Distortion theorem for locally biholomorphic Bloch mappings on the unit ball $\mathcal{B}^{n*}$

Jianfei Wang

Department of Mathematics, Zhejiang Normal University,

Jinhua, Zhejiang, 321004, P.R. China

E-mail address: wjfustc@zjnu.cn

June 16, 2014

Abstract

In this note, we establish a distortion theorem for locally biholomorphic Bloch mappings f satisfying $||f||_0 = 1$ and $\det f'(0) = \alpha \in (0,1]$, where $\|f\|_0 = \sup\{(1-|z|^2)^{\frac{n+1}{2n}} |\det f'(z)|^{\frac{1}{n}} : z \in \mathcal{B}^n\}$. This result extends the result of Bonk, Minda and Yanagihara of one complex variable to higher dimensions. Moreover, a lower estimate for the radius of the largest univalent ball in the image of f centered at $f(0)$ is given.

Keywords: distortion theorem; locally biholomorphic mappings, univalent ball; hyperbolic distance.

Mathematical Subject Classification: Primary: 32H02, Secondary: 30C65.

1. INTRODUCTION

In 1988, Bonk in [1] established the well-known Bonk Distortion Theorem for Bloch functions in the unit disk $\mathbb{D}$ in $\mathbb{C}$ which inspired further work in one and several complex variables. In one complex variable, Bonk et al. in [2] and [3] studied the general distortion theorems for locally univalent Bloch functions and Bloch functions, respectively. In several complex variables, Liu in [4] investigated the properties of Bloch functions defined in the unit ball in $\mathbb{C}^n$ and generalized the Bonk Distortion Theorem to higher dimensions. Moreover, Liu gave the following Bonk Distortion Theorem for locally biholomorphic Bloch functions with values in $\mathbb{C}^n$.

Theorem [4] *Suppose* $f \in H(\mathcal{B}^n)$, $\|f\|_0 = 1$ *and* $\det f'(0) = 1$. *If* $\det f'(z) \neq 0$ *for all* $z \in \mathcal{B}^n$, *then*

$$|\det f'(z)| \geq (1-|z|)^{-(n+1)} \exp\{\frac{-(n+1)|z|}{1-|z|}\}$$

*‡ This work was supported by the National Natural Science Foundation of China (No. 11001246, 11101139) and NSF of Zhejiang province (No. Y6090694).

for all $z \in \mathcal{B}^n$. Moreover, the above inequality is best possible.

By making use of the above distortion theorem, Liu obtained estimates of Bloch constants for various subfamilies of locally biholomorphic mappings defined in the unit ball $\mathcal{B}^n$ of $\mathbb{C}^n$ which generalized a result of Liu and Minda in [5] to higher dimensions. FitzGerald and Gong in [6] extended the above distortion theorem to the first type of the classical domains in the sense of Hua. Later, Gong and Yan in [7] and Gong in [8] established the distortion theorems for holomorphic mappings and locally biholomorphic mappings on irreducible bounded symmetric domains using Lie algebra's method, respectively. For detailed information of distortion theorems and Bloch constants the reader may consult the book of Gong, Yu and Zheng in [9].

In this article, we will establish a distortion theorem for locally biholomorphic Bloch mappings on the unit ball $\mathcal{B}^n$ which is a generalization of the above Theorem [4]. While the distortion theorem for holomorphic Bloch mappings on the unit ball $\mathcal{B}^n$ is obtained by us in [10], the general results for $\alpha-$Bloch mappings are due to Chen, Ponnusamy and Wang in [11]. As a special case of the unit disk, the distortion theorem reduces to that of Bonk, Minda and Yanagihara in [2]. As an application, we give a lower estimate for the radius of the largest univalent ball for various subfamilies of Bloch mappings defined in $\mathcal{B}^n$. In the proof we use a type of subordination lemma for horodisk in the unit disk $\mathbb{D}$. This subordination lemma enables us to obtain our distortion theorem from a unified perspective. In particular, some hyperbolic geometric properties play an important role to obtain the subordination lemma.

2. PRELIMINARIES

We will first make use of the following notation and give some definitions in this paper.

Let $\mathbb{D}$ be the unit disk in the complex plane $\mathbb{C}$. Denote by $\mathbb{C}^n$ as the n-dimensional complex Hilbert space with the inner product and the norm given by

$$< z, w >= \sum_{j=1}^{n} z_j \bar{w}_j \ , \quad |z| = (< z, z >)^{\frac{1}{2}},$$

where $z,\ w \in \mathbb{C}^n$. Let $\mathcal{B}^n$ be the open unit ball in $\mathbb{C}^n$, i.e. $\mathcal{B}^n = \{z \in \mathbb{C}^n : |z| < 1\}$. The unit sphere of $\mathbb{C}^n$ is denoted by $\partial\mathcal{B}^n = \{z \in \mathbb{C}^n : |z| = 1\}$. Denote by $\mathcal{B}^n(x, r)$ as the ball of radius r with the center x. Let $H(\mathcal{B}^n)$ be the set of all holomorphic mappings from $\mathcal{B}^n$ to $\mathbb{C}^n$. Denote by $H_{loc}(\mathcal{B}^n)$ as the set of all locally biholomorphic mappings from $\mathcal{B}^n$ to $\mathbb{C}^n$, that is, $f \in H_{loc}(\mathcal{B}^n)$ means $f \in H(\mathcal{B}^n)$ and $\det f'(z) \neq 0$ for all $z \in \mathcal{B}^n$. Throughout the paper, we will write a point $z \in \mathbb{C}^n$ as a column vector in the following $n \times 1$ matrix form

$$z = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix}.$$

For a holomorphic mapping $f \in H(\mathcal{B}^n)$, we also write f as the $n \times 1$ matrix form

$$f = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix},$$

where all of the f_i are holomorphic functions from $\mathcal{B}^n$ to $\mathbb{C}$. The derivative of the mapping $f \in H(\mathcal{B}^n)$ at a point $a \in \mathcal{B}^n$ is the complex Jacobian matrix of f given by

$$f'(a) = \left(\frac{\partial f_i}{\partial z_j} \right)_{z=a}.$$

Then $f'(a)$ is a linear mapping from $\mathbb{C}^n$ to $\mathbb{C}^n$. Denote by $\|f'(a)\|$ as the norm of complex Jacobian matrix $f'(a)$. Let $\mathrm{Aut}(\mathcal{B}^n)$ denote the group of holomorphic automorphisms of $\mathcal{B}^n$.

Definition 1 [4] A holomorphic mapping $f \in H(\mathcal{B}^n)$ is called a Bloch mapping if the family

$$F_f = \{g : g(z) = f(\varphi(z)) - f(\varphi(0)),\ \varphi \in \mathrm{Aut}(\mathcal{B}^n)\}$$

is a normal family. The Bloch semi-norm of the Bloch mapping $f(z)$ is defined as

$$\|f\|_{\mathcal{B}} = \sup\left\{ \left\| \frac{\partial(f \circ \varphi)}{\partial z}(0) \right\| : \varphi \in \mathrm{Aut}(\mathcal{B}^n) \ \right\}.$$

Denote by $\beta(K)$ as the set of Bloch mappings f with $||f||_{\mathcal{B}} \leq K$, where $1 \leq K \leq \infty$.

Definition 2 [4] Suppose $f \in H(\mathcal{B}^n)$. We define the prenorm $\|f\|_0$ of f given by

$$\|f\|_0 = \sup\{|\det g'(0)|^{\frac{1}{n}} : g \in F_f\} = \sup\{(1-|z|^2)^{\frac{n+1}{2n}} |\det f'(z)|^{\frac{1}{n}} : z \in \mathcal{B}^n\}.$$

It is easy to show that $\|f\|_0$ is invariant under the group of holomorphic automorphisms $\mathrm{Aut}(\mathcal{B}^n)$.

Definition 3 [12] If $G \subset \mathbb{C}$ is a domain including the origin and f and g are two holomorphic functions on G, then f is subordinate to g if there is a holomorphic function $\varphi : G \to G$ such that $\varphi(0) = 0$ and $f = g \circ \varphi$. We write $f \prec g$ to denote this subordination relation.

Given a holomorphic mapping $f \in H(\mathcal{B}^n)$, we denote by $r(a, f)$ as the radius of the biggest univalent ball of f centered at $f(a)$(a univalent ball $\mathcal{B}^n(f(a), r) \subset f(\mathcal{B}^n)$ means that f maps biholomorphically an open subset of $\mathcal{B}^n$ containing the point a onto this ball).

Next we recall some basic facts about hyperbolic geometry on a hyperbolic domain Ω in the complex plane $\mathbb{C}$, that is , $\mathbb{C}\backslash\Omega$ contains at least two points. For an arbitrary hyperbolic domain Ω, we have the hyperbolic metric $\lambda_\Omega(z)|dz|$ with the Gaussian curvature -4. The hyperbolic metric on the unit disk $\mathbb{D}$ is

$$\lambda_{\mathbb{D}}(z)|dz| = \frac{|dz|}{1-|z|^2}.$$

The density $\lambda_\Omega(z)|dz|$ of the hyperbolic metric on a hyperbolic domain Ω is determined from

$$\lambda_\Omega(f(z))|f'(z)| = \lambda_{\mathbb{D}}(z),$$

where $f:\mathbb{D}\to\Omega$ is a conformal mapping. We denote the hyperbolic distance on Ω by d_Ω. For $A,B\in\Omega$, the hyperbolic distance between them is determined by

$$d_\Omega(A,B)=\inf\int_\gamma \lambda_\Omega(w)|dw|,$$

where the infimum is taken over all paths γ in Ω joining A and B. For example, the hyperbolic distance $d_{\mathbb{D}}(a,u)$ between the points a and u in $\mathbb{D}$ is obtained in the following way:

$$d_{\mathbb{D}}(a,u)=\frac{1}{2}\log\{\frac{1+|\frac{a-u}{1-\bar{a}u}|}{1-|\frac{a-u}{1-\bar{a}u}|}\}=\operatorname{arctanh}(|\frac{a-u}{1-\bar{a}u}|).$$

For $r>0$, let $\Delta(1,r)$ be a horodisk in $\mathbb{D}$, that is,

$$\Delta(1,r)=\{z\in\mathbb{D}:\frac{|1-z|^2}{1-|z|^2}<r\}=D_e(\frac{1}{1+r},\frac{r}{1+r}),$$

where $D_e(\frac{1}{1+r},\frac{r}{1+r})$ is a euclidean disk with a center $\frac{1}{1+r}$ and radius $\frac{r}{1+r}$. And $\Delta(1,r)$ is a circle internally tangent to the unit circle at 1. Note that $f(z)=\frac{r}{1+r}z+\frac{1}{1+r}$ is a conformal map of $\mathbb{D}$ onto $D_e(\frac{1}{1+r},\frac{r}{1+r})$. Hence, the hyperbolic metric on $\Delta(1,r)=D_e(\frac{1}{1+r},\frac{r}{1+r})$ is

$$\lambda_{\Delta(1,r)}(w)|dw|=\frac{\frac{r}{1+r}|dw|}{(\frac{r}{1+r})^2-|w-\frac{1}{1+r}|^2}.$$

In this paper, we always assume $r>1$ for the horodisk $\Delta(1,r)$.

3. SOME LEMMAS

We present the following lemmas to establish the distortion theorem.

Lemma 1 Suppose $f\in H(\mathbb{D})$, $f(0)=a\in\mathbb{R}$ and $f(\triangle(1,r))\subset\{w:\operatorname{Re}w<s\}$. Then
(1) $f(z)\prec G_0(z)=b\frac{z+1}{z-1}+b+a$ on $\triangle(1,r)$, where $b=\frac{r(s-a)}{r-1}>0$.
(2) Re $f(x)\geq G_0(x)=\frac{2bx}{x-1}+a$ for $0<x<1$ with equality holds for some x if and only if $f=G_0$.
(3) Re $f(x)\leq G_0(-x)=\frac{2bx}{x+1}+a$ for $0<x\leq\frac{r-1}{r+1}$ with equality holds for some x if and only if $f=G_0$.

Proof. (1) Note that $G_0(0)=a=f(0)$, and

$$\operatorname{Re}G_0(z)=b\frac{|z|^2-1}{|1-z|^2}+b+a.$$

We set $\Gamma(r)=\partial\triangle(1,r)=\{z\in\mathbb{D}:\frac{|1-z|^2}{1-|z|^2}=r\}$.
Then

$$\operatorname{Re}G_0(\Gamma(r))=-\frac{b}{r}+b+a=s.$$

Hence $G_0(\triangle(1,r))=\{\omega:\operatorname{Re}\omega<s\}$. We define $\varphi(z)=G_0^{-1}\circ f(z)$. Then $\varphi\in H(\triangle(1,r),\triangle(1,r))$. It implies $f\prec G_0$ on $\triangle(1,r)$.

(2) Let $\delta_x \subset \triangle(1,r)$ be the hyperbolic circle (relative to hyperbolic geometry on $\triangle(1,r)$)with center 0 which passes through x. Then δ_x is symmetric about the real axis. By making use of $f \prec G_0$ on $\triangle(1,r)$, we have f maps the circle δ_x into the closed disk bounded by the circle $G_0(\delta_x)$. Since G_0 is decreasing on $\delta_x \bigcap \mathbb{R}$ and symmetric about $\mathbb{R}$, the point of $G_0(\delta_x)$ with the smallest real part is $G_0(x)$.

Hence for all $x \in (0,1)$, we have

$$\text{Re } f(x) \geq \min\{\text{Re } G_0(z) : z \in \delta_x\} = G_0(x) = \frac{2bx}{x-1} + a.$$

Moreover, the equality holds for some x if and only if $f = G_0$.

(3) The proof is similar to that of the inequality in part (2). For $0 < x \leq \frac{r-1}{r+1}$, let $\delta_{-x} \subset \triangle(1,r)$ be the hyperbolic circle (relative to hyperbolic geometry on $\triangle(1,r)$)with center 0 which passes through $-x$. From $f \prec G_0$ on $\triangle(1,r)$, we have f maps the circle δ_{-x} into the closed disk bounded by the circle $G_0(\delta_{-x})$. Since G_0 is decreasing on $\delta_{-x} \bigcap \mathbb{R}$ and symmetric about $\mathbb{R}$, the point of $G_0(\delta_{-x})$ with the biggest real part is $G_0(-x)$, that is,

$$\text{Re } f(x) \leq \min\{\text{Re } G_0(z) : z \in \delta_{-x}\} = G_0(-x) = \frac{-2bx}{-x-1} + a = \frac{2bx}{x+1} + a.$$

Moreover, the equality holds for some x if and only if $f = G_0$.

Lemma 2 [4] *Suppose that $A = (a_{ij})$ is an $n \times n$ complex matrix . If $\|A\| > 0$, then for any unit vector $\xi \in \partial\mathcal{B}^n$, the following inequality holds:*

$$|A\xi| \geq \frac{|\det A|}{||A||^{n-1}}.$$

Lemma 3 [4] *If f is a Bloch mapping on the unit ball $\mathcal{B}^n$, then we have*

$$\|f'(z)\| \leq \frac{\|f\|_{\mathcal{B}}}{1-|z|^2}.$$

4. MAIN THEOREMS AND PROOFS

First of all, we need to introduce the function $m(\alpha)$ to present the main results.

We define $f(t) = e^{-\frac{n+1}{2}t}(1+t)^{\frac{n+1}{2}}$. Then f is decreasing on $[0,+\infty)$ and $f(0) = 1, f(+\infty) = 0$.

Denote by m as the inverse function of $f : (0,1] \to [0,+\infty)$. Then for any $\alpha \in (0,1]$, there is the unique number $m(\alpha)$ such that

$$e^{-\frac{n+1}{2}m(\alpha)}(1+m(\alpha))^{\frac{n+1}{2}} = \alpha.$$

It is clear that $m(1) = 0$.

Now we present the following distortion theorem.

Theorem 1 *If $f \in H_{loc}(\mathcal{B}^n)$, $||f||_0 = 1$, $\det f'(0) = \alpha \in (0,1]$, then*

(1)

$$|\det f'(z)| \geq \frac{\alpha}{(1-|z|)^{n+1}} \exp\{(1+m(\alpha))\frac{-(n+1)|z|}{1-|z|}\}$$

for any $z \in \mathcal{B}^n$, where $m(\alpha)$ is the real root of an equation $e^{\frac{-(n+1)}{2}t}(1+t)^{\frac{n+1}{2}} = \alpha$ in the interval $[0, +\infty)$.

(2)

$$|\det f'(z)| \leq \frac{\alpha}{(1+|z|)^{n+1}} \exp\{(1+m(\alpha))\frac{(n+1)|z|}{1+|z|}\}$$

for $|z| \leq \frac{m(\alpha)}{2+m(\alpha)}$. Moreover, the inequalities (1) and (2) are sharp.

Proof. (1) The case $\alpha = 1$ is due to Liu [4, Theorem 7]. So we will only discuss the case $\alpha \in (0,1)$.

Fix $\xi \in \partial\mathcal{B}^n$, we define

$$g(u) = (1-u)^{n+1} \det f'(u\xi).$$

Since

$$\|f\|_0 = \sup\{(1-|z|^2)^{\frac{n+1}{2n}} |\det f'(z)|^{\frac{1}{n}} : z \in \mathcal{B}^n\}.$$

From Definition 2 and $\|f\|_0 = 1$, we have

$$(1-|z|^2)^{\frac{n+1}{2n}} |\det f'(z)|^{\frac{1}{n}} \leq 1.$$

Hence

$$|g(u)| \leq |1-u|^{n+1} |\det f'(u\xi)| \leq (\frac{|1-u|^2}{1-|u|^2})^{\frac{n+1}{2}}.$$

Note that $f \in H_{loc}(\mathcal{B}^n)$, then $g \in H(\mathbb{D})$ and $g(u) \neq 0$ for all $u \in \mathbb{D}$.

Setting $h(u) = \log g(u)$, where the branch of logarithm is chosen such that $h(0) = \log g(0) = \log \alpha$ is real. We then have

$$\text{Re } h(u) = \log|g(u)| \leq \frac{n+1}{2} \log \frac{|1-u|^2}{1-|u|^2}.$$

Note that $m(\alpha)$ is the unique root of an equation $e^{\frac{-(n+1)}{2}t}(1+t)^{\frac{n+1}{2}} = \alpha$ in the interval $[0, +\infty)$, i.e. $\frac{n+1}{2}\log(1+m(\alpha)) - \frac{n+1}{2}m(\alpha) = \log \alpha$.

If $u \in \triangle(1, 1+m(\alpha))$, then we have

$$h(\triangle(1, 1+m(\alpha))) \subset \{\omega : \text{Re } \omega < \frac{n+1}{2}\log(1+m(\alpha))\}.$$

In view of Lemma 1(1), we have $h \prec G_0$ on $\triangle(1, 1+m(\alpha))$, where

$$G_0(u) = b\frac{u+1}{u-1} + b + \log \alpha,$$

$$b = \frac{1+m(\alpha)}{m(\alpha)}(\frac{n+1}{2}\log(1+m(\alpha)) - \log \alpha) = \frac{n+1}{2}(1+m(\alpha)).$$

For any $0 < x < 1$, by Lemma 1(2), we have

$$\begin{aligned} \text{Re } h(x) &\geq \frac{n+1}{2}(1+m(\alpha))\frac{2x}{x-1} + \log \alpha \\ &= (n+1)(1+m(\alpha))\frac{x}{x-1} + \log \alpha. \end{aligned}$$

Hence

$$\log |g(x)| \geq (n+1)(1+m(\alpha))\frac{x}{x-1} + \log \alpha.$$

It means that

$$|g(x)| \geq \alpha \exp\{(n+1)(1+m(\alpha))\frac{x}{x-1}\}.$$

If $z \in \mathcal{B}^n$, we take $x = |z|$ and $\xi = \frac{z}{|z|}$.
From $g(x) = (1-x)^{n+1} \det f'(x\xi)$, we obtain

$$|\det f'(z)| \geq \frac{\alpha}{(1-|z|)^{n+1}} \exp\{(1+m(\alpha))\frac{-(n+1)|z|}{1-|z|}\}.$$

(2) If $0 < x \leq \frac{m(\alpha))}{2+m(\alpha)}$, by Lemma 1 (3), we have

$$\mathrm{Re}\, h(x) \leq G_0(-x) = (n+1)(1+m(\alpha))\frac{x}{x+1} + \log \alpha.$$

For any $z \in \mathcal{B}^n$ and $|z| \leq \frac{m(\alpha)}{2+m(\alpha)}$, we take $x = |z|$ and $\xi = \frac{z}{|z|}$.
Note that $g(x) = (1-x)^{n+1} \det f'(x\xi)$, we similarly have

$$|\det f'(z)| \leq \frac{\alpha}{(1+|z|)^{n+1}} \exp\{(1+m(\alpha))\frac{(n+1)|z|}{1+|z|}\}.$$

Finally, we shall testify the inequalities (1) and (2) are sharp.
In fact, we can take $f \in H_{loc}(\mathcal{B}^n)$ satisfying

$$f(z) = \begin{pmatrix} \int_0^{z_1} \frac{\alpha}{(1-t)^{n+1}} \exp\{(1+m(\alpha))\frac{-(n+1)t}{1-t}\}dt \\ \\ z_2 \\ \vdots \\ z_n \end{pmatrix}.$$

Then $\det f'(0) = \alpha$. Hence we will only show that $\|f\|_0 = 1$.
For any $z \in \mathcal{B}^n$, note that $e^{\frac{-(n+1)}{2}m(\alpha)}(1+m(\alpha))^{\frac{n+1}{2}} = \alpha$, we have

$$\begin{aligned} \{(1-|z|^2)^{\frac{n+1}{2}}|\det f'(z)|\}^{\frac{2}{n+1}} &= \frac{1-|z|^2}{|1-z_1|^2}\alpha^{\frac{2}{n+1}}|\exp\{-(1+m(\alpha))\frac{2z_1}{1-z_1})\}| \\ &\leq \frac{1-|z_1|^2}{|1-z_1|^2}(1+m(\alpha))\exp(-m(\alpha))|\exp\{-(1+m(\alpha))\frac{2z_1}{1-z_1})\}| \\ &\leq e(1+m(\alpha))\frac{1-|z_1|^2}{|1-z_1|^2}\exp\{-(1+m(\alpha))\frac{1-|z_1|^2}{|1-z_1|^2}\} \\ &\leq 1. \end{aligned}$$

The last inequality is obtained by the inequality

$$xe^{1-x} \leq 1$$

for any $x > 0$ with equality only for $x = 1$.

From the definition of $\|f\|_0 = \sup(1-|z|^2)^{\frac{n+1}{2n}} |\det f'(z)|^{\frac{1}{n}} : z \in \mathcal{B}^n\}$, we have $||f||_0 \leq 1$. On the other hand, we take $z = (z_1, 0, ..., 0) \in \mathcal{B}^n$ such that $\frac{|1-z_1|^2}{1-|z_1|^2} = 1 + m(\alpha)$. Then we obtain $(1-|z|^2)^{\frac{n+1}{2}} |\det f'(z)| = 1$. Thus $||f||_0 = 1$. It yields that the inequalities (1) and (2) are sharp.

Remark. As a special case of the unit disk $\mathbb{D}$, Theorem 1 reduces to Theorem 3 in [2].

By making use of the above distortion theorem, we can establish a lower estimate for the radius of the largest univalent ball in the image of f centered at $f(0)$.

Theorem 2 *If $f \in \beta(K) \bigcap H_{loc}(\mathcal{B}^n)$, $\|f\|_0 = 1$ and $\det f'(0) = \alpha \in (0, 1]$, then the largest univalent ball of f centered at $f(0)$ satisfies the following inequalities:*

$$r(0, f) \geq \alpha K^{1-n} \int_0^1 \frac{(1-t^2)^{n-1}}{(1-t)^{n+1}} \exp\{(1+m(\alpha))\frac{-(n+1)t}{1-t}\}dt \geq \frac{\alpha K^{1-n}}{(n+1)(1+m(\alpha))},$$

where $m(\alpha)$ is the real root of an equation $e^{\frac{-(n+1)}{2}t}(1+t)^{\frac{n+1}{2}} = \alpha$ in the interval $[0, +\infty)$.

Proof. By $\det f'(0) = \alpha \in (0, 1]$, there is a small ball centered at the origin such that the mapping f is biholomorphic on the small ball. If the ball in the range expands, then the preimage arrives at a point of the unit sphere $\partial \mathcal{B}^n$ or a point at which $\det f'(z) = 0$. Otherwise, we can enlarge the ball in range $f(\mathcal{B}^n)$ according to the estimate of $\det f'(z)$ in Theorem 1. In terms of the estimate of $|\det f'(z)|$ in Theorem 1, we can suppose $|\det f'(z)|$ is non-zero. Let $\Gamma \subset f(\mathcal{B}^n)$ be a straight line interval which starts at the point $f(0)$ and goes as far as it can with its preimage not running through the boundary of $\mathcal{B}^n$ or $\det f'(z) = 0$. Note that $r(0, f)$ is the largest nonnegative number r such that there exists a domain $V \subset \mathcal{B}^n$, and f maps V biholomorphically onto a ball centered at $f(0)$ with radius r. Then

$$r(0, f) \geq \left| \int_\Gamma dw \right| = \int_\Gamma |dw| = \int_\gamma \left| \frac{\partial f}{\partial z} \frac{dz}{|dz|} \right| \cdot |dz|,$$

where $\gamma = f^{-1}(\Gamma)$.
Lemma 2 implies

$$\int_\gamma \left| \frac{\partial f}{\partial z} \frac{dz}{|dz|} \right| \cdot |dz| \geq \int_\gamma \frac{|\det f'(z)|}{\|\frac{\partial f}{\partial z}\|^{n-1}} d|z|.$$

From Theorem 1 (1) and Lemma 3 we obtain the right-hand side of the preceding inequality satisfies

$$\begin{aligned} r(0,f) &\geq \alpha K^{1-n} \int_0^1 \frac{(1-t^2)^{n-1}}{(1-t)^{n+1}} \exp\{(1+m(\alpha))\frac{-(n+1)t}{1-t}\}dt \\ &\geq \alpha K^{1-n} \int_0^1 \frac{(1+t)^{n-1}}{(1-t)^2} \exp\{(1+m(\alpha))\frac{-(n+1)t}{1-t}\}dt \\ &\geq \alpha K^{1-n} \int_0^1 \frac{1}{(1-t)^2} \exp\{(1+m(\alpha))\frac{-(n+1)t}{1-t}\}dt \\ &= \frac{\alpha K^{1-n}}{(n+1)(1+m(\alpha))}. \end{aligned}$$

Therefore,

$$r(0,f) \geq \alpha K^{1-n} \int_0^1 \frac{(1-t^2)^{n-1}}{(1-t)^{n+1}} \exp\{(1+m(\alpha))\frac{-(n+1)t}{1-t}\}dt \geq \frac{\alpha K^{1-n}}{(n+1)(1+m(\alpha))}.$$

The proof of Theorem 2 is complete.

Remark. (1) When $n=1$, $\mathcal{B}^n$ is the unit disk $\mathbb{D}$. Theorem 2 implies

$$r(0,f) \geq \frac{\alpha}{2(1+m(\alpha))} = \frac{1}{2}e^{-m(\alpha)},$$

which coincides with Corollary 3 in [2].

(2) When $\alpha=1$, then $m(\alpha)=0$. Theorem 2 reduces to the lower bound for the locally biholomorphic Bloch constant obtained by Liu in [4, Theorem 8]:

$$r(0,f) \geq K^{1-n} \int_0^1 \frac{(1-t^2)^{n-1}}{(1-t)^{n+1}} \exp\{\frac{-(n+1)t}{1-t}\}dt \geq \frac{K^{1-n}}{n+1}.$$

(3) Finally, we would like to point out that all of the analogous results for locally biholomorphic Bloch mappings defined in the unit polydisc can also be obtained.

Acknowledgements The author cordially thanks to the referees for their thorough reviewing with useful suggestions and comments made to the paper.

REFERENCES

[1] Bonk, M., On Bloch's constant, *Proc. Amer. Math. Soc.*, 1990, 110(4): 889-894.

[2] Bonk, M., Minda, D., Yanagihara, H., Distortion theorems for locally univalent Bloch functions, *J. Anal. Math.* , 1996, 69: 73-95.

[3] Bonk, M., Minda, D., Yanagihara, H., Distortion theorem for Bloch functions, *Pacific J. Math.*, 1997, 179(2): 241-262.

[4] Liu, X. Y., Bloch functions of several complex variables, *Pacific J. Math.*, 1992, 152(2): 347-363.

[5] Liu, X. Y., Minda, D., Distortion theorem for Bloch functions, *Trans. Amer. Math. Soc.*, 1992, 333(1): 325-338.

[6] FitzGerald, C. H., Gong, S., the locally biholomorphic Bloch constant and Marden constant of several complex variables, *Computational methods and function theory*, 1994(Ali R M. St Ruscheweyh and Saff E B Eds)Series in approximations and decompositions, Vol 5, 1995, World Scientific Publishing Co: 147-158.

[7] Gong, S., Yan, Z. M., Bloch constant of holomorphic mappings on bounded symmetric domains, *Science in China (Series A)*, 1993, 36(3): 285-299.

[8] Gong, S., The Bloch constant of locally biholomorphic mappings on bounded symmetric domains, *Chin. Ann. Math.*, 1996, 17B(3): 271-278.

[9] Gong S., Yu Q. H., Zheng X. A., Bloch constant and Schwarzian Derivative, shanghai: Shanghai Scientific and Technical Publishers, 1998.

[10] Wang, J. F., Liu, T. S., Distortion theorem for Bloch mappings on the unit ball $\mathcal{B}^n$, *Acta Math. Sin. (Engl. Ser.)*, 2009, 25(10): 1583-1590.

[11] Chen S., Ponnusamy S., Wang X., Landau-Bloch constants for functions in α-Bloch spaces and Hardy spaces, *Complex Anal. Oper. Theory,* 2012, 6(5): 1025-1036.

[12] Conway, J. B., Functions of one complex variable II, GTM159, Spring-Verlag, 1997.