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The Neutrix Product of the Distributions $x_+^{\lambda} \ln x_+$ and $x_-^{-\lambda-r}$

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Abstract. The neutrix product of the distributions $x_+^{\lambda} \ell n\, x_+$ and $x_-^{-\lambda-r}$ is evaluated for $\lambda \neq 0, \pm 1, \pm 2, \cdots$ and $r = 1, 2, \cdots$.

In the following, we let N be the neutrix, see van der Corput [1], having domain $N = \{1, 2, \cdots, n, \cdots\}$ and range the real numbers, with negligible functions finite linear sums of the functions

$$n^{\lambda} \ell n^{r-1} n, \ \ell n^{r} n : \quad \lambda > 0, \quad r = 1, 2, \cdots$$

and all functions which converge to zero in the normal sense as n tends to infinity.

We now let $\rho(x)$ be any infinitely differentiable function having the following properties:

(i) $\rho(x) = 0$ for $|x| \geq 1$,

(ii) $\rho(x) \geq 0$,

(iii) $\rho(x) = \rho(-x)$,

(iv) $\int_{-1}^{1} \rho(x)\, dx = 1$.

Putting $\delta_n(x) = n\rho(nx)$ for $n = 1, 2, \cdots$, it follows that $\{\delta_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta(x)$.

Now let D be the space of infinitely differentiable functions with compact support and let D' be the space of distributions defined on D. Then if f is an arbitrary distribution in D', we define.

$$f_n(x) = \left(f * \delta_n \right)(x) = \left\langle f(t), \delta_n(x-t) \right\rangle$$

for $n = 1, 2, \cdots$. It follows that $\{ f_n \}$ is a regular sequence of infinitely differentiable functions converging to the distribution f.

A first extension of the product of a distribution and an infinitely differentiable function is the following, see for example [2].

Definition 1. *Let f and g be distributions in D' for which on the interval (a,b), f is the k-th derivative of a locally summable function F in $L^p(a,b)$ and $g^{(k)}$ is a locally summable function in $L^q(a,b)$ with $1/p + 1/q = 1$. Then the product $fg = gf$ of f and g is defined on the interval (a,b) by*

$$fg = \sum_{i=0}^{k} \binom{k}{i} (-1)^i \left[F g^{(i)} \right]^{(k-i)} .$$

The following definition for the non-commutative neutrix product of two distributions was given in [3] and generalizes Definition 1.

Definition 2. *Let f and g be distributions in D' and let $g_n = g * \delta_n$. We say that the neutrix product $f \circ g$ of f and g exists and is equal to the distribution h on the interval (a,b) if*

$$\underset{n \to \infty}{N - \lim} \left\langle f g_n, \phi \right\rangle = \left\langle h, \phi \right\rangle,$$

for all functions ϕ in D with support contained in the interval (a,b). Note that if

$$\lim_{n \to \infty} \left\langle f g_n, \phi \right\rangle = \left\langle h, \phi \right\rangle,$$

we simply say that the product $f \cdot g$ exists and equals h.

This definition of the neutrix product is in general non-commutative. It is obvious that if the product $f \cdot g$ exists then the neutrix product $f \circ g$ exists and $f \cdot g = f \circ g$. Further, it was proved in [3] that if the product fg exists by Definition 1 then the product $f \circ g$ exists by Definition 2 and $fg = f \circ g$.

The next two theorems were proved in [3].

Theorem 1. *Let f and g be distributions and suppose that the neutrix products $f\ o\ g$ and $f\ o\ g'$ exist on the interval (a,b). Then the neutrix product $f'o\ g$ exists and*

$$\left(f\ o\ g\right)' = f'o\,g + f\,o\,g',$$

on the interval (a,b).

Theorem 2. *The neutrix product $x_+^{\lambda}\ o\ x_-^{-\lambda-r}$ exists and*

$$x_+^{\lambda}\ o\ x_-^{-\lambda-r} = -\frac{\pi\text{cosec}\,(\pi\lambda)}{2\,(r-1)\,!}\,\delta^{(r-1)}(x) \tag{1}$$

for $\lambda \neq 0, \pm 1, \pm 2, \cdots$ and $r = 1, 2, \cdots$.

We now prove the following theorem:

Theorem 3. *The neutrix product $x_+^{\lambda}\ \ell n\ x_+\ o\ x_-^{-\lambda-r}$ exists and*

$$x_+^{\lambda}\ \ell n\ x_+\ o\ x_-^{-\lambda-r} = -\frac{\pi\text{cosec}\,(\pi\lambda)}{2\,(r-1)\,!}\left[\,2c(\rho) + \psi(\lambda+r) - \Gamma'(1)\,\right]\delta^{(r-1)}(x) \tag{2}$$

for $\lambda \neq 0, \pm 1, \pm 2, \cdots$ and $r = 1, 2, \cdots$ where Γ denotes the Gamma function and

$$\psi\left(\lambda + r\right) = \frac{\Gamma'\left(\lambda + r\right)}{\Gamma\left(\lambda + r\right)}$$

Proof. We will first of all suppose that $-1 < \lambda < 0$. Then $x_+^{\lambda}\ \ell n\ x_+$ and $x_-^{-\lambda-1}$ are locally stumble functions and

$$x_-^{-\lambda-r} = \frac{\Gamma\left(\lambda+1\right)}{\Gamma\left(\lambda+r\right)}\,(x_-^{-\lambda-1})^{(r-1)} \ .$$

Thus

$$(x_-^{-\lambda-r})_n = x_-^{-\lambda-r} * \delta_n(x) = \frac{\Gamma\left(\lambda+1\right)}{\Gamma\left(\lambda+r\right)}\int_x^{1/n}(\,t-x)^{-\lambda-1}\ \ \delta_n^{(r-1)}(t)\,dt$$

for $r = 1, 2, \cdots$ and so

$$\frac{\Gamma(\lambda+r)}{\Gamma(\lambda+1)} \int_{-\infty}^{\infty} x_+^{\lambda} \ln x_+ \left(x_-^{-\lambda-r}\right)_n x^m dx$$
$$= \int_0^{1/n} x^{\lambda+m} \ln x \int_x^{1/n} (t-x)^{-\lambda-1} \delta_n^{(r-1)}(t)\, dt\, dx$$
$$= \int_0^{1/n} \delta_n^{(r-1)}(t) \int_0^t x^{\lambda+m} \ln x (t-x)^{-\lambda-1}\, dx\, dt$$
$$= \int_0^{1/n} t^m \delta_n^{(r-1)}(t) \int_0^1 \upsilon^{\lambda+m} \ln (t\upsilon)(1-\upsilon)^{-\lambda-1}\, d\upsilon\, dt$$
$$= B(\lambda+m+1, -\lambda) \int_0^{1/n} t^m \ln t \delta_n^{(r-1)}(t)\, dt$$
$$+ B_{1,0}(\lambda+m+1, -\lambda) \int_0^{1/n} t^m \delta_n^{(r-1)}(t) dt, \qquad (3)$$

where the substitution $x = tv$ has been made, B denotes the Beta function and in general

$$B_{p,q}(\lambda, \mu) = \frac{\partial^{p+q}}{\partial^p \lambda \partial^q \mu} B(\lambda, \mu)$$

Making the substitution $nt = y$. We have

$$\int_0^{1/n} t^m \delta_n^{(r-1)}(t)\, dt = n^{r-m-1} \int_0^1 y^m \rho^{(r-1)}(y)\, dy, \qquad (4)$$

$$\int_0^{1/n} t^m \ln t\ \delta_n^{(r-1)}(t)\, dt = -n^{r-m-1} \ln n \int_0^1 y^m \rho^{(r-1)}(y)\, dy$$
$$+ n^{r-m-1} \int_0^1 y^m \ln y \rho^{(r-1)}(y)\, dy \qquad (5)$$

for $m = 0, 1, 2, \cdots$.

In particular, when $m = r-1$, it is easily proved by induction that

$$\int_0^1 y^{r-1} \rho^{(r-1)}(y)\, dy = \frac{1}{2}(-1)^{r-1}(r-1)!, \qquad (6)$$

$$\int_0^1 y^{r-1} \ln y \rho^{(r-1)}(y)\, dy = (-1)^{r-1}(r-1)!\left[\frac{1}{2}\phi(r-1) + c(\rho)\right], \qquad (7)$$

for $r = 1, 2, \cdots$, where

$$\phi(r) = \begin{cases} \sum_{i=1}^{r} i^{-1}, & r = 1, 2, \cdots \\ 0, & r = 0 \end{cases}$$

and

$$c(\rho) = \int_0^1 \ell n\ t\ \rho(t)\, dt.$$

Further, putting

$$K = -\frac{\Gamma(\lambda+1)}{\lambda\Gamma(\lambda+r)} \sup_x \left\{ \left| \rho^{(r-1)}(x) \right| \right\} > 0,$$

we have

$$\left| x_-^{-\lambda-r})_n \right| = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+r)} \left| \int_{nx}^1 n^{\lambda+1}(u-nx)^{-\lambda-1} n^{r-1} \rho^{(r-1)}(u)\, du \right|$$

$$\leq -\lambda K n^{\lambda+r} \int_{nx}^{1+nx} (u-nx)^{-\lambda-1}\, du$$

$$= K n^{\lambda+r}$$

and so when $m = r$, we have

$$\left| \int_{-\infty}^{\infty} x_+^{\lambda}\ \ell n\ x_+ \left(x_-^{-\lambda-r} \right)_n x^r\, dx \right| \leq \int_0^{1/n} \left| x^{\lambda} \ell n\ x (x_-^{-\lambda-r})_n\ x^r \right| dx \leq -\lambda^{-1} K n^{-1}\ \ell n\ n, \tag{8}$$

Now let φ be an arbitrary function in D. Then

$$\varphi(x) = \sum_{m=0}^{r-1} \frac{x^m}{m!} \varphi^{(m)}(0) + \frac{x^r}{r!} \varphi^{(r)}(\xi x),$$

where $0 < \xi < 1$ and so

$$\left\langle x_+^{\lambda}\ \ell n\ x_+, (x_-^{-\lambda-r})_n \varphi(x) \right\rangle = \sum_{m=0}^{r-1} \frac{\varphi^{(m)}(0)}{m!} \int_{-\infty}^{\infty} x_+^{\lambda}\ \ell n\ x_+ (x_-^{-\lambda-r})_n x^m\, dx$$

$$+ \frac{1}{r!} \int_{-\infty}^{\infty} x_+^{\lambda}\ \ell n\ x_+ (x_-^{-\lambda-r})_n x^r\ \varphi^{(r)}(\xi x)\, dx. \tag{9}$$

Since

$$\left| \int_{-\infty}^{\infty} x_+^{\lambda}\ \ell n\ x_+ (x_-^{-\lambda-r})_n x^r\ \varphi^{(r)}(\xi x)\, dx \right| \leq \sup_x \left\{ \left| \varphi^{(r)}(x) \right| \right\} (-\lambda^{-1}) K n^{-1}\ \ell n\ n,$$

it follows from equations (3) to (9) that

$$N-\lim_{n\to\infty}\frac{\Gamma(\lambda+r)}{\Gamma(\lambda+1)}\left\langle x_+^{\lambda}\,\ell n\, x_+,(x_-^{-\lambda-r})_n\varphi(x)\right\rangle$$
$$=(-1)^{r-1}B(\lambda+r,-\lambda)\left[\tfrac{1}{2}\phi(r-1)+c(\rho)\right]\varphi^{(r-1)}(0)$$
$$+\tfrac{1}{2}(-1)^{r-1}B_{1,0}(\lambda+r,-\lambda)\;\varphi^{(r-1)}(0). \qquad (10)$$

Differentiating the identity

$$B(\lambda,\mu)=\frac{\Gamma(\lambda)\Gamma(\mu)}{\Gamma(\lambda+\mu)}$$

partially with respect to λ, it follows that

$$B_{1,0}(\lambda+r,-\lambda)=\frac{\Gamma'(\lambda+r)\,\Gamma(-\lambda)}{(r-1)!}-\frac{\Gamma(\lambda+r)\,\Gamma(-\lambda)\Gamma'(r)}{\left[(r-1)!\right]^2} \qquad (11)$$

and taking logs and differentiating the identity

$$\Gamma(\lambda+r)=(\lambda+r-1)\cdots(\lambda+1)\,\Gamma(\lambda+1)$$

gives

$$\psi(\lambda+r)=\frac{\Gamma'(\lambda+r)}{\Gamma(\lambda+r)}=\sum_{i=1}^{r-1}(\lambda+r-i)^{-1}+\psi(\lambda+1). \qquad (12)$$

In particular, we have

$$\frac{\Gamma'(r)}{(r-1)!}=\phi(r-1)+\Gamma'(1). \qquad (13)$$

It now follows from equations (11) and (13) that

$$\frac{\Gamma(\lambda+1)}{\Gamma(\lambda+r)}B_{1,0}(\lambda+r,-\lambda)=\frac{\Gamma(\lambda+1)\,\Gamma(-\lambda)}{(r-1)!}\left[\frac{\Gamma'(\lambda+r)}{\Gamma(\lambda+r)}-\frac{\Gamma'(r)}{(r-1)!}\right]$$
$$=-\frac{\pi\operatorname{cosec}(\pi\lambda)}{(r-1)!}\left[\psi(\lambda+r)-\phi(r-1)-\Gamma'(1)\right]. \qquad (14)$$

Further,

$$\frac{\Gamma(\lambda+1)}{\Gamma(\lambda+r)}B(\lambda+r,-\lambda) = -\frac{\pi \operatorname{cosec}(\pi\lambda)}{(r-1)!} \tag{15}$$

and equation (2) now follows from equations (10), (14) and (15) for the case $-1 < \lambda < 0$.

Now let us suppose that equation (2) holds when $-k < \lambda < -k+1$ and $r = 1, 2, \cdots$, where k is a positive integer. This is true when $k = 1$. Thus if $-k-1 < \lambda < -k$, it follows from our assumption that

$$x_+^{\lambda+1}\ \ell n\, x_+\ o\ x_-^{-\lambda-1-r} = \frac{\pi \operatorname{cosec}(\pi\lambda)}{2\,(r-1)!}\left[2c(\rho) + \psi(\lambda+1+r) - \Gamma'(1)\right]\delta^{(r-1)}(x),$$

for $r = 1, 2, \cdots$. It follows from Theorem 1 that

$$\left[(\lambda+1)\,x_+^{\lambda}\ \ell n\, x_+ + x_+^{\lambda}\right] o\ x_-^{-\lambda-r-1} + (\lambda+r+1)\,x_+^{\lambda+1}\ \ell n\, x_+\ o\ x_-^{-\lambda-r-2}$$

$$= \frac{\pi \operatorname{cosec}(\pi\lambda)}{2(r-1)!}\left[2\,c\,(\rho) + \psi(\lambda+r+1) - \Gamma'(1)\right]\delta^{(r)}(x)$$

$$= (\lambda+1)\,x_+^{\lambda}\ \ell n\, x_+\ o\ x_-^{-\lambda-r-1} - \frac{\pi \operatorname{cosec}(\pi\lambda)}{2r!}\delta^{(r)}(x)$$

$$+ \frac{(\lambda+r+1)\,\pi \operatorname{cosec}(\pi\lambda)}{2r!}\left[2c(\rho) + \psi(\lambda+r+2) - \Gamma'(1)\right]\delta^{(r)}(x).$$

Thus

$$(\lambda+1)\,x_+^{\lambda}\ \ell n\, x_+ o\ x_-^{-\lambda-r-1} =$$

$$-\frac{(\lambda+1)\,\pi \operatorname{cosec}(\pi\lambda)}{2r!}\left[2c(\rho) + \psi(\lambda+r+2) - \Gamma'(1)\right]\delta^{(r)}(x) +$$

$$+\frac{\pi \operatorname{cosec}(\pi\lambda)}{2(r-1)!}\left[r^{-1} + \psi(\lambda+r+1) - \psi(\lambda+r+2)\right]\delta^{(r)}(x),$$

$$-\frac{(\lambda+1)\,\pi \operatorname{cosec}(\pi\lambda)}{2r!}\left[2c(\rho) + \psi(\lambda+r+1) - \Gamma'(1)\right]\delta^{(r)}(x),$$

since, from equation (12), we have

$$\psi(\lambda+r+2) - (\lambda+r+1)^{-1} = \psi(\lambda+r+1)$$

and so

$$r^{-1} + \psi(\lambda + r + 1) - \psi(\lambda + r + 2) = \frac{\lambda + 1}{r(\lambda + r + 1)} \cdot$$

Equation (2) now follows by induction for $\lambda < 0, \lambda \neq -1, -2, \cdots$ and $r = 2, 3, \cdots$.

To cover the case $r = 1$, we note the product $x_+^{\lambda+1} \ell n\, x_+ . x_-^{-\lambda-1}$ exists by Definition 1 and

$$x_+^{\lambda+1} \ell n\, x_+ . x_-^{-\lambda-1} = 0 \qquad (16)$$

for all λ.

Let us suppose that equation (2) holds when $-k < \lambda < -k+1$ and $r = 1$, where k is a positive integer. This is true when $k = 1$. Thus if $-k-1 < \lambda < -k$, it follows from our assumption that

$$x_+^{\lambda+1} \ell n\, x_+ \, o\, x_-^{-\lambda-2} = ½\, \pi \operatorname{cosec}(\pi\lambda) \left[2c(\rho) + \psi(\lambda+2) - \Gamma'(1) \right] \delta(x).$$

It follows from equation (16) and Theorem 1 that

$$\begin{aligned} \left[(\lambda+1) x_+^{\lambda} \ell n\, x_+ + x_+^{\lambda} \right] o\, x_-^{-\lambda-1} &+ (\lambda+1) x_+^{\lambda+1} \ell n\, x_+ o\, x_-^{-\lambda-2} = 0 \\ &= (\lambda+1) x_+^{\lambda} \ell n\, x_+ o\, x_-^{-\lambda-1} - ½\, \pi \operatorname{cosec}(\pi\lambda)\, \delta(x) \\ &+ ½(\lambda+1)\pi \operatorname{cosec}(\pi\lambda) \left[2\, c(\rho) + \psi(\lambda+2) - \Gamma'(1) \right] \delta(x) \\ &= (\lambda+1) x_+^{\lambda} \ell n\, x_+ o\, x_-^{-\lambda-1} \\ &+ ½(\lambda+1)\pi \operatorname{cosec}(\pi\lambda) \left[2\, c(\rho) + \psi(\lambda+1) - \Gamma'(1) \right] \delta(x) \end{aligned}$$

Equation (2) now follows by induction for $\lambda < 0, \lambda \neq -1, -2, \cdots$ and $r = 1$.

Now let us suppose that equation (2) holds when $k-1 < \lambda < k$ and $r = 1, 2, \cdots$, where k is a positive integer. This true when $k = 0$. Then for an arbitrary function φ in D we have

$$\left\langle x_+^{\lambda+1} \ell n\, x_+, (x_-^{-\lambda-r-1})_n \; \phi(x) \right\rangle = \left\langle x_+^{\lambda} \ell n\, x_+, (x_-^{-\lambda-r-1})_n \; \psi(x) \right\rangle,$$

where $\psi(x) = x\varphi(x)$ is also in D. It follows from our assumption with $k-1 < \lambda < k$ that

$$N-\lim_{n\to\infty}\left\langle x_+^{\lambda}\ \ell n\ x_+, (x_-^{-\lambda-r-1})_n\ \psi(x)\right\rangle$$

$$= -\frac{(-1)^r \pi \operatorname{cosec}(\pi\lambda)}{2r!}\left[2c(\rho)+\psi(\lambda+r+1)-\Gamma'(1)\right]\psi^{(r)}(0)$$

$$= -\frac{(-1)^r \pi \operatorname{cosec}(\pi\lambda)}{2(r-1)!}\left[2c(\rho)+\psi(\lambda+r+1)-\Gamma'(1)\right]\varphi^{(r-1)}(0)$$

and so

$$N-\lim_{n\to\infty}\left\langle x_+^{\lambda+1}\ \ell n\ x_+, (x_-^{-\lambda-r-1})_n\ \varphi(x)\right\rangle$$

$$= -\frac{(-1)^r \pi \operatorname{cosec}(\pi\lambda)}{2(r-1)!}\left[2c(\rho)+\psi(\lambda+r+1)-\Gamma'(1)\right]\varphi^{(r-1)}(0)$$

Equation (2) now follows by induction for $\lambda > 0,\ \lambda \neq 1,2,\cdots$ and $r = 1,2,\cdots,$ completing the proof of the theorem.

Corollary 3.1. *The neutrix product* $x_-^{\lambda}\ \ell n\ x_-\ \mathrm{o}\ x_+^{-\lambda-r}$ *exists and*

$$x_-^{\lambda}\ \ell n\ x_-\ \mathrm{o} x_+^{-\lambda-r} = \frac{(-1)^r \pi \operatorname{cosec}(\pi\lambda)}{2(r-1)!}\left[2c(\rho)+\psi(\lambda+r)-\Gamma'(1)\right]\delta^{(r-1)}(x) \qquad (17)$$

for $\lambda \neq 0, \pm 1, \pm 2,$ *and* $r = 1,2,\cdots.$

Proof. Equation (17) follows on replacing x by $-x$ in equation (2).

Theorem 4. *The neutrix product* $x_+^{\lambda}\ o\ x_-^{-\lambda-r}\ \ell n\ x_-$ *exists and*

$$x_+^{\lambda}\ \mathrm{o}\ x_-^{-\lambda-r}\ \ell n\ x_- = \frac{-\pi \operatorname{cosec}(\pi\lambda)}{2(r-1)!}\left[2c(\rho)+\Psi(-\lambda-r+1)-\Gamma'(1)\right]\delta^{(r-1)}(x) \qquad (18)$$

for $\lambda \neq 0, \pm 1, \pm 2, \cdots$ *and* $r = 1,2,\cdots.$

Proof. Differentiating equation (1) partially with respect to λ we get

$$x_+^{\lambda}\ \ \ell n\ x_+\ \mathrm{o}\ x_-^{-\lambda-r} - x_+^{\lambda}\ \mathrm{o}\ x_-^{-\lambda-r}\ \ell n\ x_- = \frac{\pi^2 \cot(\pi\lambda)\operatorname{cosec}(\pi\lambda)}{2(r-1)!}\delta^{(r-1)}(x)$$

and on using equation (2) it follows that

$$x_+^{\lambda} \circ x_-^{-\lambda-r} \ \ell n\, x_- = -\frac{\pi \operatorname{cosec}(\pi\lambda)}{2(r-1)!}\big[\pi \cot(\pi\lambda) + 2c(\rho) + \psi(\lambda + r) - \Gamma'(1)\big]\delta^{(r-1)}(x). \tag{19}$$

Taking logs and differentiating the identity

$$\Gamma(-\lambda)\Gamma(\lambda+1) = (-1)^{r-1}\Gamma(-\lambda - r + 1)\Gamma(\lambda + r) = -\pi \operatorname{cosec}(\pi\lambda)$$

gives

$$-\psi(-\lambda - r + 1) + \psi(\lambda + r) = -\pi \cot(\pi\lambda) \tag{20}$$

and equation (18) follows from equations (19) and (20).

Corollary 4.1. *The neutrix product* $x_-^{\lambda} \circ x_+^{-\lambda-r} \ \ell n\, x_+$ *exists and*

$$x_-^{\lambda} \circ x_+^{-\lambda-r} \ \ell n\, x_+ = \frac{(-1)^r \pi \operatorname{cosec}(\pi\lambda)}{2(r-1)!}\big[2c(\rho) + \psi(-\lambda - r + 1) - \Gamma'(1)\big]\,\delta^{(r-1)}(x) \tag{21}$$

for $\lambda \neq 0, \pm 1, \pm 2, \cdots$ *and* $r = 1, 2, \cdots$.

Proof. Equation (21) follows on replacing x by $-x$ in equation (18).

We finally note that if we replace λ by $-\lambda - r$ in equation (21), we get

$$x_-^{-\lambda-r} \circ x_+^{\lambda} \ \ell n\, x_+ = -\frac{\pi \operatorname{cosec}(\pi\lambda)}{2(r-1)!}\big[2c(\rho) + \psi(\lambda + 1) - \Gamma'(1)\big]\,\delta^{(r-1)}(x)\,.$$

and we see that the product of the distributions $x_+^{\lambda} \ \ell n\, x_+$ and $x_-^{-\lambda-r}$ is commutative only when $r = 1$.

References

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