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Some Weighted Estimates for Stein's Maximal Function

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Abstract. In this brief article, we prove some weighted estimates for Stein's maximal function by using interpolation techniques. In some cases, our results agree with those previously obtained in [2] and [4].

Suppose f is a Schwartz function on R^n $(n \geq 3)$. For $\mathrm{Re}(\alpha) > 0$ and $r > 0$, define the operators $M_{\alpha,r}$ by

$$M_{\alpha,r} f(x) = m_{\alpha,r} * f(x)$$

where

$$m_\alpha(x) = \begin{cases} \dfrac{\left(1-|x|^2\right)^{\alpha-1}}{\Gamma(\alpha)}, & \text{if } |x| < 1 \\ 0, & \text{otherwise} \end{cases}$$

and $m_{\alpha,r}(x) = r^{-n} m_\alpha(x/r)$ [5]. As is known,

$$\hat{m}_\alpha(\xi) = \pi^{-\alpha+1} |\xi|^{-n/2-\alpha+1} J_{n/2+\alpha-1}(2\pi|\xi|)$$

(see [6], p. 171). Thus, for $\alpha \in C$ in general, we can define $M_{\alpha,r} f$ by the relation

$$\left(M_{\alpha,r} f\right)^\wedge(\xi) = \hat{m}_\alpha(r\xi)\,\hat{f}(\xi).$$

(One may also define $M_{\alpha,r} f$ by analytic continuation; see [1] for how it works.) Observe that

$$M_{0,r} f = \frac{1}{2} M_r f$$

where $M_r f(x)$ denotes the average of f on the sphere of radius r centered at x.

Now define

$$M_\alpha f(x) = \sup_{r>0} \left| M_{\alpha,r} f(x) \right|.$$

Here $\{M_\alpha\}$ forms an analytic family of operators. Particularly we have

$$M_0 f = \frac{1}{2} M_S f$$

where $M_S f(x) = \sup_{r>0} \left| M_r f(x) \right|$ denotes Stein's maximal function. Also note that

$$M_1 f = cM_{HL} f$$

for some constant c. (Here $M_{HL} f$ denotes the well-known Hardy-Littlewood maximal function.) Stein [5] shows that

$$\| M_\alpha f \|_p \le C_{\alpha,p} \| f \|_p$$

if (a) $\mathrm{Re}(\alpha) > 1 - \frac{n}{p'}$ for $1 < p \le 2$ or (b) $\mathrm{Re}(\alpha) > \frac{2-n}{p}$ for $2 \le p \le \infty$. As a consequence of this, one can derive the estimate

$$\| M_S f \|_p \le C_p \| f \|_p$$

provided that $\frac{n}{n-1} < p \le \infty$.

In this paper, we are concerned with the weighted estimate for Stein's maximal function, namely

$$\| M_S f \|_{p,w} \le C_{p,w} \| f \|_{p,w}$$

for any possible values of $p > 1$ and weights $w \in A_p$. (Here $\| f \|_{p,w}^p = \int_{R^n} \left| f(x) \right|^p w(x)\,dx$. For definition of A_p weights, see [3].) With the above estimates for $M_\alpha f$ and the fact that $M_\alpha f$ is majorized by $M_{HL} f$ when $\mathrm{Re}(\alpha) \ge 1$, we prove by using the Stein's analytic interpolation theorem [6] (applied to the analytic family of operators $\{M_\alpha\}$) that the weighted estimate for Stein's maximal function holds for some $w \in A_p$ where $p > \frac{n}{n-1}$. Precisely, we have the following theorem:

Theorem. *The weighted estimate*

$$\|M_S f\|_{p,w} \le C_{p,w} \|f\|_{p,w}$$

holds for

(a) $\frac{n}{n-1} < p \le 2,\ w \in A_p^{1-\frac{p'}{n}}$.

(b) $2 \le p \le \infty,\ w \in A_p^{\frac{n-2}{n+p-2}}$.

(c) $\frac{n}{n-1} < p < n,\ w \in A_{\frac{p(n-2)}{n-p}}^{1-\frac{p}{n}}$.

Remark. $w \in A_p^q$ means that w can be written as $w = v^\theta$ for some $v \in A_p$ and $0 \le \theta < q$. For power weights $w(x) = |x|^a,\ w \in A_p$ if and only if $-n < a < n(p-1)$, and so $w \in A_p^q$ means that $-nq < a < n(p-1)q$.

Proof.

(a) For $\dfrac{n}{n-1} < p \le 2$, we have

$$\|M_\alpha f\|_p \le C_{\alpha,p} \|f\|_p, \quad \mathrm{Re}(\alpha) > 1 - \frac{n}{p'}$$

$$\|M_\alpha f\|_{p,w} \le C_{\alpha,p,w} \|f\|_{p,w}, \quad \mathrm{Re}(\alpha) \ge 1,\ w \in A_p.$$

By the Stein's analytic interpolation theorem,

$$\|M_\alpha f\|_{p,w^\theta} \le C_{\alpha,p,w,\theta} \|f\|_{p,w^\theta},$$
$$\mathrm{Re}(\alpha) > \theta \frac{n}{p'} + \left(1 - \frac{n}{p'}\right),\ w \in A_p.$$

In particular, when $\alpha = 0$, we have

$$\|M_S f\|_{p,w^\theta} \le C_{p,w,\theta} \|f\|_{p,w^\theta}, \quad w \in A_p,\ 0 \le \theta < 1 - \frac{p'}{n},$$

or equivalently

$$\|M_S f\|_{p,w} \le C_{p,w} \|f\|_{p,w}, \quad w \in A_p^{1-\frac{p'}{n}}.$$

(b) For $2 \le p \le \infty$, we have

$$\| M_\alpha f \|_p \le C_{\alpha,p} \| f \|_p, \quad \mathrm{Re}(\alpha) > \frac{2-n}{p}$$

$$\| M_\alpha f \|_{p,w} \le C_{\alpha,p,w} \| f \|_{p,w}, \quad \mathrm{Re}(\alpha) \ge 1, \ w \in A_p .$$

The Stein's analytic interpolation theorem gives

$$\| M_\alpha f \|_{p,w^\theta} \le C_{\alpha,p,w,\theta} \| f \|_{p,w^\theta},$$
$$\mathrm{Re}(\alpha) > \theta\left(\frac{n+p-2}{p}\right) - \left(\frac{n-2}{p}\right), \ w \in A_p .$$

When $\alpha = 0$, we have

$$\| M_S f \|_{p,w^\theta} \le C_{p,w,\theta} \| f \|_{p,w^\theta}, \quad w \in A_p, \ 0 \le \theta < \frac{n-2}{n+p-2},$$

or equivalently

$$\| M_S f \|_{p,w} \le C_{p,w} \| f \|_{p,w}, \quad w \in A_p^{\frac{n-2}{n+p-2}} .$$

(c) For $\frac{n}{n-1} < p < n$, let $q = \frac{p(n-2)}{n-p}$. Then clearly $q < p \le 2$ or $2 \le p < q$ (depending on the value of p). Now, we have

$$\| M_\alpha f \|_2 \le C_\alpha \| f \|_2, \quad \mathrm{Re}(\alpha) > 1 - \frac{n}{2}$$

$$\| M_\alpha f \|_{q,w} \le C_{\alpha,q,w} \| f \|_{q,w}, \quad \mathrm{Re}(\alpha) \ge 1, \ w \in A_q .$$

Interpolation will give

$$\| M_\alpha f \|_{r,w^\theta} \le C_{\alpha,r,w,\theta} \| f \|_{r,w^\theta},$$
$$\mathrm{Re}(\alpha) > t\frac{n}{2} + \left(1 - \frac{n}{2}\right), \ \frac{1}{r} = \frac{1-t}{2} + \frac{t}{q}, w \in A_q, \theta = \frac{rt}{q}.$$

When $\alpha = 0$, we have

$$\| M_S f \|_{r,w^\theta} \le C_{r,w,\theta} \| f \|_{r,w^\theta},$$
$$\frac{1}{r} = \frac{1-t}{2} + \frac{t}{q}, w \in A_q, \theta = \frac{rt}{q}, 0 \le t < \frac{n-2}{n}.$$

Taking t arbitrarily close to $\frac{n-2}{n}$, we get r close to p and θ close to $1-\frac{p}{n}$. Hence we conclude

$$\| M_S f \|_{p,w} \le C_{p,w} \| f \|_{p,w}, \quad w \in A_q^{1-\frac{p}{n}}.$$

Remark. Estimate (a) is sharp in θ, in the sense that θ cannot be greater than $1-\frac{p'}{n}$. This result has been previously obtained in [2] and [4]. Estimate (b) is certainly not sharp in θ, except for $p = 2$. Estimate (c) is better than (a) and (b), particularly for negative power weights. (For power weights $w(x) = |x|^a$, (c) says that the estimate holds provided that $p - n < a < np - n - p$.)

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