DIFFERENTIAL SUBORDINATIONS INVOLVING GENERALIZED BESSEL FUNCTIONS

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Abstract. In this paper our aim is to present some subordination and superordination results, by using an operator, which involves the normalized form of the generalized Bessel functions of first kind. These results are obtained by investigating some appropriate classes of admissible functions. We obtain also some sandwich-type results and we point out various known or new special cases of our main results.

1. Introduction, definitions and preliminaries

Let $\mathcal{H}(\mathbb{D})$ be the class of analytic functions in $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{H}[a, n]$ be the subclass of $\mathcal{H}(\mathbb{D})$ consisting of functions of the form $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots$, with $\mathcal{H}_0 = \mathcal{H}[0, 1]$ and $\mathcal{H}_1 = \mathcal{H}[1, 1]$. We denote by $\mathcal{A}$ the class of all functions of the form

$$f(z) = z + \sum_{n\geq 1} a_{n+1} z^{n+1}, \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{D}$. Let f and F be members of $\mathcal{H}(\mathbb{D})$. The function f is said to be *subordinate* to F, or F is said to be *superordinate* to f, written symbolically as

$$f \prec F \quad \text{or} \quad f(z) \prec F(z) \quad (z \in \mathbb{D}),$$

if there exists an analytic function $w : \mathbb{D} \to \mathbb{C}$ with $w(0) = 0$ and $|w(z)| < 1$ for $z \in \mathbb{D}$ such that $f(z) = F(w(z)),\ z \in \mathbb{D}$. In particular, if the function F is univalent in $\mathbb{D}$, then we have the following equivalence

$$f(z) \prec F(z) \ \ (z \in \mathbb{D}) \quad \Longleftrightarrow \quad f(0) = F(0) \quad \text{and} \quad f(\mathbb{D}) \subset F(\mathbb{D}).$$

Let us consider the following second-order linear homogenous differential equation (for more details see [4])

$$z^2\omega''(z) + bz\omega'(z) + \left[cz^2 - p^2 + (1-b)p\right]\omega(z) = 0, \quad (b, c, p \in \mathbb{C}). \tag{1.2}$$

The function $\omega_{p,b,c}$, which is called the generalized Bessel function of the first kind of order p, is defined as a particular solution of (1.2). The function $\omega_{p,b,c}$ has the familiar representation as follows

$$\omega_{p,b,c}(z) = \sum_{n\geq 0} \frac{(-c)^n}{n!\Gamma\left(p + n + \frac{b+1}{2}\right)} \left(\frac{z}{2}\right)^{2n+p}, \quad z \in \mathbb{C}, \tag{1.3}$$

where Γ stands for the Euler gamma function and $\kappa = p + \frac{b+1}{2} \notin \mathbb{Z}_0^- = \{0, -1, -2, \ldots\}$. The series (1.3) permits the study of Bessel, modified Bessel, spherical Bessel, modified spherical Bessel and ultraspherical Bessel functions all together. It is worth mentioning that, in particular,

• for $b = c = 1$ in (1.3), we obtain the familiar Bessel function of the first kind of order p defined by (see [21]; see also [4])

$$J_p(z) = \sum_{n\geq 0} \frac{(-1)^n}{n!\Gamma(p + n + 1)} \left(\frac{z}{2}\right)^{2n+p}, \quad z \in \mathbb{C}, \tag{1.4}$$

• for $b = 1$ and $c = -1$ in (1.3), we obtain the modified Bessel function of the first kind of order p defined by (see [21]; see also [4])

$$I_p(z) = \sum_{n\geq 0} \frac{1}{n!\Gamma(p + n + 1)} \left(\frac{z}{2}\right)^{2n+p}, \quad z \in \mathbb{C}, \tag{1.5}$$

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• for $b=2$ and $c=1$ in (1.3), the function $\omega_{p,b,c}$ reduces to $\sqrt{2}j_p/\sqrt{\pi}$ where j_p is the spherical Bessel function of the first kind of order p, defined by (see [4])

$$j_p(z)=\sqrt{\frac{\pi}{2}}\sum_{n\geq 0}\frac{(-1)^n}{n!\Gamma\left(p+n+\frac{3}{2}\right)}\left(\frac{z}{2}\right)^{2n+p},\quad z\in\mathbb{C}. \tag{1.6}$$

Now, we consider the function $\varphi_{p,b,c}:\mathbb{D}\to\mathbb{C}$, defined in terms of the generalized Bessel function $\omega_{p,b,c}$, by the transformation

$$\varphi_{p,b,c}(z)=2^p\Gamma\left(p+\frac{b+1}{2}\right)z^{1-\frac{p}{2}}\omega_{p,b,c}(\sqrt{z}). \tag{1.7}$$

By using the well-known Pochhammer symbol (or the *shifted* factorial) $(\lambda)_\mu$ defined, for $\lambda,\mu\in\mathbb{C}$ and in terms of the Euler $\Gamma-$function, by

$$(\lambda)_\mu=\frac{\Gamma(\lambda+\mu)}{\Gamma(\lambda)}=\begin{cases}1, & \mu=0,\ \lambda\in\mathbb{C}\setminus\{0\}\\ \lambda(\lambda+1)\dots(\lambda+n-1), & \mu=n\in\mathbb{N},\ \lambda\in\mathbb{C}\end{cases},$$

it being understood conventionally that $(0)_0=1$, we obtain the following series representation for the function $\varphi_{p,b,c}$ given by (1.7)

$$\varphi_{p,b,c}(z)=z+\sum_{n\geq 1}\frac{(-c)^n}{4^n(\kappa)_n}\frac{z^{n+1}}{n!}, \tag{1.8}$$

where $\kappa=p+\frac{b+1}{2}\notin\mathbb{Z}_0^-$ and $\mathbb{N}=\{1,2,3,\dots\}$. For convenience, we write $\varphi_{\kappa,c}(z)=\varphi_{p,b,c}(z)$.

For $f\in\mathcal{A}$ given by (1.1) and g given by $g(z)=z+\sum_{n\geq 1}b_{n+1}z^{n+1}$, the Hadamard product (or convolution) of f and g is defined by

$$(f*g)(z)=z+\sum_{n\geq 1}a_{n+1}b_{n+1}z^{n+1}=(g*f)(z),\quad z\in\mathbb{D}.$$

Note that $f*g\in\mathcal{A}$. Now, we consider the B_κ^c-operator, which is defined as follows

$$B_\kappa^c f(z)=\varphi_{\kappa,c}(z)*f(z)=z+\sum_{n\geq 1}\frac{(-c)^n a_{n+1}}{4^n(\kappa)_n}\frac{z^{n+1}}{n!}. \tag{1.9}$$

We note that by using the definition (1.9) we obtain that

$$z\left[B_{\kappa+1}^c f(z)\right]'=\kappa B_\kappa^c f(z)-(\kappa-1)B_{\kappa+1}^c f(z), \tag{1.10}$$

where $\kappa=p+\frac{b+1}{2}\notin\mathbb{Z}_0^-$. It is worth to mention that in fact $B_\kappa^c f$ given by (1.9) is an elementary transform of the generalized hypergeometric function, that is, we have

$$B_\kappa^c f(z)=z_0F_1\left(\kappa;\frac{-c}{4}z\right)*f(z).$$

In particular, for the B_κ^c-operator we obtain the following operators:

• Choosing $b=c=1$ in (1.9) or (1.10) we obtain the operator $\mathcal{J}_p:\mathcal{A}\to\mathcal{A}$ related with Bessel function, defined by

$$\mathcal{J}_p f(z)=\varphi_{p,1,1}(z)*f(z)=\left[2^p\Gamma(p+1)z^{1-\frac{p}{2}}J_p(\sqrt{z})\right]*f(z)=z+\sum_{n\geq 1}\frac{(-1)^n a_{n+1}}{4^n(p+1)_n}\frac{z^{n+1}}{n!}, \tag{1.11}$$

which satisfies the recursive relation

$$z\left[\mathcal{J}_{p+1}f(z)\right]'=(p+1)\mathcal{J}_p f(z)-p\mathcal{J}_{p+1}f(z).$$

• Choosing $b=1$ and $c=-1$ in (1.9) or (1.10) we obtain the operator $\mathcal{I}_p:\mathcal{A}\to\mathcal{A}$ related with modified Bessel function, defined by

$$\mathcal{I}_p f(z)=\varphi_{p,1,-1}(z)*f(z)=\left[2^p\Gamma(p+1)z^{1-\frac{p}{2}}I_p(\sqrt{z})\right]*f(z)=z+\sum_{n\geq 1}\frac{a_{n+1}}{4^n(p+1)_n}\frac{z^{n+1}}{n!}, \tag{1.12}$$

which satisfies the recursive relation

$$z\left[\mathcal{I}_{p+1}f(z)\right]'=(p+1)\mathcal{I}_p f(z)-p\mathcal{I}_{p+1}f(z).$$

• Choosing $b = 2$ and $c = 1$ in (1.9) or (1.10) we obtain the operator $\mathcal{S}_p : \mathcal{A} \to \mathcal{A}$ related with spherical Bessel function, defined by

$$\mathcal{S}_p f(z) = \left[\pi^{-\frac{1}{2}} 2^{p+\frac{1}{2}} \Gamma\left(p + \frac{3}{2}\right) z^{1-\frac{p}{2}} j_p(\sqrt{z})\right] * f(z) = z + \sum_{n\geq 1} \frac{(-1)^n a_{n+1}}{4^n \left(p + \frac{3}{2}\right)_n} \frac{z^{n+1}}{n!}, \tag{1.13}$$

which satisfies the recursive relation

$$z\left[\mathcal{S}_{p+1} f(z)\right]' = \left(p + \frac{3}{2}\right) \mathcal{S}_p f(z) - \left(p + \frac{1}{2}\right) \mathcal{S}_{p+1} f(z).$$

For further result on the transformation (1.8) of the generalized Bessel function we refer to the recent papers (see [2, 3, 4, 5, 13, 18, 19, 20]), where among other things some interesting functional inequalities, integral representations, application of admissible functions, extensions of some known trigonometric inequalities, starlikeness and convexity and univalence were established. Most of these results were motivated by the research on geometric properties of Gaussian and Kummer hypergeometric functions. For more details we refer to the papers [12, 14, 15, 16, 17] and to the references therein. We also mention that recently in [6, 7] some results like the univalence and convexity of some integral operators, defined by the normalized form of the generalized Bessel functions of the first kind given by (1.8), were deduced. For other recent results on subordinations, and other operators involving starlike and close-to-convex functions we refer to the papers [1, 8, 9] and to the references therein.

In the present paper, by making use of the differential subordination and differential superordination results of Miller and Mocanu [10, 11], we determine certain classes of admissible functions and obtain some subordination and superordination implications of analytic functions associated with the B_κ^c−operator defined by (1.9), together with some sandwich-type theorems.

To prove our main results, we need the following definitions and theorems.

Definition 1.1. [10, p. 21] *Let us denote by $\mathcal{Q}$ the class of functions q, which are analytic and injective on $\overline{\mathbb{D}} \setminus E(q)$, where*

$$E(q) = \left\{\zeta : \zeta \in \partial\mathbb{D} \text{ and } \lim_{z\to\zeta} q(z) = \infty\right\},$$

and are such that $q'(\zeta) \neq 0$ $(\zeta \in \partial\mathbb{D} \setminus E(q))$. Further let the subclass of $\mathcal{Q}$ for which $q(0) = a$ be denoted by $\mathcal{Q}(a)$, $\mathcal{Q}(0) \equiv \mathcal{Q}_0$ and $\mathcal{Q}(1) \equiv \mathcal{Q}_1$.

Definition 1.2. [10, p. 27] *Let Ω be a set in $\mathbb{C}$, $q \in \mathcal{Q}$ and $n \in \mathbb{N}$. The class of admissible functions $\Psi_n[\Omega, q]$ consists of those functions $\psi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ that satisfy the following admissibility condition*

$$\psi(r, s, t; z) \notin \Omega$$

whenever

$$r = q(\zeta), \quad s = k\zeta q'(\zeta) \quad \text{and} \quad \operatorname{Re}\left(\frac{t}{s} + 1\right) \geq k \operatorname{Re}\left(\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right)$$
$$(z \in \mathbb{D};\ \zeta \in \partial\mathbb{D} \setminus E(q);\ k \geq n).$$

We write $\Psi_1[\Omega, q]$ simply as $\Psi[\Omega, q]$.

In particular when

$$q(z) = M\frac{Mz + a}{M + \overline{a}z},$$

with $M > 0$ and $|a| < M$, then $q(\mathbb{D}) = \mathbb{D}_M = \{w : |w| < M\}$, $q(0) = a$, $E(q) = \varnothing$ and $q \in \mathcal{Q}$. In this case, we set $\Psi_n[\Omega, M, a] = \Psi_n[\Omega, q]$, and in the special case when the set $\Omega = \mathbb{D}_M$, the class is simply denoted by $\Psi_n[M, a]$.

Definition 1.3. [11, p. 817] *Let Ω be a set in $\mathbb{C}$ and $q \in \mathcal{H}[a, n]$ with $q'(z) \neq 0$. The class of admissible functions $\Psi'_n[\Omega, q]$ consist of those functions*

$$\psi : \mathbb{C}^3 \times \overline{\mathbb{D}} \to \mathbb{C}$$

that satisfy the following admissibility condition

$$\psi(r, s, t; \varsigma) \in \Omega$$

whenever

$$r = q(z), \quad s = \frac{zq'(z)}{m} \quad \text{and} \quad \operatorname{Re}\left(\frac{t}{s} + 1\right) \leq \frac{1}{m} \operatorname{Re}\left(\frac{zq''(z)}{q'(z)} + 1\right)$$
$$(z \in \mathbb{D};\ \varsigma \in \partial\mathbb{D};\ m \geq n \geq 1).$$

We write $\Psi'_1[\Omega,q]$ *simply as* $\Psi'[\Omega,q]$.

For the above two classes of admissible functions, Miller and Mocanu [10, 11] proved the following results.

Lemma 1.1. [10, p. 28] *Let* $\psi\in\Psi_n[\Omega,q]$ *with* $q(0)=a$. *If the analytic function* $p\in\mathcal{H}[a,n]$ *satisfies the following inclusion relationship*

$$\psi\left(p(z),zp'(z),z^2p''(z);z\right)\in\Omega,$$

for all $z\in\mathbb{D}$, *then* $p\prec q$.

Lemma 1.2. [11, p. 818] *Let* $\psi\in\Psi'_n[\Omega,q]$ *with* $q(0)=a$. *If* $p\in\mathcal{Q}(a)$ *and* $\psi\left(p(z),zp'(z),z^2p''(z);z\right)$ *is univalent in* $\mathbb{D}$, *then*

$$\Omega\subset\left\{\psi\left(p(z),zp'(z),z^2p''(z);z\right):z\in\mathbb{D}\right\}$$

implies that $p\prec q$.

2. Subordination results involving the B_κ^c–operator

We begin this section by proving a differential subordination theorem involving the B_κ^c–operator defined by (1.9). In the sequel the parameter c is an arbitrary complex number, while for the parameter κ we will have some assumptions in some cases. To prove our main results, we need first the following class of admissible functions.

Definition 2.1. *Let* Ω *be a set in* $\mathbb{C}$ *and* $q\in\mathcal{Q}_0\cap\mathcal{H}_0$. *The class of admissible functions* $\Phi_H[\Omega,q]$ *consists of those functions* $\phi:\mathbb{C}^3\times\mathbb{D}\to\mathbb{C}$ *that satisfy the admissibility condition*

$$\phi(u,v,w;z)\notin\Omega$$

whenever

$$u=q(\zeta),\ \ v=\frac{k\zeta q'(\zeta)+(\kappa-1)q(\zeta)}{\kappa}\qquad\left(\kappa\in\mathbb{C}\setminus\mathbb{Z}_0^-,\ \kappa\neq1\right)$$

and

$$\operatorname{Re}\left(\frac{\kappa(\kappa-1)w-(\kappa-2)(\kappa-1)u}{v\kappa-(\kappa-1)u}-(2\kappa-3)\right)\geq k\operatorname{Re}\left(\frac{\zeta q''(\zeta)}{q'(\zeta)}+1\right)$$
$$(z\in\mathbb{D};\ \zeta\in\partial\mathbb{D}\setminus E(q);\ k\geq1).$$

Theorem 2.1. *Let* $\phi\in\Phi_H[\Omega,q]$. *If* $f\in\mathcal{A}$ *satisfies the following inclusion relationship*

$$\left\{\phi\left(B_{\kappa+1}^cf(z),B_\kappa^cf(z),B_{\kappa-1}^cf(z);z\right):z\in\mathbb{D}\right\}\subset\Omega,\tag{2.1}$$

then

$$B_{\kappa+1}^cf(z)\prec q(z)\quad(z\in\mathbb{D}).$$

Proof. We define the analytic function p in $\mathbb{D}$ by

$$p(z)=B_{\kappa+1}^cf(z).\tag{2.2}$$

Then, differentiating (2.2) with respect to z and using the recursive relation (1.10), we have

$$B_\kappa^cf(z)=\frac{zp'(z)+(\kappa-1)p(z)}{\kappa}.\tag{2.3}$$

Further computations show that

$$B_{\kappa-1}^cf(z)=\frac{z^2p''(z)+2(\kappa-1)zp'(z)+(\kappa-1)(\kappa-2)p(z)}{\kappa(\kappa-1)}.\tag{2.4}$$

We now define the transformations from $\mathbb{C}^3$ to $\mathbb{C}$ by

$$u=r,\quad v=\frac{s+(\kappa-1)r}{\kappa}\quad\text{and}\quad w=\frac{t+2(\kappa-1)s+(\kappa-1)(\kappa-2)r}{\kappa(\kappa-1)}.$$

Let

$$\psi(r,s,t;z)=\phi(u,v,w;z)=\phi\left(r,\frac{s+(\kappa-1)r}{\kappa},\frac{t+2(\kappa-1)s+(\kappa-1)(\kappa-2)r}{\kappa(\kappa-1)};z\right).\tag{2.5}$$

Using equations (2.2)-(2.4), and from (2.5), we get

$$\psi(p(z),zp'(z),z^2p''(z);z)=\phi\left(B_{\kappa+1}^cf(z),B_\kappa^cf(z),B_{\kappa-1}^cf(z);z\right).\tag{2.6}$$

Hence (2.1) assumes the following form

$$\psi(p(z), zp'(z), z^2p''(z); z) \in \Omega.$$

The proof is completed if it can be shown that the admissibility condition for $\phi \in \Phi_H[\Omega, q]$ is equivalent to the admissibility condition for as given in Definition 1.2. Note that

$$\frac{t}{s} + 1 = \frac{\kappa(\kappa-1)w - (\kappa-2)(\kappa-1)u}{v\kappa - (\kappa-1)u} - (2\kappa - 3),$$

and hence $\psi \in \Psi[\Omega, q]$. By Lemma 1.1, we have

$$p(z) \prec q(z) \quad \text{or} \quad B^c_{\kappa+1} f(z) \prec q(z)$$

which completes the proof of Theorem 2.1. □

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(\mathbb{D})$ for some conformal mapping h of $\mathbb{D}$ onto Ω. In this case the class $\Phi_H[h(\mathbb{D}), q]$ is written as $\Phi_H[h, q]$.

The following result is an immediate consequence of Theorem 2.1.

Corollary 2.2. *Let $\phi \in \Phi_H[h, q]$. If $f \in \mathcal{A}$ satisfies*

$$\phi\left(B^c_{\kappa+1} f(z), B^c_{\kappa} f(z), B^c_{\kappa-1} f(z); z\right) \prec h(z), \tag{2.7}$$

then

$$B^c_{\kappa+1} f(z) \prec q(z) \quad (z \in \mathbb{D}).$$

Our next result is an extension of Theorem 2.1 to the case when the behavior of q on $\partial\mathbb{D}$ is not known.

Corollary 2.3. *Let $\Omega \subset \mathbb{C}$ and let q be univalent in $\mathbb{D}$ with $q(0) = 0$. Let $\phi \in \Phi_H[\Omega, q_\rho]$ for some $\rho \in (0, 1)$, where $q_\rho(z) = q(\rho z)$. If $f \in \mathcal{A}$ satisfies the following inclusion relationship*

$$\phi\left(B^c_{\kappa+1} f(z), B^c_{\kappa} f(z), B^c_{\kappa-1} f(z); z\right) \in \Omega,$$

then

$$B^c_{\kappa+1} f(z) \prec q(z) \quad (z \in \mathbb{D}).$$

Proof. We note from Theorem 2.1 that

$$B^c_{\kappa+1} f(z) \prec q_\rho(z) \quad (z \in \mathbb{D}).$$

The result asserted by Corollary 2.3 is now deduced from the following subordination relationship

$$q_\rho(z) \prec q(z) \quad (z \in \mathbb{D}).$$

□

Theorem 2.4. *Let h and q be univalent in $\mathbb{D}$ with $q(0) = 0$ and set $q_\rho(z) = q(\rho z)$ and $h_\rho(z) = h(\rho z)$. Let $\phi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ satisfy one of the following conditions*

(1) *$\phi \in \Phi_H[\Omega, q_\rho]$ for some $\rho \in (0, 1)$, or*
(2) *there exists $\rho_0 \in (0, 1)$ such that $\phi \in \Phi_H[h_\rho, q_\rho]$ for all $\rho \in (\rho_0, 1)$.*

If $f \in \mathcal{A}$ satisfies (2.7), then

$$B^c_{\kappa+1} f(z) \prec q(z) \quad (z \in \mathbb{D}).$$

Proof. The proof is similar to the proof of [10, Theorem 2.3d] and is therefore omitted. □

The next theorem yields the best dominant of the differential subordination (2.7).

Theorem 2.5. *Let h be univalent in $\mathbb{D}$ and $\phi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$. Suppose that the differential equation*

$$\phi\left(q(z), \frac{zq'(z) + (\kappa-1)q(z)}{\kappa}, \frac{z^2q''(z) + 2(\kappa-1)zq'(z) + (\kappa-1)(\kappa-2)q(z)}{\kappa(\kappa-1)}; z\right) = h(z) \tag{2.8}$$

has a solution q with $q(0) = 0$ and satisfies one of the following conditions

(1) *$q \in \mathcal{Q}_0$ and $\phi \in \Phi_H[h, q]$*
(2) *q is univalent in $\mathbb{D}$ and $\phi \in \Phi_H[h, q_\rho]$ for some $\rho \in (0, 1)$*
(3) *q is univalent in $\mathbb{D}$ and there exists $\rho_0 \in (0, 1)$ such that $\phi \in \Phi_H[h_\rho, q_\rho]$ for all $\rho \in (\rho_0, 1)$.*

If $f \in \mathcal{A}$ satisfies (2.7), then

$$B^c_{\kappa+1} f(z) \prec q(z) \quad (z \in \mathbb{D})$$

and q is the best dominant.

Proof. Following the same arguments as in [10, Theorem 2.3e], we deduce that q is a dominant from Corollary 2.2 and Theorem 2.4. Since q satisfies (2.8), it is also a solution of (2.7) and therefore q will be dominated by all dominants. Hence q is the best dominant. □

In the particular case when $q(z) = Mz$; $M > 0$, and in view of Definition 2.1, the class of admissible functions $\Phi_H[\Omega, q]$, denoted by $\Phi_H[\Omega, M]$, is described below.

Definition 2.2. *Let Ω be a set in $\mathbb{C}$ and $M > 0$. The class of admissible functions $\Phi_H[\Omega, M]$ consists of those functions $\phi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ such that*

$$\phi\left(Me^{i\theta}, \frac{k+(\kappa-1)}{\kappa}Me^{i\theta}, \frac{L+(\kappa-1)(2k+\kappa-2)Me^{i\theta}}{\kappa(\kappa-1)}; z\right) \notin \Omega, \tag{2.9}$$

whenever $z \in \mathbb{D}$, $\kappa \in \mathbb{C} \setminus \mathbb{Z}_0^-$ $(\kappa \neq 1)$ and $\operatorname{Re}(Le^{-i\theta}) \geq (k-1)kM$ for all $\theta \in \mathbb{R}$, $k \geq 1$.

Corollary 2.6. *Let $\phi \in \Phi_H[\Omega, M]$. If $f \in \mathcal{A}$ satisfies the following inclusion relationship*

$$\phi\left(B_{\kappa+1}^c f(z), B_\kappa^c f(z), B_{\kappa-1}^c f(z); z\right) \in \Omega,$$

then

$$B_{\kappa+1}^c f(z) \prec Mz \quad (z \in \mathbb{D}).$$

In the special case when $\Omega = \{w : |w| < M\} = q(\mathbb{D})$, the class $\Phi_H[\Omega, M]$ is simply denoted by $\Phi_H[M]$. Corollary 2.6 can now be written in the following form.

Corollary 2.7. *Let $\phi \in \Phi_H[M]$. If $f \in \mathcal{A}$ satisfies the following inequality*

$$\left|\phi\left(B_{\kappa+1}^c f(z), B_\kappa^c f(z), B_{\kappa-1}^c f(z); z\right)\right| < M,$$

then

$$\left|B_{\kappa+1}^c f(z)\right| < M.$$

Corollary 2.8. *If $M > 0$, $\operatorname{Re}(\kappa) > 0$ and $f \in \mathcal{A}$ satisfies the following inequality*

$$|B_\kappa^c f(z)| < M,$$

then

$$\left|B_{\kappa+1}^c f(z)\right| < M.$$

Proof. This follows from Corollary 2.7 by taking $\phi(u, v, w; z) = v = \frac{k+(\kappa-1)}{\kappa}Me^{i\theta}$. □

Taking into account the above results, we have the following particular cases. For $f(z) = \frac{z}{1-z}$ in Corollary 2.8 we have

$$|\varphi_{\kappa,c}(z)| < M \Rightarrow |\varphi_{\kappa+1,c}(z)| < M. \tag{2.10}$$

This result is the generalization of a result given by Prajapat [18].

Also observe that $\varphi_{\frac{3}{2},1,1}(z) = \frac{3\sin\sqrt{z}}{\sqrt{z}} - 3\cos\sqrt{z}$, $\varphi_{\frac{1}{2},1,1}(z) = \sqrt{z}\sin\sqrt{z}$ and $\varphi_{-\frac{1}{2},1,1}(z) = z\cos\sqrt{z}$, where $\varphi_{p,b,c}(z)$ is given by (1.8). Thus we can obtain some trigonometric inequalities for special cases of parameters p, b and c. For example from (2.10) for all $z \in \mathbb{D}$ and $M > 0$ we have

$$\left|z\cos\sqrt{z}\right| < M \Rightarrow \left|\sqrt{z}\sin\sqrt{z}\right| < M \Rightarrow \left|\frac{\sin\sqrt{z}}{\sqrt{z}} - \cos\sqrt{z}\right| < \frac{M}{3}.$$

Corollary 2.9. *Let $\kappa \in \mathbb{C} \setminus \mathbb{Z}_0^-$ and $M > 0$. If $f \in \mathcal{A}$ satisfies the following inequality*

$$\left|B_\kappa^c f(z) + \left(\frac{1}{\kappa} - 1\right) B_{\kappa+1}^c f(z)\right| < \frac{M}{|\kappa|},$$

then

$$\left|B_{\kappa+1}^c f(z)\right| < M.$$

Proof. Let $\phi(u, v, w; z) = v + \left(\frac{1}{\kappa} - 1\right)u$ and $\Omega = h(\mathbb{D})$ where $h(z) = \frac{Mz}{\kappa}$, $M > 0$. In order to use Corollary 2.6, we need to show that $\phi \in \Phi_H[\Omega, M]$, that is, the admissibility condition (2.9) is satisfied. This follows since

$$\left|\phi\left(Me^{i\theta}, \frac{k+(\kappa-1)}{\kappa}Me^{i\theta}, \frac{L+(\kappa-1)(2k+\kappa-2)Me^{i\theta}}{\kappa(\kappa-1)}; z\right)\right| = \left|\frac{kMe^{i\theta}}{\kappa}\right| \geq \frac{M}{|\kappa|}$$

whenever $z \in \mathbb{D}$, $\theta \in \mathbb{R}$, $\kappa \in \mathbb{C} \setminus \mathbb{Z}_0^-$ $(\kappa \neq 1)$ and $k \geq 1$. The result now follows from Corollary 2.6. □

Observe that $\varphi_{\frac{3}{2},1,-1}(z)=3\cosh\sqrt{z}-\frac{3\sinh\sqrt{z}}{\sqrt{z}}$, $\varphi_{\frac{1}{2},1,-1}(z)=\sqrt{z}\sinh\sqrt{z}$ and $\varphi_{-\frac{1}{2},1,-1}(z)=z\cosh\sqrt{z}$ where $\varphi_{p,b,c}(z)$ is given by (1.8). Moreover, if we take $f(z)=\frac{z}{1-z}$, $b=1$, $c=-1$; $p=\frac{1}{2}$ and $p=-\frac{1}{2}$ in Corollary 2.9, respectively, we have

$$\left|\frac{(z+1)\sinh\sqrt{z}}{\sqrt{z}}-\cosh\sqrt{z}\right|<\frac{2M}{3}\Rightarrow\left|\frac{\sinh\sqrt{z}}{\sqrt{z}}-\cosh\sqrt{z}\right|<\frac{M}{3}$$

and

$$\left|z\cosh\sqrt{z}+\sqrt{z}\sinh\sqrt{z}\right|<2M\Rightarrow\left|\sqrt{z}\sinh\sqrt{z}\right|<M.$$

Theorem 2.5 shows that the result is sharp. The differential equation

$$zq'(z)=Mz$$

has a univalent solution $q(z)=Mz$. It follows from Theorem 2.5 that $q(z)=Mz$ is the best dominant.

Definition 2.3. *Let Ω be a set in $\mathbb{C}$ and $q\in\mathcal{Q}_1\cap\mathcal{H}_1$. The class of admissible functions $\Phi_{H,1}[\Omega,q]$ consists of those functions $\phi:\mathbb{C}^3\times\mathbb{D}\to\mathbb{C}$ that satisfy the admissibility condition*

$$\phi(u,v,w;z)\notin\Omega$$

whenever

$$u=q(\zeta),\ \ v=\frac{1}{\kappa-1}\left(\frac{k\zeta q'(\zeta)}{q(\zeta)}+\kappa q(\zeta)-1\right)\qquad(\kappa\in\mathbb{C}\setminus\mathbb{Z}_0^-,\ \kappa\neq1,2,\ q(\zeta)\neq0),$$

and

$$\operatorname{Re}\left(\frac{(\kappa-1)v\left[(\kappa-1)(w-v)+1-w\right]}{(\kappa-1)v+1-\kappa u}-\left[2\kappa u-1-(\kappa-1)v\right]\right)\geq k\operatorname{Re}\left(\frac{\zeta q''(\zeta)}{q'(\zeta)}+1\right)$$
$$(z\in\mathbb{D};\ \zeta\in\partial\mathbb{D}\setminus E(q);\ k\geq1)\,.$$

Theorem 2.10. *Let $\phi\in\Phi_{H,1}[\Omega,q]$. If $f\in\mathcal{A}$ satisfies the following inclusion relationship*

$$\left\{\phi\left(\frac{B_\kappa^c f(z)}{B_{\kappa+1}^c f(z)},\frac{B_{\kappa-1}^c f(z)}{B_\kappa^c f(z)},\frac{B_{\kappa-2}^c f(z)}{B_{\kappa-1}^c f(z)};z\right):z\in\mathbb{D}\right\}\subset\Omega,\tag{2.11}$$

then

$$\frac{B_\kappa^c f(z)}{B_{\kappa+1}^c f(z)}\prec q(z)\quad(z\in\mathbb{D}).$$

Proof. Let us consider the analytic function $p:\mathbb{D}\to\mathbb{C}$, defined by

$$p(z)=\frac{B_\kappa^c f(z)}{B_{\kappa+1}^c f(z)}.\tag{2.12}$$

Differentiating both sides of (2.12) with respect to z and using (1.10), we have

$$\frac{B_{\kappa-1}^c f(z)}{B_\kappa^c f(z)}=\frac{1}{\kappa-1}\left(\frac{zp'(z)}{p(z)}+\kappa p(z)-1\right).\tag{2.13}$$

Further computations show that

$$\frac{B_{\kappa-2}^c f(z)}{B_{\kappa-1}^c f(z)}=\frac{1}{\kappa-2}\left(\frac{zp'(z)}{p(z)}+\kappa p(z)-2+\frac{\frac{zp'(z)}{p(z)}+\frac{z^2p''(z)}{p(z)}-\left(\frac{zp'(z)}{p(z)}\right)^2+\kappa zp'(z)}{\frac{zp'(z)}{p(z)}+\kappa p(z)-1}\right).\tag{2.14}$$

Now we define the transformation $\psi:\mathbb{C}^3\times\mathbb{D}\to\mathbb{C}$ by $\psi(r,s,t;z)=\phi(u,v,w;z)$ where

$$u=r,\quad v=\frac{1}{\kappa-1}\left(\frac{s}{r}+\kappa r-1\right)\quad\text{and}\quad w=\frac{1}{\kappa-2}\left(\frac{s}{r}+\kappa r-2+\frac{\frac{s}{r}+\frac{t}{r}-\left(\frac{s}{r}\right)^2+\kappa s}{\frac{s}{r}+\kappa r-1}\right).\tag{2.15}$$

Using equations (2.12)-(2.14), we get

$$\psi(p(z),zp'(z),z^2p''(z);z)=\phi\left(\frac{B_\kappa^c f(z)}{B_{\kappa+1}^c f(z)},\frac{B_{\kappa-1}^c f(z)}{B_\kappa^c f(z)},\frac{B_{\kappa-2}^c f(z)}{B_{\kappa-1}^c f(z)};z\right).\tag{2.16}$$

Hence (2.11) implies

$$\psi(p(z),zp'(z),z^2p''(z);z)\in\Omega.$$

The proof is completed if it can be shown that the admissibility condition for $\phi \in \Phi_{H,1}[\Omega, q]$ is equivalent to the admissibility condition for ψ as given in Definition 1.2. Note that

$$\frac{t}{s}+1=\frac{(\kappa-1)v\left[(\kappa-1)(w-v)+1-w\right]}{(\kappa-1)v+1-\kappa u}-\left[2\kappa u-1-(\kappa-1)v\right],$$

and hence $\psi \in \Psi[\Omega, q]$. By Lemma 1.1, we have

$$p(z) \prec q(z) \quad \text{or} \quad \frac{B_{\kappa}^{c}f(z)}{B_{\kappa+1}^{c}f(z)} \prec q(z)$$

which completes the proof of Theorem 2.10. □

In the case when $\Omega \neq \mathbb{C}$ is a simply connected domain with $\Omega = h(\mathbb{D})$ for some conformal mapping h of $\mathbb{D}$ onto Ω. In this case the class $\Phi_{H,1}[h(\mathbb{D}), q]$ is written as $\Phi_{H,1}[h, q]$.

The following result is an immediate consequence of Theorem 2.10.

Corollary 2.11. *Let $\phi \in \Phi_{H,1}[h, q]$. If $f \in \mathcal{A}$ satisfies*

$$\phi\left(\frac{B_{\kappa}^{c}f(z)}{B_{\kappa+1}^{c}f(z)}, \frac{B_{\kappa-1}^{c}f(z)}{B_{\kappa}^{c}f(z)}, \frac{B_{\kappa-2}^{c}f(z)}{B_{\kappa-1}^{c}f(z)}; z\right) \prec h(z), \tag{2.17}$$

then

$$\frac{B_{\kappa}^{c}f(z)}{B_{\kappa+1}^{c}f(z)} \prec q(z) \quad (z \in \mathbb{D}).$$

In the particular case when $q(z) = 1 + Mz$; $M > 0$, the class of admissible functions $\Phi_{H,1}[\Omega, q]$ is simply denoted by $\Phi_{H,1}[\Omega, M]$.

Definition 2.4. *Let Ω be a set in $\mathbb{C}$, and $M > 0$. The class of admissible functions $\Phi_{H,1}[\Omega, M]$ consists of those functions $\phi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ such that*

$$\begin{aligned}\phi\Bigg(1+Me^{i\theta}, 1+\frac{k+\kappa(1+Me^{i\theta})}{(\kappa-1)(1+Me^{i\theta})}Me^{i\theta}, 1+\frac{k+\kappa(1+Me^{i\theta})}{(\kappa-2)(1+Me^{i\theta})}Me^{i\theta}\\ +\frac{(M+e^{-i\theta})\left[Le^{-i\theta}+(1+\kappa)kM+\kappa kM^2e^{i\theta}\right]-k^2M^2}{(\kappa-2)(M+e^{-i\theta})\left[(\kappa-1)e^{-i\theta}+\kappa M^2e^{i\theta}+(1+k+2\kappa)M\right]}\Bigg) \notin \Omega\end{aligned} \tag{2.18}$$

whenever $z \in \mathbb{D}$, $\kappa \in \mathbb{C} \setminus \mathbb{Z}_0^-$ $(\kappa \neq 1, 2)$ and $\mathrm{Re}(Le^{-i\theta}) \geq (k-1)kM$ for all $\theta \in \mathbb{R}$, $k \geq 1$.

Corollary 2.12. *Let $\phi \in \Phi_{H,1}[\Omega, M]$. If $f \in \mathcal{A}$ satisfies the following inclusion relationship*

$$\phi\left(\frac{B_{\kappa}^{c}f(z)}{B_{\kappa+1}^{c}f(z)}, \frac{B_{\kappa-1}^{c}f(z)}{B_{\kappa}^{c}f(z)}, \frac{B_{\kappa-2}^{c}f(z)}{B_{\kappa-1}^{c}f(z)}; z\right) \in \Omega,$$

then

$$\frac{B_{\kappa}^{c}f(z)}{B_{\kappa+1}^{c}f(z)} - 1 \prec Mz \quad (z \in \mathbb{D}).$$

In the special case $\Omega = \{w : |w-1| < M\} = q(\mathbb{D})$, the class $\Phi_{H,1}[\Omega, M]$ is simply denoted by $\Phi_{H,1}[M]$, and Corollary 2.12 takes the following form.

Corollary 2.13. *Let $\phi \in \Phi_{H,1}[M]$. If $f \in \mathcal{A}$ satisfies the following inequality*

$$\left|\phi\left(\frac{B_{\kappa}^{c}f(z)}{B_{\kappa+1}^{c}f(z)}, \frac{B_{\kappa-1}^{c}f(z)}{B_{\kappa}^{c}f(z)}, \frac{B_{\kappa-2}^{c}f(z)}{B_{\kappa-1}^{c}f(z)}; z\right) - 1\right| < M,$$

then for all $z \in \mathbb{D}$ we have

$$\left|\frac{B_{\kappa}^{c}f(z)}{B_{\kappa+1}^{c}f(z)} - 1\right| < M.$$

Corollary 2.14. *Let $\kappa \in \mathbb{C} \setminus \mathbb{Z}_0^-$ $(\kappa \neq 1)$ and $M > 0$. If $f \in \mathcal{A}$ satisfies the following inequality*

$$\left|\frac{B_{\kappa-1}^{c}f(z)}{B_{\kappa}^{c}f(z)} - \frac{\kappa}{\kappa-1}\frac{B_{\kappa}^{c}f(z)}{B_{\kappa+1}^{c}f(z)} + \frac{1}{\kappa-1}\right| < \frac{M}{|\kappa-1|(1+M)},$$

then for all $z \in \mathbb{D}$ we have

$$\left|\frac{B_{\kappa}^{c}f(z)}{B_{\kappa+1}^{c}f(z)} - 1\right| < M.$$

Proof. This follows from Corollary 2.12 by taking $\phi(u,v,w;z) = v - \frac{\kappa}{\kappa-1}(u-1) - 1$ and $\Omega = h(\mathbb{D})$, where

$$h(z) = \frac{M}{|\kappa-1|\,(1+M)}z,$$

and $M > 0$. In order to use Corollary 2.12, we need to show that $\phi \in \Phi_{H,1}[\Omega, M]$, that is, the admissibility condition (2.18) is satisfied. This follows since

$$\begin{aligned}
|\phi\,(u,v,w;z)| &= \left|1 + \frac{k+\kappa(1+Me^{i\theta})}{(\kappa-1)(1+Me^{i\theta})}Me^{i\theta} - \frac{\kappa}{\kappa-1}\left(1+Me^{i\theta}-1\right) - 1\right| \\
&= \frac{M}{|\kappa-1|}\left|\frac{k}{1+Me^{i\theta}}\right| \geq \frac{M}{|\kappa-1|\,(1+M)}
\end{aligned}$$

when $z \in \mathbb{D}$, $\theta \in \mathbb{R}$, $\kappa \in \mathbb{C} \setminus \mathbb{Z}_0^-$ $(\kappa \neq 1)$ and $k \geq 1$. The result now follows from Corollary 2.12. □

Definition 2.5. *Let Ω be a set in $\mathbb{C}$ and $q \in \mathcal{Q}_1 \cap \mathcal{H}_1$. The class of admissible functions $\Phi_{H,2}[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ that satisfy the admissibility condition*

$$\phi(u,v,w;z) \notin \Omega$$

where

$$u = q(\zeta),\ v = \frac{k\zeta q'(\zeta) + \kappa q(\zeta)}{\kappa} \qquad \left(\kappa \in \mathbb{C} \setminus \mathbb{Z}_0^-,\ \kappa \neq 1\right)$$

and

$$\operatorname{Re}\left(\frac{(\kappa-1)(w-u)}{v-u} + (1-2\kappa)\right) \geq k\operatorname{Re}\left(\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right)$$
$$(z \in \mathbb{D};\ \zeta \in \partial\mathbb{D} \setminus E(q);\ k \geq 1)\,.$$

Theorem 2.15. *Let $\phi \in \Phi_{H,2}[\Omega, q]$. If $f \in \mathcal{A}$ satisfies the following inclusion relationship*

$$\left\{\phi\left(\frac{B_{\kappa+1}^c f(z)}{z}, \frac{B_{\kappa}^c f(z)}{z}, \frac{B_{\kappa-1}^c f(z)}{z}; z\right) : z \in \mathbb{D}\right\} \subset \Omega, \tag{2.19}$$

then

$$\frac{B_{\kappa+1}^c f(z)}{z} \prec q(z) \quad (z \in \mathbb{D}).$$

Proof. Let us define the analytic function p in $\mathbb{D}$ by

$$p(z) = \frac{B_{\kappa+1}^c f(z)}{z}. \tag{2.20}$$

By making use of (1.10) and (2.20), we get

$$\frac{B_{\kappa}^c f(z)}{z} = \frac{zp'(z) + \kappa p(z)}{\kappa}. \tag{2.21}$$

Further computations show that

$$\frac{B_{\kappa-1}^c f(z)}{z} = \frac{z^2 p''(z) + 2\kappa z p'(z) + \kappa(\kappa-1)p(z)}{\kappa(\kappa-1)}. \tag{2.22}$$

Define the transformations from $\mathbb{C}^3$ to $\mathbb{C}$ by

$$u = r, \quad v = \frac{s+\kappa r}{\kappa} \quad \text{and} \quad w = \frac{t + 2\kappa s + \kappa(\kappa-1)r}{\kappa(\kappa-1)}. \tag{2.23}$$

Let

$$\psi(r,s,t;z) = \phi(u,v,w;z) = \phi\left(r, \frac{s+\kappa r}{\kappa}, \frac{t+2\kappa s+\kappa(\kappa-1)r}{\kappa(\kappa-1)}; z\right). \tag{2.24}$$

The proof shall make use of Lemma 1.1. Using equations (2.20), (2.21) and (2.22), from (2.24) we obtain

$$\psi(p(z), zp'(z), z^2p''(z); z) = \phi\left(\frac{B_{\kappa+1}^c f(z)}{z}, \frac{B_{\kappa}^c f(z)}{z}, \frac{B_{\kappa-1}^c f(z)}{z}; z\right). \tag{2.25}$$

Hence (2.19) becomes

$$\psi(p(z), zp'(z), z^2p''(z); z) \in \Omega.$$

The proof is completed if it can be shown that the admissibility condition for $\phi \in \Phi_{H,2}[\Omega, q]$ is equivalent to the admissibility condition for as given in Definition 1.2. Note that

$$\frac{t}{s} + 1 = \frac{(\kappa - 1)(w - u)}{v - u} + (1 - 2\kappa),$$

and hence $\psi \in \Psi[\Omega, q]$. By Lemma 1.1, we have $p(z) \prec q(z)$ or

$$\frac{B_{\kappa+1}^{c} f(z)}{z} \prec q(z).$$

□

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(\mathbb{D})$ for some conformal mapping h of $\mathbb{D}$ onto Ω. In this case the class $\Phi_{H,2}[h(\mathbb{D}), q]$ is written as $\Phi_{H,2}[h, q]$.

The following result is an immediate consequence of Theorem 2.15.

Corollary 2.16. *Let $\phi \in \Phi_{H,2}[h, q]$. If $f \in \mathcal{A}$ satisfies*

$$\phi\left(\frac{B_{\kappa+1}^{c} f(z)}{z}, \frac{B_{\kappa}^{c} f(z)}{z}, \frac{B_{\kappa-1}^{c} f(z)}{z}; z\right) \prec h(z), \tag{2.26}$$

then

$$\frac{B_{\kappa+1}^{c} f(z)}{z} \prec q(z) \quad (z \in \mathbb{D}).$$

In the particular case when $q(z) = 1 + Mz$; $M > 0$, the class of admissible functions $\Phi_{H,2}[\Omega, q]$, denoted by $\Phi_{H,2}[\Omega, M]$.

Definition 2.6. *Let Ω be a set in $\mathbb{C}$ and $M > 0$. The class of admissible functions $\Phi_{H,2}[\Omega, M]$ consists of those functions $\phi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ such that*

$$\phi\left(1 + Me^{i\theta}, 1 + \frac{k + \kappa}{\kappa} Me^{i\theta}, 1 + \frac{L + \kappa(2k + \kappa - 1)Me^{i\theta}}{\kappa(\kappa - 1)}; z\right) \notin \Omega \tag{2.27}$$

whenever $z \in \mathbb{D}$, $\kappa \in \mathbb{C} \setminus \mathbb{Z}_0^-$ $(\kappa \neq 1)$ and $\mathrm{Re}(Le^{-i\theta}) \geq (k - 1)kM$ for all $\theta \in \mathbb{R}$, $k \geq 1$.

Corollary 2.17. *Let $\phi \in \Phi_{H,2}[\Omega, M]$. If $f \in \mathcal{A}$ satisfies the following inclusion relationship*

$$\phi\left(\frac{B_{\kappa+1}^{c} f(z)}{z}, \frac{B_{\kappa}^{c} f(z)}{z}, \frac{B_{\kappa-1}^{c} f(z)}{z}; z\right) \in \Omega,$$

then

$$\frac{B_{\kappa+1}^{c} f(z)}{z} - 1 \prec Mz \quad (z \in \mathbb{D}).$$

In the special case when $\Omega = \{w : \ |w - 1| < M\} = q(\mathbb{D})$, the class $\Phi_{H,2}[\Omega, M]$ is simply denoted by $\Phi_{H,2}[M]$, and Corollary 2.17 takes the following form.

Corollary 2.18. *Let $\phi \in \Phi_{H,2}[M]$. If $f \in \mathcal{A}$ satisfies the following inequality*

$$\left|\phi\left(\frac{B_{\kappa+1}^{c} f(z)}{z}, \frac{B_{\kappa}^{c} f(z)}{z}, \frac{B_{\kappa-1}^{c} f(z)}{z}; z\right) - 1\right| < M,$$

then for all $z \in \mathbb{D}$ we have

$$\left|\frac{B_{\kappa+1}^{c} f(z)}{z} - 1\right| < M.$$

Corollary 2.19. *Let $\kappa \in \mathbb{C} \setminus \mathbb{Z}_0^-$ and $M > 0$. If $f \in \mathcal{A}$ satisfies the following inequality*

$$\left|\frac{B_{\kappa}^{c} f(z)}{z} - \frac{B_{\kappa+1}^{c} f(z)}{z}\right| < \frac{M}{|\kappa|},$$

then for all $z \in \mathbb{D}$ we have

$$\left|\frac{B_{\kappa+1}^{c} f(z)}{z} - 1\right| < M.$$

Proof. This follows from Corollary 2.17 by taking $\phi(u, v, w; z) = v - u$. □

Corollary 2.20. *Let* $\mathrm{Re}(\kappa) \geq -\frac{1}{2}$, $\kappa \neq 0$ *and* $M > 0$. *If* $f \in \mathcal{A}$ *satisfy the following inequality*

$$\left| \frac{B_\kappa^c f(z)}{z} - 1 \right| < M,$$

then for all $z \in \mathbb{D}$ *we have*

$$\left| \frac{B_{\kappa+1}^c f(z)}{z} - 1 \right| < M.$$

Proof. This follows from Corollary 2.17 by taking $\phi(u, v, w; z) = v - 1$. □

For $f(z) = z/(1-z)$ in Corollary 2.20 we have

$$\left| \frac{\varphi_{\kappa,c}(z)}{z} - 1 \right| < M \Rightarrow \left| \frac{\varphi_{\kappa+1,c}(z)}{z} - 1 \right| < M. \tag{2.28}$$

This result is somewhat related to an open problem given by András and Baricz [2], which in terms of $\varphi_{\kappa,c}$ can be rewritten as follows: is it true that if $p > -1$ increase, then the image region $\varphi_{p,1,1}(\mathbb{D})$ decrease, that is, if $p > q > -1$, then $\varphi_{p,1,1}(\mathbb{D}) \subset \varphi_{q,1,1}(\mathbb{D})$?

In particular, if we take $b = c = 1$; $p = -\frac{1}{2}$ and $p = \frac{1}{2}$, in (2.28), then the following chain of implications is true

$$\left|\cos\sqrt{z} - 1\right| < M \Rightarrow \left| \frac{\sin\sqrt{z}}{\sqrt{z}} - 1 \right| < M \Rightarrow \left| \frac{\sin\sqrt{z}}{z\sqrt{z}} - \frac{\cos\sqrt{z}}{z} - \frac{1}{3} \right| < \frac{M}{3},$$

where $M > 0$.

3. Superordination and sandwich-type results involving the B_κ^c–operator

In this section we obtain some differential superordination results for functions associated with the B_κ^c–operator defined by (1.10). As in the previous section, the parameter c is an arbitrary complex number, while for the parameter κ we will have some assumptions in some cases. We consider first a class of admissible functions, which is given in the following definition.

Definition 3.1. *Let* Ω *be a set in* $\mathbb{C}$, $q \in \mathcal{H}_0$ *with* $zq'(z) \neq 0$. *The class of admissible functions* $\Phi'_H[\Omega, q]$ *consists of those functions* $\phi : \mathbb{C}^3 \times \overline{\mathbb{D}} \to \mathbb{C}$ *that satisfy the admissibility condition*

$$\phi(u, v, w; \varsigma) \notin \Omega$$

whenever

$$u = q(z), \; v = \frac{zq'(z) + m(\kappa-1)q(z)}{m\kappa}, \qquad \left(\kappa \in \mathbb{C} \setminus \mathbb{Z}_0^-, \; \kappa \neq 1\right)$$

and

$$\mathrm{Re}\left(\frac{\kappa(\kappa-1)w - (\kappa-2)(\kappa-1)u}{v\kappa - (\kappa-1)u} - (2\kappa-3) \right) \leq \frac{1}{m} \mathrm{Re}\left(\frac{zq''(z)}{q'(z)} + 1 \right)$$

$$(z \in \mathbb{D};\; \varsigma \in \partial\mathbb{D};\; m \geq 1)\,.$$

Theorem 3.1. *Let* $\phi \in \Phi'_H[\Omega, q]$. *If* $f \in \mathcal{A}$, $B_{\kappa+1}^c f \in \mathcal{Q}_0$ *and*

$$\phi\left(B_{\kappa+1}^c f(z), B_\kappa^c f(z), B_{\kappa-1}^c f(z); z\right)$$

is univalent in $\mathbb{D}$, *then*

$$\Omega \subset \left\{ \phi\left(B_{\kappa+1}^c f(z), B_\kappa^c f(z), B_{\kappa-1}^c f(z); z\right) : z \in \mathbb{D} \right\} \tag{3.1}$$

implies

$$q(z) \prec B_{\kappa+1}^c f(z) \quad (z \in \mathbb{D}).$$

Proof. Let $p(z)$ be defined by (2.2) and ψ by (2.5). Since $\phi \in \Phi'_H[\Omega, q]$, (2.6) and (3.1) yield

$$\Omega \subset \left\{ \psi(p(z), zp'(z), z^2 p''(z); z) : z \in \mathbb{D} \right\}.$$

From (2.5), we see that the admissibility condition for $\phi \in \Phi'_H[\Omega, q]$ is equivalent to the admissibility condition for ψ as given in Definition 1.3. Hence $\psi \in \Psi'[\Omega, q]$, and by Lemma 1.2, we have

$$q(z) \prec p(z) \quad \text{or} \quad q(z) \prec B_{\kappa+1}^c f(z) \quad (z \in \mathbb{D})$$

which completes the proof of Theorem 3.1. □

If $\Omega \neq \mathbb{C}$ is a simply-connected domain, then $\Omega = h(\mathbb{D})$ for some conformal mapping h of $\mathbb{D}$ onto Ω. In this case, the class $\Phi'_H[h(\mathbb{D}), q]$ is written as $\Phi'_H[h, q]$.

Proceeding similarly as in the previous section, the following result is an immediate consequence of Theorem 3.1.

Corollary 3.2. *Let $q \in \mathcal{H}_0$, h be analytic in $\mathbb{D}$ and $\phi \in \Phi'_H[h, q]$. If $f \in \mathcal{A}$, $B^c_{\kappa+1}f \in \mathcal{Q}_0$ and*

$$\phi\left(B^c_{\kappa+1}f(z), B^c_{\kappa}f(z), B^c_{\kappa-1}f(z); z\right)$$

is univalent in $\mathbb{D}$, then

$$h(z) \prec \phi\left(B^c_{\kappa+1}f(z), B^c_{\kappa}f(z), B^c_{\kappa-1}f(z); z\right) \tag{3.2}$$

implies

$$q(z) \prec B^c_{\kappa+1}f(z) \quad (z \in \mathbb{D}).$$

Theorem 3.1 and Corollary 3.2 can only be used to obtain subordinants of differential superordination of the form (3.1) or (3.2). The following theorem proves the existence of the best subordinant of (3.2) for an appropriate ϕ.

Theorem 3.3. *Let h be univalent in $\mathbb{D}$ and $\phi : \mathbb{C}^3 \times \overline{\mathbb{D}} \to \mathbb{C}$. Suppose that the differential equation*

$$\phi\left(q(z), \frac{zq'(z) + (\kappa-1)q(z)}{\kappa}, \frac{z^2q''(z) + 2(\kappa-1)zq'(z) + (\kappa-1)(\kappa-2)q(z)}{\kappa(\kappa-1)}; z\right) = h(z)$$

has a solution $q \in \mathcal{Q}_0$. If $\phi \in \Phi'_H[h, q]$, $f \in \mathcal{A}$, $B^c_{\kappa+1}f \in \mathcal{Q}_0$ and

$$\phi\left(B^c_{\kappa+1}f(z), B^c_{\kappa}f(z), B^c_{\kappa-1}f(z); z\right)$$

is univalent in $\mathbb{D}$, then

$$h(z) \prec \phi\left(B^c_{\kappa+1}f(z), B^c_{\kappa}f(z), B^c_{\kappa-1}f(z); z\right)$$

which implies

$$q(z) \prec B^c_{\kappa+1}f(z) \quad (z \in \mathbb{D})$$

and q is best subordinant.

Proof. The proof is similar to the proof of Theorem 2.5 and is therefore we omit the details. □

Combining Theorem 2.2 and Corollary 3.2, we obtain the following sandwich-type theorem.

Corollary 3.4. *Let h_1 and q_1 be analytic functions in $\mathbb{D}$, h_2 be a univalent function in $\mathbb{D}$, $q_2 \in \mathcal{Q}_0$ with $q_1(0) = q_2(0) = 1$ and $\phi \in \Phi_H[h_2, q_2] \cap \Phi'_H[h_1, q_1]$. If $f \in \mathcal{A}$, $B^c_{\kappa+1}f \in \mathcal{Q}_0 \cap \mathcal{H}_0$ and*

$$\phi\left(B^c_{\kappa+1}f(z), B^c_{\kappa}f(z), B^c_{\kappa-1}f(z); z\right)$$

is univalent in $\mathbb{D}$, then

$$h_1(z) \prec \phi\left(B^c_{\kappa+1}f(z), B^c_{\kappa}f(z), B^c_{\kappa-1}f(z); z\right) \prec h_2(z)$$

which implies

$$q_1(z) \prec B^c_{\kappa+1}f(z) \prec q_2(z) \quad (z \in \mathbb{D}).$$

Definition 3.2. *Let Ω be a set in $\mathbb{C}$, $q \in \mathcal{H}_1$ with $q(z) \neq 0$, $zq'(z) \neq 0$. The class of admissible functions $\Phi'_{H,1}[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^3 \times \overline{\mathbb{D}} \to \mathbb{C}$ that satisfy the admissibility condition*

$$\phi(u, v, w; \varsigma) \in \Omega$$

whenever

$$u = q(z),\ v = \frac{1}{\kappa-1}\left(\frac{zq'(z)}{mq(z)} + \kappa q(z) - 1\right) \quad (\kappa \in \mathbb{C} \setminus \mathbb{Z}_0^-,\ \kappa \neq 1, 2)$$

and

$$\mathrm{Re}\left(\frac{(\kappa-1)v\left[(\kappa-1)(w-v) + 1 - w\right]}{(\kappa-1)v + 1 - \kappa u} - \left[2\kappa u - 1 - (\kappa-1)v\right]\right) \leq \frac{1}{m}\,\mathrm{Re}\left(\frac{zq''(z)}{q'(z)} + 1\right)$$

$$(z \in \mathbb{D};\ \varsigma \in \partial\mathbb{D};\ m \geq 1).$$

Theorem 3.5. *Let $\phi \in \Phi'_{H,1}[\Omega, q]$. If $f \in \mathcal{A}$, $B^c_\kappa f/B^c_{\kappa+1} f \in \mathcal{Q}_1$ and*

$$\phi\left(\frac{B^c_\kappa f(z)}{B^c_{\kappa+1} f(z)}, \frac{B^c_{\kappa-1} f(z)}{B^c_\kappa f(z)}, \frac{B^c_{\kappa-2} f(z)}{B^c_{\kappa-1} f(z)}; z\right)$$

is univalent in $\mathbb{D}$, then

$$\Omega \subset \left\{\phi\left(\frac{B^c_\kappa f(z)}{B^c_{\kappa+1} f(z)}, \frac{B^c_{\kappa-1} f(z)}{B^c_\kappa f(z)}, \frac{B^c_{\kappa-2} f(z)}{B^c_{\kappa-1} f(z)}; z\right) : z \in \mathbb{D}\right\} \tag{3.3}$$

which implies

$$q(z) \prec \frac{B^c_\kappa f(z)}{B^c_{\kappa+1} f(z)} \quad (z \in \mathbb{D}).$$

Proof. Let p be defined by (2.12) and ψ by (2.16). Since $\phi \in \Phi'_{H,1}[\Omega, q]$, it follows from (2.16) and (3.3) that

$$\Omega \subset \left\{\psi(p(z), zp'(z), z^2p''(z); z) : z \in \mathbb{D}\right\}.$$

From (2.16), we see that the admissibility condition for $\phi \in \Phi'_{H,1}[\Omega, q]$ is equivalent to the admissibility condition for ψ as given in Definition 1.3. Hence $\psi \in \Psi'[\Omega, q]$, and by Lemma 1.2, we have

$$q(z) \prec p(z) \quad \text{or} \quad q(z) \prec \frac{B^c_\kappa f(z)}{B^c_{\kappa+1} f(z)} \quad (z \in \mathbb{D})$$

which completes the proof of Theorem 3.5. □

If $\Omega \neq \mathbb{C}$ is a simply-connected domain, then $\Omega = h(\mathbb{D})$ for some conformal mapping h of $\mathbb{D}$ onto Ω. In this case, the class $\Phi'_{H,1}[h(\mathbb{D}), q]$ is written as $\Phi'_{H,1}[h, q]$. Proceeding similarly in the previous section, the following result is an immediate consequence of Theorem 3.5.

Theorem 3.6. *Let $q \in \mathcal{H}_1$, h be analytic in $\mathbb{D}$ and $\phi \in \Phi'_{H,1}[h, q]$. If $f \in \mathcal{A}$, $B^c_\kappa f/B^c_{\kappa+1} f \in \mathcal{Q}_1$ and*

$$\phi\left(\frac{B^c_\kappa f(z)}{B^c_{\kappa+1} f(z)}, \frac{B^c_{\kappa-1} f(z)}{B^c_\kappa f(z)}, \frac{B^c_{\kappa-2} f(z)}{B^c_{\kappa-1} f(z)}; z\right)$$

is univalent in $\mathbb{D}$, then

$$h(z) \prec \phi\left(\frac{B^c_\kappa f(z)}{B^c_{\kappa+1} f(z)}, \frac{B^c_{\kappa-1} f(z)}{B^c_\kappa f(z)}, \frac{B^c_{\kappa-2} f(z)}{B^c_{\kappa-1} f(z)}; z\right) \tag{3.4}$$

which implies

$$q(z) \prec \frac{B^c_\kappa f(z)}{B^c_{\kappa+1} f(z)} \quad (z \in \mathbb{D}).$$

Combining Theorems 2.11 and 3.6, we obtain the following sandwich-type theorem.

Corollary 3.7. *Let h_1 and q_1 be analytic functions in $\mathbb{D}$, h_2 be a univalent function in $\mathbb{D}$, $q_2 \in \mathcal{Q}_1$ with $q_1(0) = q_2(0) = 1$ and $\phi \in \Phi_{H,1}[h_2, q_2] \cap \Phi'_{H,1}[h_1, q_1]$. If $f \in \mathcal{A}$, $B^c_\kappa f/B^c_{\kappa+1} f \in \mathcal{Q}_1 \cap \mathcal{H}_1$ and*

$$\phi\left(\frac{B^c_\kappa f(z)}{B^c_{\kappa+1} f(z)}, \frac{B^c_{\kappa-1} f(z)}{B^c_\kappa f(z)}, \frac{B^c_{\kappa-2} f(z)}{B^c_{\kappa-1} f(z)}; z\right)$$

is univalent in $\mathbb{D}$, then

$$h_1(z) \prec \phi\left(\frac{B^c_\kappa f(z)}{B^c_{\kappa+1} f(z)}, \frac{B^c_{\kappa-1} f(z)}{B^c_\kappa f(z)}, \frac{B^c_{\kappa-2} f(z)}{B^c_{\kappa-1} f(z)}; z\right) \prec h_2(z)$$

which implies

$$q_1(z) \prec \frac{B^c_\kappa f(z)}{B^c_{\kappa+1} f(z)} \prec q_2(z) \quad (z \in \mathbb{D}).$$

Definition 3.3. *Let Ω be a set in $\mathbb{C}$, $q \in \mathcal{H}_1$ with $q(z) \neq 0$, $zq'(z) \neq 0$. The class of admissible functions $\Phi'_{H,2}[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^3 \times \overline{\mathbb{D}} \to \mathbb{C}$ that satisfy the admissibility condition*

$$\phi(u, v, w; \varsigma) \in \Omega$$

whenever

$$u = q(z),\ v = \frac{zq'(z) + \kappa m q(z)}{\kappa m} \quad (\kappa \in \mathbb{C} \setminus \mathbb{Z}^-_0,\ \kappa \neq 1)$$

and
$$\operatorname{Re}\left(\frac{(\kappa-1)(w-u)}{v-u}+(1-2\kappa)\right)\leq\frac{1}{m}\operatorname{Re}\left(\frac{zq''(z)}{q'(z)}+1\right)$$
$$(z\in\mathbb{D};\ \varsigma\in\partial\mathbb{D};\ m\geq 1).$$

Theorem 3.8. *Let $\phi\in\Phi'_{H,2}[\Omega,q]$. If $f\in\mathcal{A}$, $\frac{B^c_{\kappa+1}f(z)}{z}\in\mathcal{Q}_1$ and*
$$\phi\left(\frac{B^c_{\kappa+1}f(z)}{z},\frac{B^c_{\kappa}f(z)}{z},\frac{B^c_{\kappa-1}f(z)}{z};z\right)$$
is univalent in $\mathbb{D}$, then
$$\Omega\subset\left\{\phi\left(\frac{B^c_{\kappa+1}f(z)}{z},\frac{B^c_{\kappa}f(z)}{z},\frac{B^c_{\kappa-1}f(z)}{z};z\right):z\in\mathbb{D}\right\}\tag{3.5}$$
which implies
$$q(z)\prec\frac{B^c_{\kappa+1}f(z)}{z}\quad(z\in\mathbb{D}).$$

Proof. From (2.25) and (3.5) we obtain that
$$\Omega\subset\left\{\psi(p(z),zp'(z),z^2p''(z);z):z\in\mathbb{D}\right\}.$$
From (2.23), we see that the admissibility condition for $\phi\in\Phi'_{H,2}[\Omega,q]$ is equivalent to the admissibility condition for ψ as given in Definition 1.3. Hence $\psi\in\Psi'[\Omega,q]$, and by Lemma 1.2, we have
$$q(z)\prec p(z)\quad\text{or}\quad q(z)\prec\frac{B^c_{\kappa+1}f(z)}{z}\quad(z\in\mathbb{D}).$$
□

If $\Omega\neq\mathbb{C}$ is a simply-connected domain, then $\Omega=h(\mathbb{D})$ for some conformal mapping h of $\mathbb{D}$ onto Ω. In this case, the class $\Phi'_{H,2}[h(\mathbb{D}),q]$ is written as $\Phi'_{H,2}[h,q]$. Proceeding similarly in the previous section, the following result is an immediate consequence of Theorem 3.8.

Corollary 3.9. *Let $q\in\mathcal{H}_1$, h be analytic in $\mathbb{D}$ and $\phi\in\Phi'_{H,2}[h,q]$. If $f\in\mathcal{A}$, $\frac{B^c_{\kappa+1}f(z)}{z}\in\mathcal{Q}_1$ and*
$$\phi\left(\frac{B^c_{\kappa+1}f(z)}{z},\frac{B^c_{\kappa}f(z)}{z},\frac{B^c_{\kappa-1}f(z)}{z};z\right)$$
is univalent in $\mathbb{D}$, then
$$h(z)\prec\phi\left(\frac{B^c_{\kappa+1}f(z)}{z},\frac{B^c_{\kappa}f(z)}{z},\frac{B^c_{\kappa-1}f(z)}{z};z\right)\tag{3.6}$$
which implies
$$q(z)\prec\frac{B^c_{\kappa+1}f(z)}{z}\quad(z\in\mathbb{D}).$$

Combining Corollaries 2.16 and 3.9, we obtain the following sandwich-type theorem.

Corollary 3.10. *Let h_1 and q_1 be analytic functions in $\mathbb{D}$, h_2 be a univalent function in $\mathbb{D}$, $q_2\in\mathcal{Q}_1$ with $q_1(0)=q_2(0)=1$ and $\phi\in\Phi_{H,2}[h_2,q_2]\cap\Phi'_{H,2}[h_1,q_1]$. If $f\in\mathcal{A}$, $\frac{B^c_{\kappa+1}f(z)}{z}\in\mathcal{Q}_1\cap\mathcal{H}_1$ and*
$$\phi\left(\frac{B^c_{\kappa+1}f(z)}{z},\frac{B^c_{\kappa}f(z)}{z},\frac{B^c_{\kappa-1}f(z)}{z};z\right)$$
is univalent in $\mathbb{D}$, then
$$h_1(z)\prec\phi\left(\frac{B^c_{\kappa+1}f(z)}{z},\frac{B^c_{\kappa}f(z)}{z},\frac{B^c_{\kappa-1}f(z)}{z};z\right)\prec h_2(z)$$
which implies
$$q_1(z)\prec\frac{B^c_{\kappa+1}f(z)}{z}\prec q_2(z)\quad(z\in\mathbb{D}).$$

We note that in particular the above main results reduce to results for the operators $\mathcal{J}_pf$, $\mathcal{I}_pf$ and $\mathcal{S}_pf$, which are defined by (1.11), (1.12) and (1.13), respectively.

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References

[1] R.M. Ali, S. Nagpal and V. Ravichandran, Second-order differential subordination for analytic functions with fixed initial coefficient, *Bull. Malays. Math. Sci. Soc. (2)*, **34**(3) (2011), 611–629.

[2] S. András and Á. Baricz, Monotonicity property of generalized and normalized Bessel functions of complex order, *Complex Var. Elliptic Equ.*, **54**(7) (2009), 689–696.

[3] Á. Baricz, Geometric properties of generalized Bessel functions, *Publ. Math. Debrecen*, **73** (2008), 155–178.

[4] Á. Baricz, *Generalized Bessel functions of the first kind*, Lecture Notes in Mathematics, vol. 1994, Springer Verlag, Berlin, 2010.

[5] Á. Baricz and S. Ponnusamy, Starlikeness and covexity of generalized Bessel function, *Integral Transform Spec. Funct.*, **21**(9) (2010), 641–651.

[6] E. Deniz, H. Orhan and H.M. Srivastava, Some sufficient conditions for univalence of certain families of integral operators involving generalized Bessel functions, *Taiwansese J. Math.*, **15**(2) (2011), 883–917.

[7] E. Deniz, Convexity of integral operators involving generalized Bessel functions, *Integral Transform Spec. Funct.*, **24**(3) (2013), 201–216.

[8] B. Kowalczyk and A. Lecko, Fekete-Szegő problem for a certain subclass of close-to-convex functions, *Bull. Malays. Math. Sci. Soc. (2)*, (in press).

[9] J.-L. Liu, Certain sufficient conditions for strongly starlike functions associated with an integral operator, *Bull. Malays. Math. Sci. Soc. (2)*, **34**(1) (2011), 21–30.

[10] S.S. Miller and P.T. Mocanu, *Differential subordinations*, Marcel Dekker, New York, 2000.

[11] S.S. Miller and P.T. Mocanu, Subordination of differential superordinations, *Complex Var. Theory Appl.*, **48**(10) (2003), 815–826.

[12] S.S. Miller and P.T. Mocanu, Univalence of Gaussian and confluent hypergeometric functions, *Proc. Amer. Math. Soc.*, **110**(2) (1990), 333–342.

[13] S.R. Mondal and A. Swaminathan, Geometric properties of generalized Bessel functions, *Bull. Malays. Math. Sci. Soc. (2)*, **35**(1) (2012), 179–194.

[14] S. Owa and H.M. Srivastava, Univalent and starlike generalized hypergeometric functions, *Canad. J. Math.*, **39**(5) (1987), 1057–1077.

[15] S. Ponnusamy and F. Ronning, Geometric properties for convolutions of hypergeometric functions and functions with the derivative in a halfplane, *Integral Transform. Spec. Funct.*, **8** (1999), 121–138.

[16] S. Ponnusamy and M. Vuorinen, Univalence and convexity properties for confluent hypergeometric functions, *Complex Var. Theory Appl.*, **36**(1) (1998), 73–97.

[17] S. Ponnusamy and M. Vuorinen, Univalence and convexity properties for Gaussian hypergeometric functions, *Rocky Mountain J. Math.*, **31**(1) (2001), 327–353.

[18] J.K. Prajapat, Certain geometric properties of normalized Bessel functions, *Appl. Math. Lett.*, **24**(12) (2011), 2133–2139.

[19] R. Szász, On starlikeness of Bessel functions of the first kind, In: Proceedings of the 8th Joint Conference on Mathematics and Computer Science, Komárno, Slovakia, 2010, 9pp.

[20] R. Szász and P.A. Kupán, About the univalence of the Bessel functions, *Stud. Univ. Babeş-Bolyai Math.*, **54**(1) (2009), 127–132.

[21] G.N. Watson, *A Treatise on the Theory of Bessel Functions*, Cambridge University Press, Cambridge, 1944.

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