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Some Families of *S*-Graphs

CHONG-YUN CHAO AND BRUCE MONTGOMERY
Department of Mathematics, University of Pittsburgh, Pittsburgh, PA 15260, U.S.A.

Abstract. An *S*-graph is a graph G which is isomorphic to the complement of G^2. Here, we present three infinite families of *S*-graphs.

1. Introduction

In [2], Schuster proposed the problem of determining the graphs G satisfying $G^2 \cong \overline{G}$, or $\overline{G^2} \cong G$, where G is a finite, undirected, and simple graph without loops, G^2 is the graph with vertex set $V(G^2) = V(G)$ for which (u, v) is an edge in G^2 if and only if the distance between u and v is 1 or 2 in G, i.e., $(u, v) \in E(G^2)$ if and only if $1 \le d_G(u, v) \le 2$, and $\overline{G}$ is the complementary graph of G. Such graphs are called *S*-graphs.

To determine all *S*-graphs seems to be very difficult. Zhou and Liu stated in [1] that a long search for *S*-graphs strongly suggested symmetry as a necessary property. This led them to consider circulant graphs. A circulant graph is a (undirected) graph whose adjacency matrix is a symmetric circulant. Let $Z_n = \{0, 1, \cdots, n-1\}$ be the additive abelian groups of integers modulo n, and H be a subset of Z_n with $0 \notin H$. Then the Cayley graph $C_{Z_n, H}$ is the undirected graph with $V(C_{Z_n, H}) = Z_n$ and $E(C_{Z_n, H}) = \{(x,\ x+y) \ : \ x \in Z_n,\ y \in H$ and the addition is taken modulo $n\}$. Thus, the adjacency matrix of $X_{Z_n, H}$ is an $n \times n$ symmetric circulant with entries 0 and 1. Since a circulant matrix is determined by the entries on the 0-th row and a symmetric circulant is determined by the entries of the first part of the 0-th row, we use $C_n = \langle j_1, j_2, \cdots, j_r \rangle$ where $j_1, j_2, \cdots, j_r$ are either 0 or 1, and $0 \le r \le [n/2]$, to denote the adjacency matrix. For instance, the 9-cycle with the vertex set $\{0, 1, \cdots, 8\}$ and the edge set $\{(0,1), (1,2), \cdots, (8,0)\}$ is a Cayley graph $C_{Z_9, H}$ with $H = \{1\}$, and the 0-th row of its adjacency matrix a 9×9 symmetric circulant, is $(0,1,0,0,0,0,0,0,1)$. We denote this 9-cycle by $C_9\langle 1 \rangle$.

In [1], Zhou and Liu showed that if G is a circulant graph with n vertices and degree $2r$, and G is an S-graph, i.e., $G = C_n\langle j_1, j_2, \cdots, j_r\rangle$, then

$$4r+1 < n \leq 2r^2 + 4r + 1.$$

They also determined that, up to isomorphism, $C_7\langle 1\rangle$ is the only S-graph among all graphs $C_n\langle j_1\rangle$, and there are only two non-isomorphic S-graphs, $C_{14}\langle 1, 6\rangle$ and $C_{17}\langle 1, 4\rangle$ among all graphs $C_n\langle j_1, j_2\rangle$. Here we present three infinite families of S-graphs.

2. Three infinite families of *S*-graphs

Theorem 1.

(a) For each integer $m \geq 1$, there exists an S-graph $C_{7(2m+1)}\langle 1, 7-1, 7+1, \cdots, 7k-1, 7k+1, \cdots, 7m+1\rangle$ with $7(2m+1)$ vertices and degree $2(2m+1)$.
(b) For each integer $m \geq 1$, there exists an S-graph $C_{7(2m)}\langle 1, 7-1, 7+1, \cdots, 7k-1, 7k+1, \cdots, 7m-1\rangle$ with $7(2m)$ vertices and degree $2(2m)$.

Proof. We know that if $G = C_n\langle j_1, j_2, \cdots, j_r\rangle$, then, by using the definition of G^2, we have

$$G^2 = C_n\langle j_1, j_2, \cdots, j_r, 2j_1, 2j_2, \cdots, 2j_r, j_2 \mp j_1, j_3 \mp j_1, \cdots, j_r \mp j_1, j_3 \mp j_2, j_4 \mp j_2, \cdots, j_r \mp j_2, \cdots, j_r \mp j_{r-1}\rangle \quad (1)$$

where the operations are taken modulo n.

Since G is a vertex-transitive circulant graph, G^2 and $\overline{G^2}$ are vertex-transitive circulant graphs.

(a) The operations in the following are taken modulo $7(2m+1)$. For each integer $m \geq 0$, let $G = C_{7(2m+1)}\langle 1, 7-1, 7+1, \cdots, 7k-1, 7k+1, \cdots, 7m+1\rangle$ with $7(2m+1)$ vertices and degree $2(2m+1)$. The edges incident with the vertex 0 in G are $(0, 7k \mp 1)$ for $k = 1, 2, \cdots, 2m+1$.

In G^2, by using (1), the edges incident with the vertex 0 are:

$$(0, 7k \mp 1) \text{ for } k = 1, 2, \cdots, 2m+1,$$
$$(0, 7k \mp 2) \text{ for } k = 1, 2, \cdots, 2m+1, \text{ and}$$
$$(0, 7k) \text{ for } k = 1, 2, \cdots, 2m.$$

Then, in $\overline{G^2}$, the edges incident with the vertex 0 are:

$$(0, 7k \mp 3) \text{ for } k = 1, 2, \cdots, 2m+1.$$

Thus, $\overline{G^2} = C_{7(2m+1)}\langle 3, 7-3, 7+3, \cdots, 7k-3, 7k+3, \cdots, 7m+3\rangle$.

We claim that $G \cong \overline{G^2}$. For $m = 0, C_7\langle 1\rangle$ and $C_7\langle 3\rangle$ are 7-cycles and they are isomorphic. For $m \geq 1$, we define $\sigma : V(G) \to V(\overline{G^2})$ by $\sigma(j) = 4j = (7-3)j$ for $j = 0, 1, \cdots, 7(2m+1)-1$. Since $7(2m+1)$ and 4 are relatively prime, σ is bijective.

Let e be any edge in G. Since G is vertex-transitive, there is an automorphism θ of G such that $\theta e = (0, 7k-1)$ or $\theta e = (0, 7k+1)$, for some $k = 1, 2, \cdots, 2m+1$, is an edge in G. Then $(\sigma(0), \sigma(7k \mp 1)) = (0, (7-3)(7k \mp 1)) = (0, 7(4k \pm 1) \mp 3)$ is an edge in $\overline{G^2}$.

Let e' be a non-edge in G. Then for any automorphism of G such that $\theta(0) = 0$, $\theta e'$ is a non-edge in G, i.e., $\theta e' = (0, 7k-t)$ or $\theta e' = (0, 7k+t)$, for some $t = 0, 2, 3, 4, 5$, and for some $k = 1, 2, \cdots, 2m+1$, is a non-edge in G. Then $(\sigma 0, \sigma(7k \mp 1)) = (0, (7-3)(7k \mp t)) = (0, 7(4k \mp t) \pm 3t)$ is a non-edge in $\overline{G^2}$. Hence, σ is an isomorphism, and G is an S-graph.

(b) The operations in the following are taken modulo $7(2m)$. For $m \geq 1$, let $H = C_{7(2m)}\langle 1, 7-1, 7+1, \cdots, 7k-1, 7k+1, \cdots, 7(2m)-1\rangle$ with $7(2m)$ vertices and degree $2(2m)$. The edges incident with the vertex 0 in H are $(0, 7k \mp 1)$ for $k = 1, 2, \cdots, 2m$.

In H^2, by using (1), the edges incident with the vertex 0 are:

$$(0, 7k \mp 1) \text{ for } k = 1, 2, \cdots, 2m,$$
$$(0, 7k \mp 2) \text{ for } k = 1, 2, \cdots, 2m, \text{ and}$$
$$(0, 7k) \text{ for } k = 1, 2, \cdots, 2m-1.$$

Then, in $\overline{H^2}$, the edges incident with the vertex 0 are:

$$(0, 7k \mp 3) \text{ for } k = 1, 2, \cdots, 2m.$$

Thus, $\overline{H^2} = C_{7(2m)}\langle 3, 7-3, 7+3, \cdots, 7k-3, 7k+3, \cdots, 7m-3\rangle$.

We claim that $H \cong \overline{H^2}$. For $m \geq 1$, we write $7(2m)$ as $7(2^{i_1} p_2^{i_2} p_3^{i_3} \cdots p_d^{i_d})$ where $p_2, p_3, \cdots, p_d$ are primes, $2 < p_2 < p_3 < \cdots p_d$, and $2^{i_1} p_2^{i_2} p_3^{i_3} \cdots p_d^{i_d}$ is the prime power decomposition of $2m$. There are two cases to be considered.

Case 1. $p_2 = 3$. We define $\sigma : V(H) \to V(\overline{H^2})$ by $\sigma(j) = (7q+3)j$ for $j = 0, 1, \cdots, 2m$, where $q = 2^{i_1} p_3^{i_3} \cdots p_d^{i_d}$. Since $7(2m) = 7(2^{i_1} 3^{i_2} p_3^{i_3} \cdots p_d^{i_d})$ and $7q+3 = 7(2^{i_1} p_3^{i_3} \cdots p_d^{i_d}) + 3$ are relatively prime, σ is bijective.

Let e be any edge in H. Since H is vertex-transitive, there is an automorphism θ of H such that $\theta e = (0, 7k-1)$ or $\theta e = (0, 7k+1)$, for some $k = 1, 2, \cdots, 2m$, is an edge in H. Then $(\sigma(0), \sigma(7k \mp 1)) = (0, (7q+3)(7k \mp 1)) = (0, 7(7qk \mp q + 3k) \mp 3)$ is an edge in $\overline{H^2}$.

Let e' be any non-edge in H. Then for any automorphism θ of H such that $\theta(0) = 0$, $\theta e'$ is a non-edge in H, i.e., $\theta e' = (0, 7k - t)$ or $\theta e' = (0, 7k + t)$ for some $t = 2, 3, 4, 5$, and some $k = 1, 2, \cdots, 2m$. Then $(\sigma 0, \sigma(7k \mp t)) = (0, (7q+3)(7k \mp t))$ $= (0, 7(7qk \mp qt + 3k) \mp 3t)$ is a non-edge in $\overline{H^2}$ for all $t = 2, 3, 4, 5$ and all $k = 1, 2, \cdots, 2m$. Hence, σ is an isomorphism and H is an S-graph.

Case 2. $p_2 \neq 3$. We define $\sigma : V(H) \to V(\overline{H^2})$ by $\sigma(j) = 3j$ for $j = 0, 1, \cdots, 2m-1$. Since $7(2m)$ and 3 are relatively prime, σ is bijective. Similar to the proof in Case 1, σ is an isomorphism and H is an S-graph.

Example 1. All operations are taken modulo 21. Let $G = C_{7(2(1)+1)}\langle 1, 7-1, 7+1\rangle = C_{21}\langle 1, 6, 8\rangle$. Then G is a circulant graph with 21 vertices and degree 6. We shall use $\{1, 6, 8, 13, 15, 20\}$ to denote the set of vertices in G such that the edges $(0,1), (0,6), (0,8), (0,13), (0,15), \text{and } (0,20)$ are incident with the vertex 0. Then in G^2, by using (1), we have $\{1, 6, 8, 13, 15, 20, 2, 12, 16, 5, 9, 19, 7, 14\}$. In $\overline{G^2}$, we have $\{3, 4, 10, 11, 17, 18\} = \{3\cdot 7+3, 7-3, 7+3, 2\cdot 7-3, 2\cdot 7+3, 3\cdot 7-3\}$ $= \{7k \mp 3, k = 1, 2, 3\}$. Also, $\{3, 4, 10, 11, 17, 18\} = \{7-4,\ 3\cdot 7+4,\ 2\cdot 7-4,\ 7+4, 3\cdot 7-4,\ 2\cdot 7+4\}$ $= \{7k \mp 4,\ k = 1, 2, 3\}$. Since 4 and 21 are relatively prime, we define $\sigma : V(G) \to V(\overline{G^2})$ by $\sigma(i) = 4i$ for $i = 0, 1, \cdots, 20$, i.e.,

$$\sigma = (0)(1\ 4\ 16)(2\ 8\ 11)(3\ 12\ 6)(5\ 20\ 17)(7)(9\ 15\ 18)(10\ 19\ 13)(14)$$

Clearly, σ is an isomorphism of G onto $\overline{G^2}$. Thus, $C_{21}\langle 1, 6, 8\rangle \cong C_{21}\langle 3, 4, 10\rangle$, and G is an S-graph. In fact, if $H = C_{21}\langle 3, 4, 10\rangle$, then H is an S-graph with $\overline{H^2} = C_{21}\langle 2, 5, 9\rangle$. Also, if $K = C_{21}\langle 2, 5, 9\rangle$, then K is an S-graph with $\overline{K^2} = C_{21}\langle 1, 6, 8\rangle$.

Example 2. All operations are taken modulo 28. Let $G = C_{7(2(2))}\langle 1, 7-1, 7+1, 2\cdot 7-1\rangle$ $= C_{28}\langle 1, 6, 8, 13\rangle$. Then G is a circulant graph with 28 vertices and degree 8. In G, we have $\{1, 6, 8, 13, 15, 20, 22, 27\}$. Then in G^2, by using (1), we have

$\{1, 6, 8, 13, 15, 20, 22, 27, 2, 12, 16, 26, 5, 7, 9, 14, 19, 21, 23\}$. In $\overline{G^2}$, we have $\{3, 4, 10, 11, 17, 18, 24, 25\}$.

Since 3 and 28 are relatively prime, we define $\tau : V(G) \to V(\overline{G^2})$ by $\tau(i) = 3i$ for $i = 0, 1, \cdots, 27$, i.e.,

$$\tau = (0)(1\ 3\ 9\ 27\ 25\ 19)(2\ 6\ 18\ 26\ 22\ 10)(4\ 12\ 8\ 24\ 16\ 20)$$
$$(5\ 15\ 17\ 23\ 13\ 11)(7\ 21)(14).$$

Clearly, τ is an isomorphism of G onto $\overline{G^2}$. Thus, $C_{28}\langle 1, 6, 8, 13\rangle \cong C_{28}\langle 3, 4, 10, 11\rangle$, and G is an S-graph. We also have that if $H = C_{28}\langle 3, 4, 10, 11\rangle$, then H is an S-graph with $\overline{H^2} = C_{28}\langle 2, 5, 9, 12\rangle$. If $K = C_{28}\langle 2, 5, 9, 12\rangle$, then K is an S-graph with $\overline{K^2} = C_{28}\langle 1, 6, 8, 13\rangle$.

Theorem 2.

(a) *For each integer* $m \geq 0$, *there exists an S-graph*

$$C_{17(2m+1)}\langle 1, 4, \cdots, 17k-4, 17k-1, 17k+1, 17k+4, \cdots, 17m+1, 17m+4\rangle$$

with $17(2m+1)$ *vertices and degree* $4(2m+1)$.

(b) *For each integer* $m \geq 1$, *there exists an S-graph*

$$C_{17(2m)}\langle 1, 4, \cdots, 17k-4, 17k-1, 17k+1, 17k+4, \cdots, 17m-4, 17m-1\rangle$$

with $17(2m)$ *vertices and degree* $4(2m)$.

Proof.

(a) The operations in the following are taken modulo $17(2m+1)$. For $m \geq 0$, let $G = C_{17(2m+1)}\langle 1, 4, \cdots, 17k-4, 17k-1, 17k+1, 17k+4, \cdots, 17m+1, 17m+4\rangle$ with $17(2m+1)$ vertices and degree $4(2m+1)$. Then the edges incident with the vertex 0 in G are:

$$(0, 17k \mp 4) \text{ and } (0, 17k \mp 1) \quad \text{for} \quad k = 1, 2, \cdots, 2m+1.$$

In G^2, by using (1), the edges incident with the vertex 0 are:

$$(0, 17k \mp 1), (0, 17k \mp 2), (0, 17k \mp 3), (0, 17k \mp 4), (0, 17k \mp 5) \text{ and } (0, 17k \mp 8)$$

for $k = 1, 2, \cdots, 2m+1$ and $(0, 17k)$ for $k = 1, 2, \cdots, 2m$. Then, in $\overline{G^2}$, the edges incident with the vertex 0 are:

$$(0, 17k \mp 6) \text{ and } (0, 17\text{k} \mp 7) \text{ for } k = 1, 2, \cdots, 2m+1.$$

We claim that $G \cong \overline{G^2}$. For $m \geq 0$, we write $17(2m+1)$ as $17(p_1^{i_1}\, p_2^{i_2} \cdots\, p_d^{i_d})$, where $p_1^{i_1}, p_2^{i_2}, \cdots, p_d^{i_d}$ are distinct primes and $p_1^{i_1}\, p_2^{i_2} \cdots p_d^{i_d}$ is the prime power decomposition of $2m+1$. (If $2m+1 = 1$, then the prime power decomposition of $2m+1$ is taken as 1.) There are two cases to be considered.

Case 1. $17(2m+1)$ and 6 are relatively prime. We define $\sigma : V(G) \to V(\overline{G^2})$ by $\sigma(j) = 6j$ for $j = 0, 1, \cdots, 17(2m+1)-1$. Since $17(2m+1)$ and 6 are relatively prime, σ is bijective.

Let e be any edge in G. Since G is vertex-transitive, there is an automorphism θ of G such that $\theta e = (0, 17k-4), (0, 17k-1), (0, 17k+1)$ or $(0, 17k+4)$ for some $k = 1, 2, \cdots, 2m+1$. Then

$$\begin{aligned}
&(\sigma 0, \sigma(17k-4)) = (0, 17(6k) - 24) = (0, 17(6k-1) - 7),\\
&(\sigma 0, \sigma(17k-1)) = (0, 17(6k) - 6),\\
&(\sigma 0, \sigma(17k+1)) = (0, 17(6k) + 6), \quad \text{or}\\
&(\sigma 0, \sigma(17k+4)) = (0, 17(6k) + 24) = (0, 17(6k+1) + 7).
\end{aligned}$$

In any case, it is an edge in $\overline{G^2}$.

Let e' be any non-edge in G. Then for any automorphism θ of G such that $\theta(0) = 0$, $\theta e'$ is a non-edge in G, i.e., $\theta e' = (0, 17k \mp t)$ for some $t = 2, 3, 5, 6, 7, 8$, and for some $k = 1, 2, \cdots, 2m+1$. Then $(\sigma 0, \sigma(17k \mp t)) = (0, 17(6k) \mp 6t)$ is a non-edge in $\overline{G^2}$ for all $t = 2, 3, 5, 6, 7, 8$, and for all $k = 1, 2, \cdots, 2m+1$. Hence, σ is an isomorphism, and G is an S-graph.

Case 2. $17(2m+1) = 17(2^{i_1}\, 3^{i_2}\, p_3^{i_3} \cdots p_d^{i_d})$ and 6 are not relatively prime. We define $\sigma : V(G) \to V(\overline{G^2})$ by $\sigma(j) = (17q+6)j$ for $j = 0, 1, \cdots, 17(2m+1)-1$ where $q = p_3^{i_3}\, p_4^{i_4} \cdots p_d^{i_d}$. Since $17(2m+1)$ and $(17q+6)$ are relatively prime, σ is bijective.

Similar to Case 1, σ takes an edge in G to an edge in $\overline{G^2}$, and σ takes a non-edge in G to a non-edge in $\overline{G^2}$. Hence, σ is an isomorphism and G is an S-graph.

(b) The operations in the following are taken modulo $17(2m)$ with $m \geq 1$. Let $G = C_{17(2m)}\langle 1, 4, \cdots, 17k-4, 17k-1, 17k+1, 17k+4, \cdots, 17m-4, 17m-1 \rangle$ with $17(2m)$ vertices and degree $4(2m)$. Then the edges incident with the vertex 0 in G are:

$$(0, 17k \mp 4) \text{ and } (0, 17k \mp 1) \text{ for } k = 1, 2, \cdots, 2m.$$

In G^2, using (1), the edges incident with the vertex 0 are: $(0,17k \mp 1)$, $(0,17k \mp 2)$, $(0,17k \mp 3)$, $(0,17k \mp 4)$, $(0,17k \mp 5)$, and $(0,17k \mp 8)$ for $k = 1, 2, \cdots, 2m$, and $(0, 17k)$ for $k = 1, 2, \cdots, 2m-1$.

Then, in $\overline{G^2}$, the edges incident with the vertex 0 are:

$$(0, 17k \mp 6) \text{ and } (0, 17k \mp 7) \text{ for } k = 1, 2, \cdots, 2m.$$

We claim that $G \cong \overline{G^2}$. There are two cases to be considered:

Case 1. $17(2m)$ and 7 are relatively prime. We define $\sigma : V(G) \to V(\overline{G^2})$ by $\sigma(j) = 7j$ for $j = 0, 1, \cdots, 17(2m)-1.$. Since $17(2m)$ and 7 are relatively prime, σ is bijective.

Let e be an edge in G. Since G is vertex-transitive, there is an automorphism θ of G such that $\theta e = (0,17k-4)$, $(0,17k-1)$, $(0,17k+1)$, or $(0,17k+4)$ for some $k = 1, 2, \cdots, 2m$. Then

$$\begin{aligned}
(\sigma 0, \sigma(17k-4)) &= (0, 17(7k) - 28) = (0, 17(7k-2) + 6), \\
(\sigma 0, \sigma(17k-1)) &= (0, 17(7k) - 7), \\
(\sigma 0, \sigma(17k+1)) &= (0, 17(7k) + 7), \text{ or} \\
(\sigma 0, \sigma(17k+4)) &= (0, 17(7k) + 28) = (0, 17(7k+2) - 6).
\end{aligned}$$

In any case, it is an edge in $\overline{G^2}$.

Let e' be any non-edge in G. Then for any automorphism θ of G such that $\theta(0) = 0$, $\theta e'$ is a non-edge in G, i.e., $\theta e' = (0, 17k \mp t)$ for some $t = 2, 3, 5, 6, 7, 8$ and some $k = 1, 2, \cdots, 2m$. Then $(0, 17k \mp t) = (0, 17(7k) \mp 7t)$ is a non-edge in $\overline{G^2}$ for all $t = 2, 3, 5, 6, 7, 8$, and for all $k = 1, 2, \cdots, 2m$. Hence, σ is an isomorphism, and G is an S-graph.

Case 2. $17(2m) = 17(p_1^{i_1}\ p_2^{i_2}\ p_3^{i_3}\ 7^{i_4} \ldots p_d^{i_d})$ and 7 are not relatively prime where the prime power decomposition of $2m$ is $p_1^{i_{11}}\ p_2^{i_2}\ p_3^{i_3}\ 7^{i_4} \cdots p_d^{i_d}$. Let $q = (2m / 7^{i_4})$. We define $\sigma : V(G) \to V(\overline{G^2})$ by $\sigma(j) = (17q+7)j$ for $j = 0, 1, \cdots, 17(2m)-1$. Since $17(2m)$ and $(17q+7)$ are relatively prime, σ is bijective.

Similar to Case 1, σ takes an edge in G to an edge $\overline{G^2}$, and σ takes a non-edge in G to a non-edge in $\overline{G^2}$. Hence, σ is an isomorphism and G is an S-graph.

Theorem 3. *For each integer* $m \geq 0$, *there exists an S-graph*

$$C_{12(2^{m+1})}\langle 3, 5, 7, \cdots, 12k+3, 12k+5, 12k+7, \cdots, \\ 12(2^m-1)+3, 12(2^m-1)+5, 12(2^m-1)+7\rangle$$

with $12(2^{m+1})$ *vertices and degree* $3(2^{m+1})$.

Proof. The following operations are taken modulo $12(2^{m+1})$. For each integer $m \geq 0$, let $G = C_{12(2^{m+1})}\langle 3, 5, 7, \cdots, 12k+3, 12+5, 12k+7, \cdots, 12(2^m-1)+3, 12(2^m-1)+5, 12(2^m-1)+7\rangle$ with $12(2^{m+1})$ vertices and degree $3(2^{m+1})$. Then the edges incident with 0 in G are: For $k = 0, 1, \cdots, 2^m - 1$,

$$(0, \pm(12k+3)),\ (0, \pm\ (12k+5)), \text{ and } (0, \pm(12k+7)).$$

In G^2, by using (1) the edges incident with the vertex 0 are: For $k = 0, 1, \cdots, 2^m - 1$,

$$(0, \pm(12k+3)), (0, \pm(12k+5)), (0, \pm(12k+7)), \\ (0, \pm(12k+6)), (0, \pm(12k+10)), (0, \pm(12k+2)), \\ (0, \pm(12k+8)), \text{ and } (0, \pm(12k+4)).$$

Then, in $\overline{G^2}$, the edges incident with the vertex 0 are: For $k = 0, 1, \cdots, 2^m - 1$,

$$(0, \pm(12k+1)),\ (0, \pm(12k+9)), \text{ and } (0, \pm(12k+11)).$$

We claim that $\overline{G^2} \cong G$. There are two cases to be considered.

(a) The case of m being even. We define $\sigma: V(\overline{G^2}) \to V(G)$ by $\sigma(j) = (2^{m+2}+1)j$ for $j = 0, 1, \cdots, 12(2^{m+1}) - 1$. Since m is even, $2^{m+2}+1$ and $12(2^{m+1})$ are relatively prime. Thus, σ is bijective. Let e be any edge in $\overline{G^2}$. Since G is vertex-transitive, G^2 and $\overline{G^2}$ are vertex-transitive. Hence, there is an automorphism θ of $\overline{G^2}$ such that $\theta e = (0, 12k+i)$ or $(0, -(12k+i))$ for some $i = 1, 9, 11$, and for some $k = 0, 1, \cdots, 2^m - 1$.

We show that $\sigma(\theta e)$ is an edge in G. For each $k = 0, 1, \cdots, 2^m - 1$, we have, since 3 divides $(2^m - 1)$,

$$(\sigma 0,\ \sigma(12k+1)) = (0,\ (12k+1)(2^{m+2}+1)) = (0, 12(k+((2^m-1)/3))+5) \in E(G),$$
$$(\sigma 0, \sigma(12k+9)) = (0, (12k+9)(2^{m+2}+1)) = (0, 12(k+1-2^m)-3) \in E(G),$$
$$(\sigma 0, \sigma(12k+11)) = (0, (12k+11)(2^{m+2}+1)) = (0, 12(k-((2^m-1)/3))+7) \in E(G),$$
$$(\sigma 0, \sigma(-12k+1)) = (0, -12(k+((2^m-1)/3))+5) \in E(G),$$
$$(\sigma 0, \sigma(-12k+9)) = (0, -12(k+1)-3) \in E(G), \text{ and}$$
$$(\sigma 0, \sigma(-(12k+11))) = (0,\ -12(k-((2^m-1)/3))+7) \in E(G).$$

We claim that $\sigma e_1 \neq \sigma e_2$ where e_1 and e_2 are any two different edges incident with the vertex 0 in $\overline{G^2}$. Say, $e_1 = (0, 12k+i)$ and $e_2 = (0, 12k'+j)$ for some $i, j = 1, 9, 11$ and $0 \le k,\ k' \le 2^m - 1$. Suppose that $\sigma e_1 = \sigma e_2$. Then $(\sigma 0, \sigma(12k+i)) = (\sigma 0, \sigma(12k'+j))$, i.e., $\sigma(12k+i) = \sigma(12k'+j)$, and $(12k+i)\ (2^{m+2}+1) = (12k'+j)(2^{m+2}+1)$. Since $2^{m+2}+1$ and $12(2^{m+1}-1)$ are relatively prime, $12k+i = 12k'+j$. If $i = j$, then $12k = 12k'$, $12(k-k') = 0$, $k = k'$ and $e_1 = e_2$. That is a contradiction. If $i \neq j$, then we have the following cases:

$$\sigma(12k+1) = \sigma(12k'+9),\ 12(k+((2^m-1)/3)) + 5 = 12(k'+1-2^m) - 3,$$
$$16(2^m) - 8 = 12(k'-k),\ (2(2^m-1)/3) + ((2^{m+1})/3) = k'-k$$

where two of the three terms are integers and the third is not. That is a contradiction.

$$\sigma(12k+1) = \sigma(12k'+11),\ 12(k+((2^m-1)/3))+5 = 12(k'-((2^m-1)/3)) + 7,$$
$$12k+4(2^m)-1 = 12k'-4(2^m)+11,\ 8(2^m)-12 = 12(k'-k),\ (2^{m+1}/3)-1 = k'-k$$

where, again, two of the terms are integers and the third is not. That is a contradiction.

$$\sigma(12k+9) = \sigma(12k'+11),\ 12(k+1-2^m)-3 = 12(k'-((2^m-1)/3))+7,$$
$$8(2^m)+2 = 12(k-k'),\ (k-k') = (4(2^m-1)/6) + 5/6$$

where, again, two of the terms are integers and the third is not. That is a contradiction. Thus, σ maps the $3(2^{m+1})$ edges incident with the vertex 0 in $\overline{G^2}$ injectively onto the $3(2^{m+1})$ edges incident with the vertex 0 in G. Consequently, σ takes an edge to an edge, and a non-edge to a non-edge, and σ is an isomorphism.

(b) The case of m being odd. We define $\tau : V(\overline{G^2}) \to V(G)$ by $\tau(j) = (2^{m+2}-1)j$ for $j = 0, 1, \cdots, 12(2^{m+1})-1$. Since m is odd, $2^{m+2}-1$ and $12(2^{m+1})$ are relatively prime, and τ is bijective. The proof for τ being an isomorphism of $\overline{G^2}$ onto G is analogous to the proof in (a) with $2^{m+2} - 1$ and $(12(2^{m+2} + 2)/3) + 7$ replacing $2^{m+2} + 1$ and $(12(2^{m+2} - 1)/3) + 5$ respectively.

References

1. Y. Zhou and B. Liu, The realization of Schuster property in circulant graphs, *Mathematica Applicata* **7** (1994), 481-486.
2. S. Schuster, The theory and applications of graphs, *Proceedings of the Fourth International Conference*, Wiley (1981), 611.