CHARACTERIZATIONS OF FINITE GROUPS WITH p-FUSION OF SQUAREFREE TYPE

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ABSTRACT. For G a finite group and p a prime this paper proves two theorems under hypotheses that restrict the index of the subgroup generated by every p-element x in certain subgroups generated by pairs of its conjugates. Under one set of hypotheses G is shown to be supersolvable. Simple groups satisfying a complementary fusion-theoretic hypothesis are classified.

1. INTRODUCTION

The role of fusion of elements of prime power order in finite group theory has a long and rich history, punctuated with highlights such as the Focal Subgroup Theorem, the Glauberman Z^*-Theorem and the Baer–Suzuki Theorem. The latter result states that in a finite group G, for p a prime and x any p-element in G, if $\langle x, x^g \rangle$ is a p-group for all conjugates x^g of x, then $x \in O_p(G)$, the largest normal p-subgroup of G. In their paper [8], Li and Xu consider finite groups G in which, for any p-element x and any $g \in G$, the index of $\langle x \rangle$ in $\langle x, x^g \rangle$ is squarefree, for all primes p dividing the order of G. Under this hypothesis they prove G must be supersolvable. The first main result of this note gives an independent, self-contained and elementary proof of a generalization of their result.

Theorem 1. *Let G be a finite group. Assume for all pairs of distinct primes p, q that q^2 does not divide the order of $\langle x, x^y \rangle$, for all p-elements x and q-elements y in G. Then G is supersolvable.*

The techniques used in the proof of Theorem 1 also indicate how one might further investigate the structure of (now supersolvable) groups satisfying its fusion hypotheses. In particular, the result and its proof illuminate the p, q-Hall nature that turns out to be the key fusion ingredient.

It is generally quite difficult to determine the structure or order of the subgroup generated by two arbitrary conjugates of a p-element, so it seems natural to impose restrictions on such subgroups to make them more tractable. The second main result of this paper therefore examines a "complementary" fusion hypothesis, which is even more closely aligned with the classical theorems mentioned at the outset, and which may be easily transported to the very active area of modern fusion systems (which we say more about at the end of the Introduction).

For any finite group G and prime p we say G is *p-fusion modular* if for every p-element x in G and every $g \in G$ such that $\langle x, x^g \rangle$ is a p-group, the index of $\langle x \rangle$ in $\langle x, x^g \rangle$ is 1 or p. In this situation the p-group $\langle x, x^g \rangle$ has a cyclic subgroup of

index at most p, so it is cyclic, abelian of type (p^m, p) or modular; when $p = 2$, it may also be quaternion of order 8 (see Lemma 3.1). Recall that the modular p-group of order p^{n+1} is

$$M_{n+1}(p) = \langle\, a, b \mid a^{p^n} = b^p = 1,\ b^{-1}ab = a^{1+p^{n-1}} \,\rangle,$$

so these are the "generic" non-abelian subgroups generated by pairs of fused p-elements, which motivates our choice of terminology. Any group whose Sylow p-subgroups are either cyclic or elementary abelian is necessarily p-fusion modular (such as $L_2(2^n)$, for all primes p). A well-known transfer theorem, however, shows that for odd p, a Sylow p-subgroup of a simple group cannot be isomorphic to $M_{n+1}(p)$ for any $n \geq 2$ (see [7], Theorem 10.12).

It is an exercise that the p-fusion modular property inherits to both subgroups and quotient groups, so, in the spirit of Theorem 1, we are naturally led to examine the possible composition factors of such groups. For odd p the simple groups that are p-fusion modular are quite common for the aforementioned reasons (the Sylow p-subgroups of the simple groups of Lie type are nicely described in Sections 3.3 and 4.10 of [6]). Our second main result classifies the 2-fusion modular simple groups.

Theorem 2. *If G is a non-abelian finite simple group that is 2-fusion modular, then G has abelian Sylow 2-subgroups. More specifically, $G \cong L_2(2^n)$, $L_2(q)$ for some prime power $q \equiv \pm 3 \pmod 8$, $Re(3^n)$, or J_1.*

The groups $Re(3^n)$ are not 3-fusion modular (Lemma 3.1(g)), whereas all the remaining Goldschmidt groups in the conclusion are p-fusion modular for all primes p.

Results that are similar in nature to the above appear in [1] and [10].

Finally, we observe that the arguments in the proof of Theorem 2 primarily involve fusion and transfer, hence are highly adaptable to the theory of fusion systems. We therefore surmise that a corresponding result in this realm could be achieved, perhaps even classifying simple p-fusion modular saturated fusion systems for all primes p. For the sake of cohesiveness we have chosen not to pursue this but to keep all our results couched in finite group theory.

2. The Proof of Theorem 1

The notation in the proofs is standard, and familiar results from finite group theory, which may all be found for instance in Gorenstein's book [4], are invoked without specific reference.

Before starting the proof, recall first that a finite group G is supersolvable if it has a chief series all of whose chief factors are of prime order. Moreover, a chief series may be constructed so that as the terms ascend from the identity to G, the sequence of orders of these chief factors is nonincreasing. Let $p_1 < p_2 < \cdots < p_r$ be the distinct primes dividing $|G|$ and let $\pi_i = \{p_r, p_{r-1}, \ldots, p_{r-i}\}$. Supersolvability then implies, but is not equivalent to, the existence of a normal π_i-Hall subgroup for $0 \leq i < r$ (and these are necessarily characteristic and form an increasing tower).

Proof of Theorem 1: Let G be a counterexample of minimal order. Since the hypotheses clearly inherit to subgroups and quotient groups, every proper subgroup and quotient group of G is supersolvable.

Using the same notation as above, we first show that

(1) G has a an increasing tower of normal π_i-Hall subgroups, for $0 \leq i < r$.

This follows by induction once we prove G has a normal p_1-complement. To see the latter let P be any nontrivial p_1-subgroup. If $N_G(P)$ is proper in G, it is supersolvable, hence has a normal p_1-complement. By Frobenius' Theorem, if this is true for all P, then G has a normal p_1-complement too, as claimed. By way of contradiction we may assume P is normal and G does not have a normal p_1-complement. Thus some element x of prime power order q^α with $q \neq p_1$ acts nontrivially on P. By Burnside's Basis Theorem and Maschke, x has a nontrivial irreducible submodule in its action on $P/\Phi(P)$. It follows from the minimality of G that $\Phi(P) = 1$ with x acting irreducibly on P, and $G = P\langle x \rangle$. Since $q > p_1$, $\dim_{\mathbb{F}_{p_1}} P \geq 2$. For any nonidentity $y \in P$, $1 \neq [y, x] \in P \cap \langle x, x^y \rangle$. By the irreducible action, $P \leq \langle x, x^y \rangle$, contrary to p_1^2 not dividing this order. This contradiction establishes (1).

Next let E be any minimal normal subgroup of G and let $q = p_r$. By (1) we may assume E is an elementary abelian q-subgroup. If $|E| = q$, then since G/E is supersolvable, so too is G, a contradiction. Thus $\dim_{\mathbb{F}_q} E \geq 2$. Since a Sylow q-subgroup is normal in G, the irreducible action of G on E forces E to be in the center of the Sylow q-subgroup. Let H be a q'-Hall subgroup, i.e., a complement to the normal Sylow q-subgroup. Thus H acts irreducibly on E, hence by minimality of G it follows easily that

$$G = EH \qquad \text{and} \qquad C_H(E) = 1.$$

Now H is supersolvable so it possesses a normal subgroup M of prime index p. Let $H = M\langle x \rangle$, where x is a p-element of H. By minimality, EM is supersolvable and so M stabilizes some 1-dimensional subspace of E; in particular, M does not act irreducibly on E. By Clifford's Theorem therefore

$$E = E_1 \oplus E_2 \oplus \cdots \oplus E_p$$

where each E_i is an irreducible $\mathbb{F}_q M$-module and $\langle x \rangle$ permutes these constituents transitively by conjugation. As noted, the irreducible $\mathbb{F}_q M$-submodules of E are 1-dimensional, i.e.

(2)
$$E_i = \mathbb{F}_q e_i \quad 1 \leq i \leq p \text{ and}$$
M is represented by diagonal matrices with respect to the basis $e_1, \ldots, e_p$.

Now E is a cyclic $\mathbb{F}_q\langle x \rangle$-module generated by e_1, and so the $\mathbb{F}_q\langle x \rangle$-submodule generated by $[e_1, x] = (x - 1)e_1$ is a homomorphic image of the augmentation ideal of that ring, which is of codimension one. As before, $[e_1, x] \in \langle x, x^{e_1} \rangle$; and since q^2 does not divide the order of the latter group, $[e_1, x]$ must span a 1-dimensional $\mathbb{F}_q\langle x \rangle$-submodule of E. This proves

$$\dim_{\mathbb{F}_q} E = 2 \qquad \text{and} \qquad p = 2.$$

Suppose first that we may choose the 2-element x of order > 2 in $H - M$. For arbitrary first basis vector $e_1 \in E_1$ let $e_2 = e_1^x \in E_2$. The matrix, $[x]$, of x with respect to the basis e_1, e_2 is

$$[x] = \begin{pmatrix} 0 & a \\ 1 & 0 \end{pmatrix},$$

where $a \neq 1$ because $|x| > 2$. Thus $[(e_1 + e_2), x] = (a - 1)e_1 \in \langle x, x^{e_1 + e_2} \rangle \cap E$; so e_1 lies in this intersection. But now the action of $\langle x \rangle$ forces $E \leq \langle x, x^{e_1 + e_2} \rangle$,

again contradicting the hypothesis that q^2 does not divide the latter group order. This proves

(3)   every 2-element in $H - M$ has order 2.

If M acted entirely as scalar matrices on E, then $H = M\langle x \rangle$ would be abelian. In this case x would fix the vector $e_1 + e_1^x$, and this would span the H-stable subspace $C_E(x)$, contrary to the irreducibility of H. Thus M contains non-scalar matrices (and H is non-abelian).

Let h be any nontrivial element of prime power order in M represented as in (2) by the diagonal matrix $\operatorname{diag}(a, b)$ on E, with h not a scalar (i.e., $a \neq b$). Note that the only h-stable 1-dimensional subspaces of E are E_1 and E_2. Then

$$[e_1 + e_2, h] = (a - 1)e_1 + (b - 1)e_2 = v.$$

Since $[e_1 + e_2, h] \in \langle h, h^{e_1 + e_2} \rangle$ whereas q^2 does not divide the latter group's order, v must be an eigenvector for h. This forces one of a, b to equal 1. Choosing notation so that $\langle h \rangle$ acts (faithfully) as $\langle \operatorname{diag}(a, 1) \rangle$, a must be a primitive $|h|$ root of unity. But in this case h^x acts as $\operatorname{diag}(1, a')$ for some primitive $|h|$ root of unity a'. Thus M contains the subgroup of all $\operatorname{diag}(a, a')$ for all combinations of $|h|$ roots of unity a, a'. Unless $|h| = 2$, since h was arbitrary, there is always a choice of diagonal matrix h where $a \neq a'$ and neither a nor a' is 1. Since every element of M is a product of prime power elements, this forces

the only non-scalar matrices in M are

$\operatorname{diag}(-1, 1)$ and $\operatorname{diag}(1, -1)$, and both are in M.

Since x has order 2, $D = \langle x, \operatorname{diag}(-1, 1) \rangle$ is dihedral of order 8, with $D \cap M$ a 4-group. This contradicts (3), and so completes the proof. $\square$

3. p-Fusion Modular Groups

Throughout this section p is a prime and G is a finite group. Before proving Theorem 2 we make some observations that, in particular, restrict the structure of the Sylow p-subgroups of p-fusion modular groups.

Lemma 3.1. *Let G be a p-fusion modular group and let P be a p-subgroup of G. Assume $x, x^g \in P$ for some $g \in G$, so that*

$$|\langle x, x^g \rangle : \langle x \rangle| = p \text{ or } 1.$$

Then the following hold:

(a) *If p is odd then $\langle x, x^g \rangle$ is either cyclic, abelian of type (p^k, p) or modular.*

(b) *If $p = 2$ then $\langle x, x^g \rangle$ is either cyclic, abelian of type $(2^k, 2)$, modular, or quaternion of order 8. Moreover, P does not contain dihedral or semidihedral subgroups of order ≥ 16, or generalized quaternion subgroups of order ≥ 32.*

(c) *If x has order p, then $\langle x^G \cap P \rangle$ is a weakly closed elementary abelian subgroup of P.*

(d) *The element g normalizes $\langle x^p \rangle$. Hence for all $x \in P$, $\langle x^p \rangle \trianglelefteq P$, and $\Omega_1(\langle x^p \rangle) \leq Z(P)$.*

(e) *P' centralizes $\langle x^p \mid x \in P \rangle$. Hence if $p = 2$, P' centralizes $\Phi(P)$.*

(f) *If $p = 2$ and P has exponent ≤ 4, then $\Phi(P) \leq \Omega_1(Z(P))$ and P has class at most 2.*

(g) *The simple groups $Sz(2^{2m-1})$ and $U_3(2^m)$ for $m \geq 2$ are not 2-fusion modular. The simple groups $Re(3^m)$ for $m \geq 3$ are not 3-fusion modular.*

Proof. The structure of p-groups that possess a cyclic subgroup of index p is established in Section 5.4 of [4], which gives (a) immediately. Assume $p = 2$ and $\langle x, x^g \rangle$ is not cyclic. When $|x| \geq 8$, the same reference yields the same families as in (a). If $|x| = 4$, then $|\langle x, x^g \rangle| \leq 8$, so the latter subgroup must be abelian of type (4,2) or quaternion. If $|x| = 2$, then the dihedral group $\langle x, x^g \rangle$ must be a Klein 4-group. Note further that semidihedral and dihedral 2-groups of order ≥ 16 each possess conjugate involutions x, x^g that generate a group of order 8, hence these cannot be subgroups of P. Likewise a generalized quaternion group of order ≥ 32 contains conjugate elements of order 4 that generate a generalized quaternion group of order 16, so P cannot contain such subgroups. This establishes (b).

If $|x| = p$ then by (a) and (b) any two G-conjugates of x lying in P generate an abelian subgroup of P of order at most p^2, which gives (c).

If $x, x^g \in P$ then in all cases $\langle (x^g)^p \rangle = \langle x^p \rangle$; and so g normalizes $\langle x^p \rangle$. Applying this for every $g \in P$ then gives (d).

Since P acts on each cyclic normal subgroup $\langle x^p \rangle$ for every x, P' centralizes these subgroups. When $p = 2$ the squares generate the Frattini subgroup. This proves (e), as well as (f) for P of exponent 4. If P has exponent 2, then $\Phi(P) = 1$, proving (f) in all cases.

Each of the simple groups $Sz(2^{2m-1})$ or $U_3(2^m)$ for $m \geq 2$ possesses elements x_1, x_2 of order 4 that are conjugate under the action of a Cartan element and with $x_1^2 \neq x_2^2$ (see [6], Section 6.5). Thus $|\langle x_1, x_2 \rangle| > 8$, a contradiction. By [9], the Ree groups likewise possess a pair of conjugate elements of order 9 that generate a 3-subgroup of order > 27, hence they are not 3-fusion modular. $\qquad\square$

Proposition 3.2. *Let G be a 2-fusion modular finite group and let T be a Sylow 2-subgroup of G. If T possesses an element y of order 8, then $Z = \langle (y^4)^G \cap T \rangle$ is a strongly closed elementary abelian subgroup of G in which all involutions are conjugate. Moreover, $|Z : C_Z(y^2)| \leq 2$.*

Proof. Under the hypotheses of the lemma let $x = y^2$, $z = y^4$ and $Z = \langle z^G \cap T \rangle$. By Lemma 3.1(c),

$$Z \text{ is abelian and weakly closed in } T.$$

It follows that two elements of Z are conjugate in G if and only if they are conjugate in $N_G(Z)$.

Let $N = N_G(Z)$ and let $\overline{N} = N/C_G(Z)$. By Frattini's Argument $N_N(T \cap C_G(Z))$ is transitive on $z^G \cap T$. If $x \in C_G(Z)$ then it follows from Lemma 3.1(d) that $z^G \cap T = \{z\}$, and so z is isolated in G as desired. Assume that $x \notin C_G(Z)$. By Lemma 3.1(d) applied to y we have $\langle x \rangle \trianglelefteq T$. Thus $[T, \langle x \rangle] \leq \langle z \rangle$, so by duality

$$x \text{ centralizes a hyperplane of } Z.$$

(Since x acts as an $\mathbb{F}_2$-transvection on Z, the conjugacy class of $\overline{x}$ in $\overline{N}$ is a set of 3-transpositions; we exploit an elementary consequence of this.) Let z^g be an arbitrary N-conjugate of z in Z different from z. The involutions $\overline{x}$ and $\overline{x}^g$ then generate a dihedral group that is not a 2-group. Thus $\overline{x}$ and $\overline{x}^g$ invert an element $\overline{h}$ of $\langle x, x^g \rangle$ of odd prime order. Since $\overline{x}$ and $\overline{x}^g$ both centralize hyperplanes of Z, $\overline{h}$ centralizes a subspace of codimension 2, so $\overline{h}$ has order 3. Moreover $\langle \overline{h} \rangle$ acts

transitively on the three involutions in the 2-dimensional space $[Z, \overline{h}]$, namely on $\{z, z^g, zz^g\}$. Since z^g was arbitrary, it follows that

> for any distinct $z_1, z_2 \in z^G \cap T$, the involution $z_1 z_2$ is conjugate to z.

Finally, if $w = z_1 z_2 \cdots z_k$ is any involution in Z written as a commuting product of G-conjugates z_i of z, it follows by induction on k that w is conjugate to z. Thus $z^G \cap T = Z - \{1\}$ and Z is strongly closed as desired. This completes the proof of the proposition. $\square$

Under the hypotheses of Theorem 2, if G possesses a strongly closed abelian subgroup, then by Goldschmidt's Theorem [3], G is a Goldschmidt simple group, and the theorem follows via Lemma 3.1(g). By Proposition 3.2, it remains to consider when a Sylow 2-subgroup T has exponent 4. By Lemma 3.1(f), T is class 2.

At this point we could invoke the Gorenstein-Gilman classification of simple groups with Sylow 2-subgroups of class 2, [5], noting that all the simple groups in their conclusion contain dihedral of order 8 subgroups generated by two conjugate involutions, hence are not 2-fusion modular. But a more self-contained, elementary argument circumvents this deep result.

Lemma 3.3. *Let N be a finite group, let $T \in Syl_2(N)$, let D be a normal 2-subgroup of N and let $\overline{N} = N/D$. Assume further that the following conditions hold:*

(a) $C_T(D) \le D$, *and*
(b) $\overline{N}/O_{2'}(\overline{N}) \cong L_2(2^n)$ *for some $n \ge 2$.*

Then N is not 2-fusion modular.

Proof. By way of contradiction assume N is 2-fusion modular and of minimal order with these properties. Since the 2-fusion modular property carries over to subgroups and quotient groups, we may obtain a contradiction by showing that some section of N satisfying hypotheses (a) and (b) is not 2-fusion modular. By minimality therefore $O_{2'}(N) = 1$; and it follows from (a) and (b) that $D = O_2(N) = F^*(N)$. Every odd order element of N acts faithfully on D, hence also acts faithfully on $D/\Phi(D)$. Replacing N by $N/\Phi(D)$, by minimality we may assume $\Phi(D) = 1$.

Let H be a $2'$-Hall complement in $N_N(T)$ and let $h \in H$ map to a generator for a Cartan subgroup in the $L_2(2^n)$ quotient $\overline{N}/O_{2'}(\overline{N})$. Let w be a 2-element of $N_N(\langle h \rangle)$ mapping to a Weyl element inverting the image of h in this $L_2(2^n)$ quotient. Then h also normalizes T^w. Since $\langle \overline{T}, \overline{T}^w \rangle$ maps onto $\overline{N}/O_{2'}(\overline{N})$, by minimality we may assume

$$(4) \qquad\qquad\qquad N = \langle T, T^w \rangle.$$

Since N is 2-fusion modular it follows as usual that h centralizes every element in $\Phi(T)$ hence it centralizes its subgroup $[D, T]$. Thus h also centralizes $[D, T^w]$, and hence centralizes $D_0 = [D, T] + [D, T^w]$. By construction both T and T^w normalize D_0, so by (4), $D_0 \trianglelefteq N$. Likewise, both T and T^w act trivially on D/D_0, hence so too does N. But now h acts trivially on both D_0 and D/D_0, and so acts trivially on $D = F^*(N)$. This contradiction completes the proof of the lemma. $\square$

Proof of Theorem 2: As before let G be a 2-fusion modular non-abelian simple group, let $T \in Syl_2(G)$, and assume T is not elementary abelian. By Lemma 3.1 and Proposition 3.2, T is of nilpotence class 2 and exponent 4. Let $x \in T$ be

an element of order 4, let $z = x^2$ and let $Z = \langle z^G \cap T \rangle$. By Lemma 3.1(c), Z is elementary abelian. Since z is not isolated in T, there is some member of Goldschmidt's conjugation family [2] where a nontrivial conjugation of z into T takes place. Moreover, since Z is a weakly closed abelian subgroup of T, $N_G(Z)$ is transitive on $z^G \cap T$. We may therefore take all members of the Goldschmidt conjugation family to lie in $N_G(Z)$. Thus by Goldschmidt's Theorem, [2], Z lies in some subgroup D of T with $N = N_G(D)$ satisfying the following properties:

(1) $N_T(D) \in Syl_2(N)$,
(2) $C_T(D) \leq D$,
(3) either $D = T$ or N/D is 2-isolated, and
(4) $z \neq z^g$ for some $g \in N$.

By Lemma 3.1(f) and hypothesis (2), $\Phi(T) \leq D$. Hence $T \leq N$ and N/D has elementary abelian Sylow 2-subgroups. Note also that because $z \neq z^g$, if x belonged to D then $\langle x, x^g \rangle$ would have order ≥ 16, a contradiction. Indeed, by the same reasoning

(5) no G-conjugate of x lies in D.

In particular, $D \neq T$. Since then N/D is 2-isolated, by Bender's Theorem (see [2]) it either has cyclic Sylow 2-subgroups (of order 2) or it possesses a normal subgroup L of odd index such that $L/O_{2,2'}(N) \cong L_2(2^n)$, for some $n \geq 2$. (The other Bender groups are not permissible by Lemma 3.1(g).) By Lemma 3.3 applied to L in place of N, the latter configuration does not hold, so

$$|T : D| = 2 \text{ and } N/D \text{ has a normal 2-complement.}$$

Let N_0 be the subgroup of N of index 2 containing D, so that $N = N_0\langle x \rangle$. Let $V : G \to N/N_0$ be the transfer homomorphism. We compute the value of $V(x)$ by familiar means ([4], Theorem 7.3.3):

$$(6) \qquad\qquad V(x) = \prod_{i=1}^{r} g_i^{-1} x^{n_i} g_i N_0$$

where $g_1, \ldots, g_r$ are representatives of the distinct orbits of $\langle x \rangle$ acting by left multiplication on the left cosets of N in G, and n_i is the cardinality of the ith orbit.

Since $x^4 = 1$, the factors on the right hand side of (6) coming from orbits of size 4 contribute the identity to the product. If g_i is a representative of an orbit of size 2, then $z^{g_i} = (x^2)^{g_i} \in N$. In this case $\langle z^{g_i} \rangle D$ is a 2-group whereas Z is a maximal 2-group generated by conjugates of z, hence $z^{g_i} \in Z \leq N_0$. Thus the orbits of size 2 contribute only the identity to right hand product in (6) as well. Finally, g_i is a representative of an orbit of size 1 if and only if $g_i^{-1} x g_i \in N$. We may adjust the representative by multiplying it on the right by an element of N so that $g_i^{-1} x g_i \in T$ as well. As usual, by the 2-fusion modular property of $\langle x, x^{g_i} \rangle$ we obtain that $g_i \in C_G(z)$. Thus the number of orbits of size 1 of $\langle x \rangle$ acting on G/N is the same as the number of distinct left cosets in the action of $C_G(z)$ on the set of left cosets of N, namely $|C_G(z) : C_G(z) \cap N|$. Since $T \leq C_G(z) \cap N$, the latter index is odd. By (5) each orbit of size 1 contributes a factor of xN_0 to the transfer product in (6). This proves $V(x)$ equals xN_0 raised to an odd power, hence is not the identity in N/N_0. This violates the simplicity of G, and so completes the proof. $\qquad\square$

Finally, we close with an easy consequence of Proposition 3.2.

Corollary 3.4. *Let G be a 2-fusion modular finite group with $O_{2'}(G) = 1$, and let T be a Sylow 2-subgroup of G. If T possesses an element y of order 8, then $y^4 \in O_2(G)$.*

Proof. Let $x = y^2$ and $z = y^4$. By Proposition 3.2, $Z = \langle z^G \cap T \rangle$ is a strongly closed elementary abelian subgroup all of whose involutions are conjugate in G. By Goldschmidt's Theorem [3], $G_1 = \langle Z^G \rangle$ is a direct product of an elementary abelian subgroup with simple Goldschmidt groups. Since all involutions of Z are conjugate, if G_1 had any non-abelian simple component, then G_1 would necessarily be a single simple component containing Z. By Lemma 3.1(g), G_1 is not a Suzuki or unitary group. The groups $L_2(q)$ with $q \equiv \pm 3 \pmod 8$, $Re(3^{2n+1})$, $L_2(4)$ and J_1 do not possess automorphisms of order 8. The groups $L_2(2^m)$ for $m \geq 3$ have elementary abelian Sylow 2-subgroups, and so x would induce an outer automorphism on G_1. In this case $m = 2n$ and x would lie in the field automorphism of order 2 coset (and all elements in this coset act the same way on the abelian Sylow 2-subgroup Z of G_1, i.e., as field automorphisms of order 2). It follows that if $G_1 \cong L_2(2^{2n})$ then x would centralize only an elementary abelian subgroup of order 2^n in Z, contrary to Proposition 3.2. This proves G_1 is a 2-group, as desired. □

Acknowledgment

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