be odd, contradicting (1)(b).

Thus, we see that Z' must contain a vertex e'_i which corresponds to an edge e_i in $L^{k-1}(S)$ which is not on any cycle of $L^{k-1}(S)$, $k \geq 1$, but incident to a vertex x in $L^{k-1}(S)$ such that its degree $d_{k-1}(x) \geq 3$ in $L^{k-1}(S)$. Let e'_{i-1} and e'_{i+1} be the vertices adjacent to e'_i in Z'. Suppose $Z' = (e'_{i-1}, e'_i, e'_{i+1}, e'_{i-1})$ is of length three. Since e'_i does not belong to any cycle in $L^{k-1}(S)$, the edges e_{i-1} and e_{i+1} in $L^{k-1}(S)$ corresponding to vertices e'_{i-1} and e'_{i+1} in Z', respectively, must both be incident at x in $L^{k-1}(S)$. Since Z' is unbalanced, it has either one negative edge or all its three edges are negative. If Z' is all-negative then it follows that $d^-_{k-1}(x) \geq 3$. Thus, by definition of L if $k = 1$ then $L^{k-1}(S) = L^0(S) = S$ has a vertex $x := v$ such that if $d^-_{k-1}(v) \geq 3$, contradicting (2). But, again, due to a result of Menon [22], it follows that if $d^-_{k-1}(v) \geq 3$ then S has an all-negative edge-induced subgraph with at least one vertex of negative degree three, contradicting (2). Hence, Z' must be a triangle in $L^k(S)$ with just one negative edge. Since the length of a negative section gets reduced by one in every iterated operation of L, it must be due to two negative edges incident at x in $L^{k-1}(S)$ such that $d_{k-1}(x) \geq 3$. Therefore, length of a negative section through x is at least two. Again, if $k = 1$ then S would have a vertex $x := v$ of $d(v) \geq 3$ such that $d^-(v) = 2$, contradicting (2). So $k \geq 2$. By definition of L, S would then have a vertex $x := v$ on Z such that $d(v) \geq 3$ through which there would be a negative section of length greater than $k + 1$ (Menon [22]), contradicting (2). Thus, it follows that Z' is of length at least four. Let $Z' = (e'_1, e'_2, ..., e'_{i-1}, e'_i, e'_{i+1}, ...e'_r, e'_1), r \geq 4$, where e_i does not belong to any cycle in $L^{k-1}(S)$. Therefore, there exists a vertex e' in Z' corresponding to an edge e on a cycle Z in $L^{k-1}(S)$. Clearly, the vertex x at which e_i is incident lies on Z. Now by condition (2) and the definition of $L(S)$, in $(k-1)^{th}$ iteration $d^-_{k-1}(x) \leq 1$, which implies that the triangles in the clique formed in $L^k(S)$ due to $d_{k-1}(x) \geq 3$ edges will all be positive and also by condition (1) the cycle in $L^k(S)$ due to the edges of the cycle Z in $L^{k-1}(S)$ is balanced. Thus Z' being the symmetric difference of the edge sets of the positive triangles and that of the balanced cycle Z is also balanced due to Lemma 3, contradicting our assumption that Z' is unbalanced. Thus, by contraposition, it follows that $L^k(S)$ must be balanced and the proof is complete. □

Corollary 8. *$L^k(S)$ is balanced for all k if and only if S satisfies following conditions:*

(1) *For every cycle Z in S,*

- (a) *if Z is all-negative then Z is of even length; and*
- (b) *if Z is heterogeneous then the number of negative sections of each length greater than one is even; and*

(2) *if for a vertex v, $d(v) > 2$, then there is at most one negative edge incident at v, $v \in V(S)$.*

Corollary 9. *For a heterogeneous signed tree S, $L^k(S)$ is balanced for every $k \in N$ if and only if for any vertex of degree greater or equal to 3, its negative degree is at most 1.*

Theorem 10. *For a heterogeneous signed graph S, $L^k(S)$ is all-positive, for $k \in N$ if and only if*

(*i*) *$d^-(v) \leq 2$, $\forall\ v \in V(S)$, and*

(*ii*) *Each cycle Z of S is either all-positive or heterogeneous.*

Moreover, the value of k is at least as large as the length of maximum negative section in S.

Proof. ***Necessity***: Suppose $L^k(S)$ is all-positive, for $k \in N$. Now we shall show that both the conditions are satisfied by S. On the contrary, condition (i) or condition (ii) does not hold. First suppose S does not satisfies condition (i). That means there is a vertex $v \in V(S)$ such that $d^-(v) \geq 3$. It follows that $L(S)$ has all-negative cycle of length 3. That means $L^k(S)$ has a all-negative cycle of length 3, $\forall\ k \in N$, contradicting our assumption that $L^k(S)$ is all-positive for $k \in N$.

Next suppose that condition (ii) does not holds. That means there exists an all-negative cycle $\forall\ k \in N$, contradicting our assumption. This completes the necessity.

Sufficiency: Conversely, let S be a heterogeneous signed graph and it satisfies above given conditions (i) and (ii). Now we shall show that $L^k(S)$ is all-positive, for $k \in N$. It is given that $d^-(v) \leq 2$, $\forall\ v \in V(S)$. Now let us suppose that there is a vertex $v \in V(S)$, of degree greater or equal to 3. Then by the definition of signed line graph its signed line graph generates clique, but its clique never all negative *i.e.*, every edge of clique is not negative. Because by the condition (i) negative degree of every vertex of signed graph is less or equal to 2. Now let us suppose that the signed graph contains a negative section of length q. Then by the definition of signed line graph, length of negative section is reduced by one in each iteration. Its line signed graph has negative section of length $q-1$, now repeating this process q times, then its iterated signed line graph $L^q(S)$ has negative section of length of $q - q =$

0, therefore $L^k(S)$ is all positive when $k = q$.

Now let Z is a cycle of S. Then by the condition (ii), If Z is all positive, then theorem is trivially true.

If Z is heterogeneous, then it contains negative sections, let us suppose that the signed graph S has a negative section of maximum length p. Then again by the similar procedure $L^k(S)$ is all positive if $k = p$. Hence, if S is a signed graph which satisfies conditions (i) and (ii), then $L^k(S)$ is all-positive for a natural number k. Moreover, the value of k is the length of the maximal negative section in S. □

3 S-consistent iterated signed line graphs

Invoking the results of obtained in Theorem 5 and Theorem 7 in the foregoing section, we derive the following main result of this paper.

Theorem 11. *For any isolate-free signed graph S of order p, and any positive integer k, $L^k(S)$ is S-consistent if and only if the following conditions are satisfied by S:*

(1) *for any cycle Z in S;*

- (a) *if Z is all-negative then Z is of even length; and*
- (b) *if Z is heterogeneous then the number of negative sections of even (odd) length $\geq k$ is even if k is even (odd); and*

(2) *for any vertex v_i, $1 \leq i \leq p$, in S of $d(v_i) \geq 3$, $d^-(v_i) < 3$ and exactly one of the following conditions is satisfied by S:*

- (a) *$k = 1$ and there exists a vertex v_i with $d(v_i) = 3$ such that it either lies on no cycle and $d^-(v_i) = 2$, or it lies on an all-negative cycle of even length and there is no other vertex whose total degree is at least three and negative degree is exactly one, or it lies on a heterogeneous cycle, $d^-(v_i) = 2$ and this negative degree is due to the edges of the heterogeneous cycle, whence again there is no other vertex whose total degree is at least three and negative degree is exactly one;*
- (b) *$k >$ max. $\{$lengths of negative sections through $v_i\}$.*

4 Conclusions and scope

In this paper, we have given a characterization of a signed graph S whose iterated signed line graph $L^k(S)$ is $S-$consistent for some (all) natural number(s) k. Hence, given any signed graph H, let $\mathcal{L}_H = \{S : S$ is a signed graph for which $L(S) \cong H\}$. Call H a *signed line graph* if $\mathcal{L}_H \neq \phi$ and then any $S \in \mathcal{L}_H$ a *line-root* of H (cf.: Acharya and Sinha [7]). At the very outset, minimization (maximization) of k such that $L^k(S)$ remains $\mathcal{S}-$consistent for some $S \in \mathcal{L}_H$ might be of some interest. Further, it can be easily shown that if H is not an all-negative signed line graph then it must have at least one heterogeneous line root. Hence, some problems of immediate interest could be to minimize (maximize) the number of negative edges in a marker signed graph M, here a line-root, for a given $\mathcal{S}-$consistent signed line graph H, develop algorithms for attainment of these bounds and for identification and/or enumeration of line-root markers with the corresponding number of negative edges. For instance, if H is an $\mathcal{S}-$consistent all-negative signed line graph, then since its line-root is a unique all-negative signed graph both the minimum and the maximum of the number of negative edges in a line-root marker of H are equal to the order of H; in general, the same conclusion holds for any $\mathcal{S}-$consistent signed line graphs having unique line-root markers and there do exist such heterogeneous signed graphs (characterization of these signed graphs might itself be an interesting problem).

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