THE BETTI NUMBERS FOR A FAMILY OF SOLVABLE LIE ALGEBRAS

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ABSTRACT. We give a property of symplectic quadratic Lie algebras that their Lie algebra of inner derivations has an invertible derivation. A family of symplectic quadratic Lie algebras is introduced to illustrate this situation. Finally, we calculate explicitly the Betti numbers of a family of solvable Lie algebras in two ways: using the cohomology of quadratic Lie algebras and applying a Pouseele's result on extensions of the one-dimensional Lie algebra by Heisenberg Lie algebras.

0. INTRODUCTION

Let $\mathfrak{g}$ be a complex Lie algebra endowed with a non-degenerate invariant symmetric bilinear form B, $\{X_1,...,X_n\}$ be a basis of $\mathfrak{g}$ and $\{\omega_1,...,\omega_n\}$ be its dual basis. Denote by $\{Y_1,...,Y_n\}$ the basis of $\mathfrak{g}$ defined by $B(Y_i,.) = \omega_i$, $1 \leq i \leq n$. Pinczon and Ushirobira discovered in [5] that the differential ∂ on $\bigwedge(\mathfrak{g}^*)$, the space of antisymmetric forms on $\mathfrak{g}$, is given by $\partial := -\{I,.\}$ where I is defined by:

$$I(X,Y,Z) = B([X,Y],Z), \ \forall X, Y, Z \in \mathfrak{g}$$

and $\{\ ,\ \}$ is the super Poisson bracket on $\bigwedge(\mathfrak{g}^*)$ defined by

$$\{\Omega,\Omega'\} = (-1)^{k+1}\sum_{i,j} B(Y_i,Y_j)\,\iota_{X_i}(\Omega)\wedge\iota_{X_j}(\Omega'), \ \forall\, \Omega \in \bigwedge^k(\mathfrak{g}^*), \ \Omega' \in \bigwedge(\mathfrak{g}^*).$$

In Section 1, by using this, we detail a result of Medina and Revoy in [4] that there is an isomorphism between the second cohomology group $H^2(\mathfrak{g},\mathbb{C})$ and $\mathrm{Der}_a(\mathfrak{g})/\operatorname{ad}(\mathfrak{g})$ where $\mathrm{Der}_a(\mathfrak{g})$ is the vector space of skew-symmetric derivations of $\mathfrak{g}$ and $\operatorname{ad}(\mathfrak{g})$ is its subspace of inner ones.

A well-known theorem of Jacobson says that a Lie algebra over a field of characteristic zero is nilpotent if it admits an invertible derivation, but on the contrary, there exist nilpotent Lie algebras that have no invertible derivation. It is also well-known that symplectic quadratic Lie algebras are nilpotent and hence so too are their algebras of inner derivations. We prove that the latter algebras have an invertible derivation. In particular, we have the following (Proposition 1.5).

THEOREM 1. *Let* $(\mathfrak{g},B,\omega)$ *be a symplectic quadratic Lie algebra. Consider the mapping* $\mathscr{D}:\operatorname{ad}(\mathfrak{g})\to\operatorname{ad}(\mathfrak{g})$ *defined by* $\mathscr{D}(\operatorname{ad}(X)) = \operatorname{ad}\left(\phi^{-1}\left(\iota_X(\omega)\right)\right)$ *with* $\phi:\mathfrak{g}\to\mathfrak{g}^*$, $\phi(X) = B(X,.)$*, then* $\mathscr{D}$ *is an invertible derivation of* $\operatorname{ad}(\mathfrak{g})$.

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The reader is referred to [2] for futher information about symplectic quadratic Lie algebras. A family of such algebras is given to illustrate this situation.

In Section 2, motivated by Corollary 4.4 in [4], we give the Betti numbers for a family of solvable quadratic Lie algebras defined as follows. For each $n \in \mathbb{N}$, let $\mathfrak{g}_{2n+2}$ denote the Lie algebra with basis $\{X_0, ..., X_n, Y_0, ..., Y_n\}$ and non-zero Lie brackets $[Y_0, X_i] = X_i$, $[Y_0, Y_i] = -Y_i$, $[X_i, Y_i] = X_0$, $1 \leq i \leq n$. Denote by $B^k(\mathfrak{g}_{2n+2}) = B^k(\mathfrak{g}_{2n+2}, \mathbb{C})$, $Z^k(\mathfrak{g}_{2n+2}) = Z^k(\mathfrak{g}_{2n+2}, \mathbb{C})$, $H^k(\mathfrak{g}_{2n+2}) = H^k(\mathfrak{g}_{2n+2}, \mathbb{C})$ and $b_k = b_k(\mathfrak{g}_{2n+2}, \mathbb{C})$. By computing on super Poisson brackets, our second result is the following.

THEOREM 2. *The k^{th} Betti numbers of $\mathfrak{g}_{2n+2}$ are given as follows:*

(1) *If k is even then one has*

$$b_k = \left| \binom{n}{\frac{k}{2}} \binom{n}{\frac{k}{2}} - \binom{n}{\frac{k-2}{2}} \binom{n}{\frac{k-2}{2}} \right|.$$

(2) *If k is odd then one has*
- *if $k < n+1$ then*

$$b_k = \binom{n}{\frac{k-1}{2}} \binom{n}{\frac{k-1}{2}} - \binom{n}{\frac{k-3}{2}} \binom{n}{\frac{k-3}{2}},$$

- *if $k = n+1$ then*

$$b_{n+1} = 2\binom{n}{\frac{n}{2}} \binom{n}{\frac{n}{2}} - 2\binom{n}{\frac{n+2}{2}} \binom{n}{\frac{n+2}{2}},$$

- *if $k > n+1$ then*

$$b_k = \binom{n}{\frac{k-1}{2}} \binom{n}{\frac{k-1}{2}} - \binom{n}{\frac{k+1}{2}} \binom{n}{\frac{k+1}{2}}.$$

Our method is direct and different from Pouseele's method given in [6] that we shall recall in Appendix 1. In Pouseele's method, the Betti numbers of $\mathfrak{g}_{2n+2}$ are derived from the Betti numbers of the $2n+1$-dimensional Lie algebra $\mathfrak{f}$ defined by $[Y, X_i] = X_i$ and $[Y, Y_i] = -Y_i$ for all $1 \leq i \leq n$.

For other results of Betti numbers for some families of nilpotent Lie algebras, we refer the reader to [1], [6] or [7].

1. A CHARACTERIZATION OF SYMPLECTIC QUADRATIC LIE ALGEBRAS

Let $\mathfrak{g}$ be a complex Lie algebra endowed with a non-degenerate invariant symmetric bilinear form B. In this case, we call the pair $(\mathfrak{g}, B)$ a *quadratic* Lie algebra. Denote by $\mathrm{Der}_a(\mathfrak{g})$ the vector space of skew-symmetric derivations of $\mathfrak{g}$, that is the vector space of derivations D satisfying $B(D(X), Y) = -B(X, D(Y))$ for all $X,\ Y \in \mathfrak{g}$, then $\mathrm{Der}_a(\mathfrak{g})$ is a Lie subalgebra of $\mathrm{Der}(\mathfrak{g})$.

Proposition 1.1. *There exists a Lie algebra isomorphism T between $\mathrm{Der}_a(\mathfrak{g})$ and the space $\{\Omega \in \bigwedge^2(\mathfrak{g}^*) \mid \{I, \Omega\} = 0\}$. This isomorphism induces an isomorphism from $\mathrm{ad}(\mathfrak{g})$ onto $\iota_{\mathfrak{g}}(I) = \{\iota_X(I) \in \bigwedge^2(\mathfrak{g}^*) \mid X \in \mathfrak{g}\}$.*

Proof. Let $D \in \mathrm{Der}_a(\mathfrak{g})$ and set $\Omega \in \bigwedge^2(\mathfrak{g}^*)$ by $\Omega(X, Y) = B(D(X), Y)$ for all $X,\ Y \in \mathfrak{g}$. Then D is a derivation of $\mathfrak{g}$ if and only if

$$\Omega([X, Y], Z) + \Omega([Y, Z], X) + \Omega([Z, X], Y) = 0$$

for all $X,\ Y,\ Z \in \mathfrak{g}$. It means $\{I,\Omega\}=0$. Define the map T from $\mathrm{Der}_a(\mathfrak{g})$ onto $\{\Omega \in \bigwedge^2(\mathfrak{g}^*) \mid \{I,\Omega\}=0\}$ by $T(D)=\Omega$ then T is a one-to-one correspondence.

Now we shall show that $T([D,D'])=\{T(D),T(D')\}$ for all $D,\ D' \in \mathrm{Der}_a(\mathfrak{g})$. Indeed, set $\Omega = T(D)$, $\Omega' = T(D')$ and fix an orthonormal basis $\{X_j\}_{j=1}^n$ of $\mathfrak{g}$. One has

$$\begin{aligned}
\{\Omega,\Omega'\}(X,Y) &= -\left(\sum_{j=1}^{n} \iota_{X_j}(\Omega)\wedge\iota_{X_j}(\Omega')\right)(X,Y)\\
&= -\sum_{j=1}^{n}\big(\Omega(X_j,X)\Omega'(X_j,Y)-\Omega(X_j,Y)\Omega'(X_j,X)\big)\\
&= -\sum_{j=1}^{n} B\big(B(D(X_j),X)D'(X_j)-B(D'(X_j),X)D(X_j),Y\big)\\
&= -\sum_{j=1}^{n} B\big(D'(D(X))-D(D'(X)),Y\big) = -B([D',D](X),Y).
\end{aligned}$$

That means $T([D,D'])=\{T(D),T(D')\}$ and then T is a Lie algebra isomorphism.

If $D=\mathrm{ad}(X_0)$ then $T(D)(X,Y)=B([X_0,X],Y)=I(X_0,Y,Z)=\iota_{X_0}(I)(X,Y)$. Therefore, $T(D)=\iota_{X_0}(I)$. □

Corollary 1.2. $\{\iota_X(I),\iota_Y(I)\}=\iota_{[X,Y]}(I)$.

Corollary 1.3. [4]
The cohomology group $H^2(\mathfrak{g},\mathbb{C}) \simeq \mathrm{Der}_a(\mathfrak{g},B)/\mathrm{ad}(\mathfrak{g})$.

Definition 1.4. A non-degenerate skew-symmetric bilinear form $\omega:\mathfrak{g}\times\mathfrak{g}\to\mathbb{C}$ is called a *symplectic structure* on $\mathfrak{g}$ if it satisfies

$$\omega([X,Y],Z)+\omega([Y,Z],X)+\omega([Z,X],Y)=0$$

for all $X,\ Y,\ Z\in\mathfrak{g}$.

A symplectic structure ω on a quadratic Lie algebra $(\mathfrak{g},B)$ is corresponding to a skew-symmetric invertible derivation D defined by $\omega(X,Y)=B(D(X),Y)$, for all $X,\ Y\in\mathfrak{g}$. As above, a symplectic structure is exactly a non-degenerate 2-form ω satisfying $\{I,\omega\}=0$. If $\mathfrak{g}$ has a such ω then we call $(\mathfrak{g},B,\omega)$ a *symplectic* quadratic Lie algebra.

For symplectic quadratic Lie algebras, the reader can refer to [2] for more details. Here we give a following property.

Proposition 1.5. *Let* $(\mathfrak{g},B,\omega)$ *be a symplectic quadratic Lie algebra. Consider the mapping* $\mathscr{D}:\mathrm{ad}(\mathfrak{g})\to\mathrm{ad}(\mathfrak{g})$ *defined by* $\mathscr{D}(\mathrm{ad}(X))=\mathrm{ad}\left(\phi^{-1}\left(\iota_X(\omega)\right)\right)$ *with* $\phi:\mathfrak{g}\to\mathfrak{g}^*$, $\phi(X)=B(X,.)$*, then* $\mathscr{D}$ *is an invertible derivation of* $\mathrm{ad}(\mathfrak{g})$.

Proof. As above we have $\{I,\omega\}=0$ and then $\iota_X(\{I,\omega\})=0$ for all $X\in\mathfrak{g}$. It implies $\{\iota_X(I),\omega\}=\{I,\iota_X(\omega)\}$ for all $X\in\mathfrak{g}$. Note that if X is nonzero, since ω is non-degenerate then $\iota_X(\omega)$ is non trivial. Set $Y=\phi^{-1}(\iota_X(\omega))$ then $\{I,\iota_X(\omega)\}=\iota_Y(I)$ and therefore this defines an inner derivation. Let D be the derivation corresponding to ω then one has $[\mathrm{ad}(X),D]=\mathrm{ad}(Y)$.

Let $\mathrm{ad}(X)\in\mathrm{ad}(\mathfrak{g})$. Set $\alpha=\phi(X)$. Since ω is non-degenerate then there exists an element $Y\in\mathfrak{g}$ such that $\alpha=\iota_Y(\omega)$. In this case, $\mathscr{D}(\mathrm{ad}(Y))=\mathrm{ad}(X)$. That means $\mathscr{D}$ onto and therefore it is bijective. □

Next, we give a family of symplectic quadratic Lie algebras that has been defined in [3] as follows.

Example 1.6. Let $p \in \mathbb{N} \setminus \{0\}$. We denote the *Jordan block of size* p by $J_1 := (0)$ and for $p \geq 2$,

$$J_p := \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \dots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

For $p \geq 2$, we consider $\mathfrak{q} = \mathbb{C}^{2p}$ with a basis $\{X_i,\ Y_i\}$, $1 \leq i \leq p$, and equipped with a bilinear form B satisfying $B(X_i, X_j) = B(Y_i, Y_j) = 0$ and $B(X_i, Y_j) = \delta_{ij}$. Let $C : \mathfrak{q} \to \mathfrak{q}$ with matrix

$$C = \begin{pmatrix} J_p & 0 \\ 0 & -{}^tJ_p \end{pmatrix}$$

in the given basis. Then $C \in \mathfrak{o}(2p)$.

Let $\mathfrak{h} = \mathbb{C}^2$ and $\{X_0, Y_0\}$ be a basis of $\mathfrak{h}$. Define on the vector space $\mathfrak{j}_{2p} = \mathfrak{q} \oplus \mathfrak{h}$ the Lie bracket $[Y_0, X] = C(X)$, $[X, Y] = B(C(X), Y)X_0$ and the bilinear form $\overline{B}(X_0, Y_0) = 1$, $\overline{B}(X_0, X_0) = \overline{B}(Y_0, Y_0) = \overline{B}(X_0, X) = \overline{B}(Y_0, X) = 0$ and $\overline{B}(X, Y) = B(X, Y)$ for all $X,\ Y \in \mathfrak{q}$. So $\mathfrak{j}_{2p}$ is a nilpotent Lie algebra and it will be called a $2p+2$-dimensional *nilpotent Jordan-type* Lie algebra.

Denote by $\{\alpha,\ \alpha_1, ...,\ \alpha_p,\ \beta,\ \beta_1, ...,\ \beta_p\}$ the dual basis of $\{X_0, ..., X_p, Y_0, ..., Y_p\}$ then $I = \beta \wedge \sum_{i=1}^{p-1} \alpha_{i+1} \wedge \beta_i$. In this case, we choose $\omega = \alpha \wedge \beta + \sum_{i=1}^{p} i\alpha_i \wedge \beta_i$ then $\{I, \omega\} = 0$ and therefore $(\mathfrak{j}_{2p}, B, \omega)$ is a symplectic quadratic Lie algebra. Notice that if we define $\mathscr{D}(\mathrm{ad}(Y_0)) = -\mathrm{ad}(Y_0)$, $\mathscr{D}(\mathrm{ad}(X_i)) = i\,\mathrm{ad}(X_i)$ and $\mathscr{D}(\mathrm{ad}(Y_i)) = -i\,\mathrm{ad}(Y_i)$ then $\mathscr{D}$ is an invertible derivation of $\mathrm{ad}(\mathfrak{j}_{2p})$.

2. The Betti numbers for a family of solvable quadratic Lie algebras

For each $n \in \mathbb{N}$, let $\mathfrak{g}_{2n+2}$ denote the Lie algebra with basis $\{X_0, ..., X_n, Y_0, ..., Y_n\}$ and non-zero Lie brackets $[Y_0, X_i] = X_i$, $[Y_0, Y_i] = -Y_i$, $[X_i, Y_i] = X_0$, $1 \leq i \leq n$. Then $\mathfrak{g}$ is quadratic with invariant bilinear form B given by $B(X_i, Y_i) = 1$, $0 \leq i \leq n$, zero otherwise.

Let $\{\alpha,\ \alpha_1, ...,\ \alpha_n,\ \beta,\ \beta_1, ...,\ \beta_n\}$ be the dual basis of $\{X_0, ..., X_n, Y_0, ..., Y_n\}$ and set $V = \mathrm{span}\{\alpha_i\}$, $W = \mathrm{span}\{\beta_i\}$, $1 \leq i \leq n$. It is easy to check that the associated 3-form of $\mathfrak{g}_{2n+2}$:

$$I = \beta \wedge \sum_{i=1}^{n} \alpha_i \wedge \beta_i.$$

Denote by $\Omega_n := \sum_{i=1}^{n} \alpha_i \wedge \beta_i$ then one has

$$B^2(\mathfrak{g}_{2n+2}) = \{\iota_X(I) |\ X \in \mathfrak{g}_{2n+2}\} = \mathrm{span}\{\beta \wedge \alpha_i, \beta \wedge \beta_i, \Omega_n \mid 1 \leq i \leq n\}.$$

If $n = 1$ then by we can directly calculate that $H^2(\mathfrak{g}_4) = \{0\}$. If $n > 1$, we have the non-zero super Poisson brackets:

(i) $\{I, \alpha \wedge \alpha_i\} = \alpha_i \wedge \Omega_n - \alpha \wedge \beta \wedge \alpha_i$ and $\{I, \alpha \wedge \beta_i\} = \beta_i \wedge \Omega_n + \alpha \wedge \beta \wedge \beta_i$,
(ii) $\{I, \alpha \wedge \beta\} = I$,
(iii) $\{I, \alpha_i \wedge \alpha_j\} = 2\beta \wedge \alpha_i \wedge \alpha_j$ and $\{I, \beta_i \wedge \beta_j\} = -2\beta \wedge \beta_i \wedge \beta_j$.

It results that $Z^2(\mathfrak{g}_{2n+2}) = \mathrm{span}\left\{\beta \wedge \alpha_i, \beta \wedge \beta_i, \alpha_i \wedge \beta_j \mid 1 \leq i, j \leq n\right\}$ and then the second cohomology group $H^2(\mathfrak{g}_{2n+2}) = \mathrm{span}\left\{[\alpha_i \wedge \beta_j]\right\} / \mathrm{span}\left\{[\sum_{i=1}^{n} \alpha_i \wedge \beta_i]\right\}$, where $1 \leq i, j \leq n$. So we recover the result of Medina and Revoy in [4] obtained by describing the space $\mathrm{Der}_a(\mathfrak{g}_{2n+2})$ that $b_2 = n^2 - 1$.

To get the Betti numbers b_k for $k \geq 3$, we need the following lemma.

Lemma 2.1. *The map* $\{\Omega_n, .\} : \bigwedge^k(V) \otimes \bigwedge^m(W) \to \bigwedge^k(V) \otimes \bigwedge^m(W)$ *with* $k, m \geq 0$ *is a vector space isomorphism if* $k \neq m$ *and* $\{\Omega_n, \bigwedge^k(V) \otimes \bigwedge^k(W)\} = \{0\}$.

Proof. We have $\{\Omega_n, \alpha_{i_1} \wedge \ldots \wedge \alpha_{i_k}\} = k\alpha_{i_1} \wedge \ldots \wedge \alpha_{i_k}$, $\{\Omega_n, \beta_{i_1} \wedge \ldots \wedge \beta_{i_m}\} = -m \wedge \beta_{i_1} \wedge \ldots \wedge \beta_{i_m}$ and $\{\Omega_n, \alpha_{i_1} \wedge \ldots \wedge \alpha_{i_k} \wedge \beta_{j_1} \wedge \ldots \wedge \beta_{j_m}\} = (k-m)\alpha_{i_1} \wedge \ldots \wedge \alpha_{i_k} \wedge \beta_{j_1} \wedge \ldots \wedge \beta_{j_m}$ then the result follows. □

By a straightforward computation on super Poisson brackets we have the following corollary.

Corollary 2.2. *The restrictions of the differential* ∂ *from* $\alpha \wedge \bigwedge^i(V) \otimes \bigwedge^j(W)$ *onto* $\Omega_n \wedge \bigwedge^i(V) \otimes \bigwedge^j(W) \oplus \alpha \wedge \beta \wedge \bigwedge^i(V) \otimes \bigwedge^j(W)$ *and from* $\bigwedge^i(V) \otimes \bigwedge^j(W)$ *onto* $\beta \wedge \bigwedge^i(V) \otimes \bigwedge^j(W)$ *with* $i, j \geq 0$, $i \neq j$ *are vector space isomorphisms.*

Let us now give the cases for which $\ker(\partial)$ can be obtained. The following lemma is easy:

Lemma 2.3. *We have* $\partial\left(\bigwedge^i(V) \otimes \bigwedge^i(W)\right) = \partial\left(\beta \wedge \bigwedge^i(V) \otimes \bigwedge^j(W)\right) = \{0\}$ *with* $i, j \geq 0$. *Moreover,* $\partial\left(\alpha \wedge \beta \wedge \bigwedge^i(V) \otimes \bigwedge^j(W)\right) \subset \partial\left(\bigwedge^{i+1}(V) \otimes \bigwedge^{j+1}(W)\right)$ *for all* $i,\ j \geq 0$, $i \neq j$ *and*

(i) $\partial\left(\alpha \wedge \beta \wedge \bigwedge^i(V) \otimes \bigwedge^i(W)\right) = \beta \wedge \Omega_n \wedge \bigwedge^i(V) \otimes \bigwedge^i(W)$,
(ii) $\partial\left(\alpha \wedge \bigwedge^i(V) \otimes \bigwedge^i(W)\right) = \Omega_n \wedge \bigwedge^i(V) \otimes \bigwedge^i(W)$.

By the reason shown in (i) and (ii) of Lemma 2.3 we set the map

$$\phi_{k_1,k_2,n} : \bigwedge^{k_1}(\alpha_1, \ldots, \alpha_n) \otimes \bigwedge^{k_2}(\beta_1, \ldots, \beta_n) \to \bigwedge^{k_1+1}(\alpha_1, \ldots, \alpha_n) \otimes \bigwedge^{k_2+1}(\beta_1, \ldots, \beta_n)$$

defined by $\phi_{k_1,k_2,n}(\omega) = \Omega_n \wedge \omega$ then we have the following result.

Proposition 2.4.

(i) *If k is even then*

$$\dim\ker(\partial_k) = \binom{n}{\frac{k}{2}}\binom{n}{\frac{k}{2}} + \sum_{i=0}^{k-1}\binom{n}{i}\binom{n+1}{k-1-i} + \dim\ker\phi_{\frac{k-2}{2},\frac{k-2}{2},n} - \binom{n}{\frac{k-2}{2}}\binom{n}{\frac{k-2}{2}}.$$

(ii) *If k is odd then*

$$\dim\ker(\partial_k) = \dim\ker\phi_{\frac{k-1}{2},\frac{k-1}{2},n} + \sum_{i=0}^{k-1}\binom{n}{i}\binom{n+1}{k-1-i}.$$

Using the formula $b_k(\mathfrak{g}_{2n+2}) = \dim\ker(\partial_k) + \dim\ker(\partial_{k-1}) - \binom{2n+2}{k-1}$, the binomial identity

$$\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$$

and the formula

$$\sum_{i=0}^{k}\binom{n}{i}\binom{n}{k-i} = \binom{2n}{k}$$

we obtain the following corollary.

Corollary 2.5. *The k^{th} Betti numbers of $\mathfrak{g}_{2n+2}$ are given as follows:*

(i) *If k is even then*

$$b_k(\mathfrak{g}_{2n+2}) = \binom{n}{\frac{k}{2}}\binom{n}{\frac{k}{2}} + 2\dim\ker\phi_{\frac{k-2}{2},\frac{k-2}{2},n} - \binom{n}{\frac{k-2}{2}}\binom{n}{\frac{k-2}{2}}.$$

(ii) *If k is odd then*

$$b_k(\mathfrak{g}_{2n+2}) = \binom{n}{\frac{k-1}{2}}\binom{n}{\frac{k-1}{2}} + \dim\ker\phi_{\frac{k-1}{2},\frac{k-1}{2},n} + \dim\ker\phi_{\frac{k-3}{2},\frac{k-3}{2},n} - \binom{n}{\frac{k-3}{2}}\binom{n}{\frac{k-3}{2}}.$$

Hence, it remains to compute $\dim\ker\left(\phi_{k,k,n}\right)$. Consider the power $\phi^m_{k_1,k_2,n}$ of the map $\phi_{k_1,k_2,n}$ and let

$$K(m,k_1,k_2,n) = \dim\ker\left(\phi^m_{k_1,k_2,n}\right)$$

then one has:

Lemma 2.6.

(i) *The map*

$$\theta^m_{k_1,k_2,n+1} : \ker\left(\phi^{m+1}_{k_1-1,k_2-1,n}\right) \oplus \ker\left(\phi^m_{k_1-1,k_2,n}\right) \oplus \ker\left(\phi^m_{k_1,k_2-1,n}\right)$$
$$\oplus \ker\left(\phi^{m-1}_{k_1,k_2,n}\right) \to \ker\left(\phi^m_{k_1,k_2,n+1}\right)$$

defined by

$$\theta^m_{k_1,k_2,n+1}(\omega_1,\omega_2,\omega_3,\omega_4) = \alpha_{n+1}\wedge\beta_{n+1}\wedge\omega_1 + \alpha_{n+1}\wedge\omega_2 + \beta_{n+1}\wedge\omega_3$$
$$+\omega_4 - \frac{1}{m}\phi_{k_1-1,k_2-1,n}(\omega_1)$$

is a vector space isomorphism.

(ii) $K(m,k_1,k_2,n) = K(m+1,k_1-1,k_2-1,n-1) + K(m,k_1-1,k_2,n-1) + K(m,k_1,k_2-1,n-1) + K(m-1,k_1,k_2,n-1)$.

Proof.

(i) The map $\theta^m_{k_1,k_2,n+1}$ is clearly injective. To prove $\theta^m_{k_1,k_2,n+1}$ surjective, let us consider $\omega \in \bigwedge^{k_1}(\alpha_1,\dots,\alpha_{n+1}) \otimes \bigwedge^{k_2}(\beta_1,\dots,\beta_{n+1})$ such that $\Omega^m_{n+1}\wedge\omega = 0$. Observe that ω can be written in the form $\omega = \alpha_{n+1}\wedge\beta_{n+1}\wedge\omega_1 + \alpha_{n+1}\wedge\omega_2 + \beta_{n+1}\wedge\omega_3 + \omega_4$ where $\omega_1 \in \bigwedge^{k_1-1}(\alpha_1,\dots,\alpha_n)\otimes\bigwedge^{k_2-1}(\beta_1,\dots,\beta_n)$, $\omega_2 \in \bigwedge^{k_1-1}(\alpha_1,\dots,\alpha_n)\otimes\bigwedge^{k_2}(\beta_1,\dots,\beta_n)$, $\omega_3 \in \bigwedge^{k_1}(\alpha_1,\dots,\alpha_n)\otimes\bigwedge^{k_2-1}(\beta_1,\dots,\beta_n)$ and $\omega_4 \in \bigwedge^{k_1}(\alpha_1,\dots,\alpha_n)\otimes\bigwedge^{k_2}(\beta_1,\dots,\beta_n)$. By $\Omega^m_{n+1}\wedge\omega = 0$, we obtain $\Omega^m_n\wedge\omega_2 = \Omega^m_n\wedge\omega_3 = \Omega^m_n\wedge\omega_4 = 0$, $\Omega^m_n\wedge\omega_1 = -m\Omega^{m-1}_n\wedge\omega_4$. It implies $\Omega^{m+1}_n\wedge\omega_1 = 0$ and then $\omega_1 \in \ker\left(\phi^{m+1}_{k_1-1,k_2-1,n}\right)$. Moreover, $\Omega_n\wedge\omega_1 + m\omega_4 \in \ker\left(\phi^{m-1}_{k_1,k_2,n}\right)$ means

$$\theta^m_{k_1,k_2,n+1}\left(\omega_1,\omega_2,\omega_3,\omega_4 + \frac{1}{m}\phi_{k_1-1,k_2-1,n}(\omega_1)\right) = \omega.$$

(ii) The assertion (2) follows (1).

□

To calculate $K(m,k_1,k_2,n)$, we use the following boundary conditions from the definition of $\phi^m_{k_1,k_2,n}$ in which we assume $\phi^0_{k_1,k_2,n}$ is the identity map:

(1) $K(0,k_1,k_2,n) = 0$ for all $k_1,\ k_2,\ n \geq 0$.

(2) $K(m,0,0,n) = \begin{cases} 0, & \text{if } m \leq n, \\ 1, & \text{if } m > n. \end{cases}$

(3) $K(m,0,1,n) = K(m,1,0,n) = \begin{cases} 0, & \text{if } m = 0 \text{ or } n > m, \\ n, & \text{if } 1 \leq n \leq m. \end{cases}$

(4) $K(m,k_1,k_2,0) = \begin{cases} 1, & \text{if } m \geq 1, k_1 = k_2 = 0, \\ 0, & \text{otherwise.} \end{cases}$

By the condition (2) we extend $K(m,k_1,k_2,n) = 0$ for negative k_1 or k_2 and by the condition (1) we set the condition (5) by $K(-m,k_1,k_2,n) = -K(m,k_1-m,k_2-m,n)$.

Lemma 2.7.

$$K(m,k,k,n) = \sum_{p=0}^{n}\sum_{q=0}^{n} \binom{n}{p}\binom{n}{q} K(m+n-p-q, k-n+p, k-n+q, 0).$$

Proof. By induction on l, we prove that

$$K(m,k,k,n) = \sum_{p=0}^{l}\sum_{q=0}^{l} \binom{l}{p}\binom{l}{q} K(m+l-p-q, k-l+p, k-l+q, n-l).$$

Let $l = n$ to get the lemma. □

The Betti numbers of $\mathfrak{g}_{2n+2}$ is in the case $m = 1$. By the conditions (4) and (5) we reduce the following.

Corollary 2.8.

$$K(1,k,k,n) = \begin{cases} 0, & \text{if } k < \frac{1}{2}n, \\ \binom{n}{k}\binom{n}{k} - \binom{n}{k+1}\binom{n}{k+1}, & \text{if } k \geq \frac{1}{2}n. \end{cases}$$

Finally, by applying this formula we obtain the Betti number of $\mathfrak{g}_{2n+2}$ according to Corollary 2.5.

3. Appendix 1: Another way to get the Betti numbers of $\mathfrak{g}_{2n+2}$

In this part, we shall give another way to get the Betti numbers of $\mathfrak{g}_{2n+2}$. It is based on the following result.

Proposition 3.1. [6]

Let $\mathfrak{g}$ be an extension of the one-dimensional Lie algebra $\langle Z \rangle$ by the Heisenberg Lie algebra $\mathfrak{h}_{2n+1}$, for some n,

$$1 \longrightarrow \mathfrak{h}_{2n+1} \longrightarrow \mathfrak{g} \longrightarrow \langle Z \rangle \longrightarrow 0$$

such that $\mathfrak{g}$ acts trivially on the center $\mathfrak{z} = \langle W \rangle$ of $\mathfrak{h}_{2n+1}$. Let $\mathfrak{f} = \mathfrak{g}/\mathfrak{z}$. Then

$$b_k(\mathfrak{g}) = \begin{cases} b_k(\mathfrak{f}) & \text{for } k = 0 \text{ or } k = 1, \\ b_k(\mathfrak{f}) - b_{k-2}(\mathfrak{f}) & \text{for } 2 \leq k \leq n, \\ 2\,[b_{n+1}(\mathfrak{f}) - b_{n-1}(\mathfrak{f})] & \text{for } k = n+1, \\ b_{k-1}(\mathfrak{f}) - b_{k+1}(\mathfrak{f}) & \text{for } n+2 \leq k \leq 2n, \\ b_{k-1}(\mathfrak{f}) & \text{for } k = 2n+1 \text{ or } k = 2n+2. \end{cases}$$

It is easy to see that $\mathfrak{g}_{2n+2}$ is an extension of the one-dimensional Lie algebra $\langle Y_0 \rangle$ by $\mathfrak{h}_{2n+1}$. To calculate the Betti numbers of $\mathfrak{g}_{2n+2}$ it needs to find the Betti numbers of the $2n+1$-dimensional Lie algebra $\mathfrak{f}$ with a basis $\{Y, X_1, \ldots, X_n, Y_1, \ldots, Y_n\}$ and the Lie bracket

$$[Y, X_i] = X_i, \quad [Y, Y_i] = -Y_i$$

for all $1 \leq i \leq n$.

Let $\{Y^*, X_1^*, \ldots, X_n^*, Y_1^*, \ldots, Y_n^*\}$ be the dual basis of $\{Y, X_1, \ldots, X_n, Y_1, \ldots, Y_n\}$.

Proposition 3.2.

(1) *One has*

$$\partial_k\left(Y^*\wedge\left(\bigwedge^{k-1}(X_1^*,\dots,X_n^*,Y_1^*,\dots,Y_n^*)\right)\right)=0.$$

(2) *Assume $j+l=k$ then we have*

- *if $j=l$ then*

$$\partial_k\left(\bigwedge^{j}(X_1^*,\dots,X_n^*)\otimes\bigwedge^{l}(Y_1^*,\dots,Y_n^*)\right)=0,$$

- *if $j\neq l$ then*

$$\partial_k\left(\bigwedge^{j}(X_1^*,\dots,X_n^*)\otimes\bigwedge^{l}(Y_1^*,\dots,Y_n^*)\right)=Y^*\wedge\left(\bigwedge^{j}(X_1^*,\dots,X_n^*)\otimes\bigwedge^{l}(Y_1^*,\dots,Y_n^*)\right).$$

Proof. The assertion (1) is obvious. For (2), we use the following computation:

$$\partial_k\left(X_{i_1}^*\wedge\dots\wedge X_{i_j}^*\wedge Y_{r_1}^*\wedge\dots\wedge Y_{r_l}^*\right)=(j-k)Y^*\wedge X_{i_1}^*\wedge\dots\wedge X_{i_j}^*\wedge Y_{r_1}^*\wedge\dots\wedge Y_{r_l}$$

for all $1\le i_1<\dots<i_j\le n$ and $1\le r_1<\dots<r_l\le n$. □

The following corollary results.

Corollary 3.3. *The Betti numbers of $\mathfrak{f}$ are given as follows:*

$$b_k(\mathfrak{f})=\binom{n}{\left[\frac{k}{2}\right]}\binom{n}{\left[\frac{k}{2}\right]}$$

where $[x]$ denotes the integer part of x.

Applying this corollary, we have

$$b_k(\mathfrak{g}_{2n+2})=\begin{cases}1 & \text{for } k=0 \text{ or } k=1,\\ \binom{n}{\left[\frac{k}{2}\right]}\binom{n}{\left[\frac{k}{2}\right]}-\binom{n}{\left[\frac{k-2}{2}\right]}\binom{n}{\left[\frac{k-2}{2}\right]} & \text{for } 2\le k\le n,\\ 2\binom{n}{\left[\frac{n+1}{2}\right]}\binom{n}{\left[\frac{n+1}{2}\right]}-2\binom{n}{\left[\frac{n-1}{2}\right]}\binom{n}{\left[\frac{n-1}{2}\right]} & \text{for } k=n+1,\\ \binom{n}{\left[\frac{k-1}{2}\right]}\binom{n}{\left[\frac{k-1}{2}\right]}-\binom{n}{\left[\frac{k+1}{2}\right]}\binom{n}{\left[\frac{k+1}{2}\right]} & \text{for } n+2\le k\le 2n,\\ 1 & \text{for } k=2n+1 \text{ or } k=2n+2.\end{cases}$$

and then Theorem 2 is obtained.

4. Appendix 2: The second cohomology group of a family of nilpotent Lie algebras

In this appendix, in the progress of our work, we give the second cohomology of a family of nilpotent Lie algebras that are double extensions of an Abelian Lie algebra (see [3] for more details about these Lie algebras).

Let us denote $\mathfrak{g}_{4n+2}$ a 2-nilpotent quadratic Lie algebra of dimension $4n+2$ spanned by $\{X,X_1,\dots,X_{2n},Y,Y_1,\dots,Y_{2n}\}$ where the Lie bracket is defined by $[Y,Y_{2i-1}]=X_{2i}$, $[Y,Y_{2i}]=-X_{2i-1}$, $[Y_{2i-1},Y_{2i}]=X$ and the bilinear form is given by $B(X,Y)=B(X_i,Y_i)=1$, zero otherwise. Let $\{\alpha,\alpha_i,\beta,\beta_i\}$ be the dual basis of $\{X,X_i,Y,Y_i\}$. We can check that the associated 3-form I of $\mathfrak{g}_{4n+2}$ is $I=\beta\wedge\Omega$ where $\Omega=\beta_1\wedge\beta_2+\beta_3\wedge\beta_4+\dots+\beta_{2n-1}\wedge\beta_{2n}$. Therefore, it is easy to see that $\iota_{\mathfrak{g}_{4n+2}}(I)=\mathrm{span}\{\Omega,\beta\wedge\beta_i\}$ for all $1\le i\le 2n$. We have the following proposition.

Proposition 4.1. $\dim(H^2(\mathfrak{g}_{4n+2},\mathbb{C})) = 8$ *if* $n = 1$ *and* $\dim(H^2(\mathfrak{g}_{4n+2},\mathbb{C})) = 5n^2 + n$ *if* $n > 1$.

Proof. First we need describe $\ker(\partial_2)$. Let V be the space spanned by $\{\beta,\beta_1,...,\beta_{2n}\}$ then $\{I,\omega\} = 0$ for all $\omega \in V \wedge V$. By a straightforward computation, we have

(1) $\{I,\beta\wedge\alpha_i\} = \{I,\alpha_{2i-1}\wedge\beta_{2i}\} = \{I,\alpha_{2i}\wedge\beta_{2i-1}\} = 0$,
(2) $\{I,\alpha\wedge\beta\} = I$,
(3) $\{I,\alpha\wedge\beta_{2i-1}\} = \beta_{2i-1}\wedge\Omega$, $\{I,\alpha\wedge\beta_{2i}\} = \beta_{2i}\wedge\Omega$,
(4) $\{I,\alpha\wedge\alpha_{2i-1}\} = \alpha_{2i-1}\wedge\Omega+\beta\wedge\beta_{2i}\wedge\alpha$, $\{I,\alpha\wedge\alpha_{2i}\} = \alpha_{2i}\wedge\Omega-\beta\wedge\beta_{2i-1}\wedge\alpha$,
(5) $\{I,\alpha_{2i-1}\wedge\alpha_{2j}\} = -\beta\wedge\beta_{2i}\wedge\alpha_{2j}-\beta\wedge\beta_{2j-1}\wedge\alpha_{2i-1}$, $\{I,\alpha_{2i}\wedge\alpha_{2j}\} = \beta\wedge\beta_{2i-1}\wedge\alpha_{2j}-\beta\wedge\beta_{2j-1}\wedge\alpha_{2i}$,
(6) $\{I,\alpha_{2i-1}\wedge\beta_{2j}\} = -\{I,\alpha_{2j-1}\wedge\beta_{2i}\} = -\beta\wedge\beta_{2i}\wedge\beta_{2j}$, $i\neq j$,
(7) $\{I,\alpha_{2i-1}\wedge\beta_{2j-1}\} = \{I,\alpha_{2j}\wedge\beta_{2i}\} = -\beta\wedge\beta_{2i}\wedge\beta_{2j-1}$,
(8) $\{I,\alpha_{2i}\wedge\beta_{2j-1}\} = -\{I,\alpha_{2j}\wedge\beta_{2i-1}\} = \beta\wedge\beta_{2i-1}\wedge\beta_{2j-1}$, $i\neq j$.

As a consequence, if $n = 1$ then it is direct that

$$\ker(\partial_2) = V\wedge V\oplus\mathrm{span}\{\beta\wedge\alpha_1,\beta\wedge\alpha_2,\alpha\wedge\beta-\alpha_1\wedge\beta_1,\alpha_1\wedge\beta_2,\alpha_1\wedge\beta_1-\alpha_2\wedge\beta_2,\alpha_2\wedge\beta_1\}.$$

Therefore, we obtain $\dim(H^2(\mathfrak{g}_{4n+2},\mathbb{C})) = 8$.

In the case $n > 1$ then Ω is indecomposable. Hence,

$$\ker(\partial_2) = V\wedge V\oplus\mathrm{span}\{\beta\wedge\alpha_{2i-1},\beta\wedge\alpha_{2i},\alpha\wedge\beta-\sum_{i=1}^{n}\alpha_{2i-1}\wedge\beta_{2i-1},$$
$$\alpha_{2i-1}\wedge\beta_{2j}+\alpha_{2j-1}\wedge\beta_{2i},\alpha_{2i-1}\wedge\beta_{2j-1}-\alpha_{2j}\wedge\beta_{2i},\alpha_{2i}\wedge\beta_{2j-1}+\alpha_{2j}\wedge\beta_{2i-1}\}$$

with $1\leq i,j\leq n$ and it is easy to check that $\dim(H^2(\mathfrak{g}_{4n+2},\mathbb{C})) = 5n^2+n$. □

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