

On Pre-Urysohn Spaces

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Abstract. In this paper some basic properties of pre-Urysohn space [15] have been studied and also two types of new functions termed as mapping with p - θ -closed graph and pre- θ -closed graph have been introduced. The conditions when a pre-Urysohn space coincides with Urysohn space have also been investigated.

1. Introduction

In 1982, Mashhour *et al.* [11] introduced the notion of a preopen set in a topological space and obtained its various properties. Pal and Bhattacharyya defined pre-Urysohn spaces [15] with the aid of preopen sets. The purpose of the present paper is to study some basic properties of this space. The class of pre-Urysohn spaces contains the class of Urysohn spaces. We also introduce two types of new functions termed as mapping with p - θ -closed graph and pre- θ -closed graph. The interrelationship of the Urysohn spaces defined here with these new mappings have also been investigated. In Section 2, pre-Urysohn spaces are defined and their relationships with other known spaces are studied. Section 3 deals with the basic properties of this space. Section 4 is concerned with product spaces of pre-Urysohn spaces while the last section deals with the properties of functions with p - θ -closed graph and pre- θ -closed graph and their relationships with this new Urysohn space. In this section, the condition when a pre-Urysohn space coincides with Urysohn space has also been studied.

Throughout this paper (X, τ) , (Y, σ) etc. (or simply X, Y etc.) will always denote topological spaces. If A is a subset of a space (X, τ) then the closure of A is denoted by $Cl_X(A)$ or simply $Cl(A)$ if there is no possibility of confusion. A subset A of a space is called preopen [11] (resp. semi-open [10], regular open [18]) briefly p.o. (resp. s.o., r.o.) if $A \subset \text{Int}(Cl(A))$ (resp. $A \subset Cl(\text{Int}(A))$, $A = \text{Int}(Cl(A))$). The family of all preopen (resp. s.o., r.o.) sets in X is denoted by $PO(X)$ (resp. $SO(X), RO(X)$). For each $x \in X$, the family of all p.o. sets containing x is denoted by $PO(x)$. If there is scope for confusion we write $PO(X, x)$ instead of $PO(x)$. The complement of a p.o. (resp. s.o., r.o.) set is called preclosed [11] (resp. semi-closed [2], regular closed [5]). The family of all preclosed (resp. semi-closed, regular-closed) sets in X is denoted by $PC(X)$ (resp. $SC(X), RC(X)$). The intersection of all preclosed sets containing A is said to be the preclosure [4] of A and is denoted by $pcl_X(A)$ or simply by $pcl(A)$. In (X, τ) for a

set $A \subset X$, the family $\{U \in \tau : A \subset U\}$ is denoted by $\Sigma(A)$ and for a point $x \in X$, $\Sigma(x) = \{U \in \tau : x \in U\}$.

2. Pre-Urysohn space and its basic properties

Definition 2.1. X is called

- (i) *Urysohn* [18] if for every pair of points $x, y \in X, x \neq y$ there exist $U \in \Sigma(x), V \in \Sigma(y)$ such that $Cl(U) \cap Cl(V) = \emptyset$,
- (ii) *pre-Urysohn* [15] if for every pair of points $x, y \in X, x \neq y$ there exist $U \in PO(x), V \in PO(y)$ such that $pcl(U) \cap pcl(V) = \emptyset$.

Remark 2.1. A Urysohn space is pre-Urysohn. But the converse is not, in general, true as shown by

Example 2.1. Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, X, \{c, d\}\}$. Then (X, τ) is pre-Urysohn but not Urysohn.

We require the following known definitions in the sequel.

Definition 2.2. X is

- (i) *pre- T_1* [9] if and only if for $x, y \in X$ such that $x \neq y$, there exist a p.o. set containing x but not y and a p.o. set containing y but not x ,
- (ii) *pre- T_2* [9] if and only if for $x, y \in X$ such that $x \neq y$ there exist p.o. sets $U \in PO(X), V \in PO(Y)$, with $U \cap V = \emptyset$.

Theorem 2.1. A pre-Urysohn space is pre- T_1 .

Proof. Let x and y be two distinct points of X . Since X is pre-Urysohn, there exist $U \in PO(X), V \in PO(Y)$ such that $pcl(U) \cap pcl(V) = \emptyset$. This indicates that $x \notin pcl(V)$ and $y \notin pcl(U)$. Now, $pcl(U), pcl(V) \in PC(X)$. Therefore $X - pcl(U), X - pcl(V) \in PO(X)$ and are such that $x \in X - pcl(V)$ and $y \in X - pcl(U)$ while $x \notin X - pcl(U)$ and $y \notin X - pcl(V)$. Thus X is pre- T_1 .

Definition 2.3. A subset A of X is called an α -set [14] if $A \subset \text{Int}(Cl(\text{Int}(A)))$. The family of all α -sets is denoted by τ^α .

Lemma 2.1. *In a topological space (X, τ) , if $A \in PO(X)$, $B \in \tau^\alpha$ then $A \cap B \in PO(X)$.*

Proof. The preopenness of A gives $A \subset \text{Int}(Cl(A))$. Again B being an α -set, $B \subset \text{Int}(Cl(\text{Int}(A)))$. Therefore

$$\begin{aligned} A \cap B &\subset \text{Int}(Cl(A)) \cap \text{Int}(Cl(\text{Int}(B))) \\ &= \text{Int}(\text{Int}(Cl(A))) \cap \text{Int}(Cl(\text{Int}(B))) \\ &= \text{Int}[\text{Int}(Cl(A)) \cap Cl(\text{Int}(B))] \subset \text{Int}[Cl(\text{Int}(B) \cap \text{Int}(Cl(A)))] \\ &\subset \text{Int}(Cl(\text{Int}(B) \cap Cl(A))) \subset \text{Int}(Cl(Cl(A \cap \text{Int}(B)))) \\ &= \text{Int}(Cl(A \cap \text{Int}(B))) \subset \text{Int}(Cl(A \cap B)) \\ &\Rightarrow A \cap B \in PO(X). \end{aligned}$$

Lemma 2.2. [9] *If $A \subset Y \subset X$ and $Y \in PO(X)$, then $A \in PO(X)$ if and only if $A \in PO(Y)$.*

Lemma 2.3. [4] *Let $x \in X$. Then $x \in pcl(A)$ if and only if $A \cap V \neq \emptyset$, $\forall V \in PO(X, x)$.*

Lemma 2.4. *If $B \subset Y \subset X$ and $Y \in \tau^\alpha$, then $pcl_Y(B) = pcl_X(B) \cap Y$.*

Proof. Let $y \in pcl_Y(B)$, so that $y \in Y$. Let $V \in PO(X, y)$. By Lemma 2.1, $V \cap Y \in PO(X, y)$. Since every α -set is a p.o. set and $V \cap Y \subset Y \subset X$, Lemma 2.2 gives that $V \cap Y \in PO(Y, y)$. Consequently, $(V \cap Y) \cap B \neq \emptyset$ whence $V \cap B \neq \emptyset$. So, by Lemma 2.3, $y \in pcl_X(B)$ which implies that $y \in pcl_X(B) \cap Y$. So, $pcl_Y(B) \subset pcl_X(B) \cap Y$.

To establish the reverse inclusion, let $y \in pcl_X(B) \cap Y$. Then $y \in pcl_X(B)$, $y \in Y$. Take any $V_0 \in PO(Y, y)$. Pursuing the same reasoning as above we obtain $V_0 \in PO(X, y)$. Hence by Lemma 2.3, $V_0 \cap B \neq \emptyset$. This assures that $y \in pcl_Y(B)$. Therefore $pcl_X(B) \cap Y \subset pcl_Y(B)$.

Remark 2.2. The property of being a pre-Urysohn space is not hereditary as shown by

Example 2.2. Let X be the same topological space of Example 2.1. Then X is pre-Urysohn but the subspace $\{b, c\}$ of X is not pre-Urysohn. However, we do have

Theorem 2.2. Every α -subspace of a pre-Urysohn space (X, τ) is pre-Urysohn.

Proof. Let $Y \subset X$ and $Y \in \tau^\alpha$. Let $x, y \in Y$ and $x \neq y$. Since $Y \subset X$, x, y are also the distinct points of X . Since X is pre-Urysohn, there exist $U \in PO(X, x)$, $V \in PO(X, y)$ such that $pcl_X(U) \cap pcl_X(V) = \emptyset$. Since $Y \in \tau^\alpha$, by Lemma 2.1, $U \cap Y \in PO(X, x)$ and $V \cap Y \in PO(X, y)$. Also by Lemma 2.4,

$$\begin{aligned} pcl_Y(U \cap Y) \cap pcl_Y(V \cap Y) &= (pcl_X(U \cap Y) \cap Y) \cap (pcl_X(V \cap Y) \cap Y) \\ &= pcl_X(U \cap Y) \cap pcl_X(V \cap Y) \cap Y \\ &\subset pcl_X(U \cap Y) \cap pcl_X(V \cap Y) \\ &\subset pcl_X(U) \cap pcl_X(V) = \emptyset. \end{aligned}$$

Therefore $pcl_Y(U \cap Y) \cap pcl_Y(V \cap Y) = \emptyset$. This implies that Y is pre-Urysohn.

Definition 2.4. A function $f : X \rightarrow Y$ is called p -open [8] if $f[A] \in PO(Y)$ for all $A \in PO(X)$.

Lemma 2.5. Let the bijection $f : X \rightarrow Y$ be p -open. Then for any $F \in PC(X)$, $f[F] \in PC(Y)$.

Proof. Obvious.

Pre-Urysohn spaces remain invariant under certain bijective mapping as is shown in the next theorem.

Theorem 2.3. If the bijection $f : X \rightarrow Y$ is p -open and X is pre-Urysohn then Y is pre-Urysohn.

Proof. Let $y_1, y_2 \in Y$ and $y_1 \neq y_2$. Since f is bijective $f^{-1}(y_1), f^{-1}(y_2) \in X$ and $f^{-1}(y_1) \neq f^{-1}(y_2)$. The pre-Urysohn property of X gives the existence of sets $U \in PO(f^{-1}(y_1))$, $V \in PO(f^{-1}(y_2))$ with $pcl_X(U) \cap pcl_X(V) = \emptyset$. By [1, Lemma 2.3], $pcl_X(U)$ is a pre-closed set in X . The bijectivity and p -openness of f together then indicate, by Lemma 2.5, that $f[pcl_X(U)] \in PC(Y)$. Again from $U \subset pcl_X(U)$ it follows that $f[U] \subset f[pcl_X(U)]$. Since preclosure respects inclusion, $pcl_Y(f[U]) \subset pcl_Y(f[pcl_X(U)]) = f[pcl_X(U)]$. In like manner $pcl_Y(f[V]) \subset f[pcl_X(V)]$. Therefore, by the injectivity of f , $pcl_Y(f[U]) \cap pcl_Y(f[V]) \subset f[pcl_X(U)] \cap f[pcl_X(V)] = f[pcl_X(U)] \cap [pcl_X(V)] = f[\emptyset] = \emptyset$.

Thus p -openness of f gives the existence of two sets $f[U] \in PO(Y, y_1)$, $f[V] \in PO(Y, y_2)$, with $pcl_Y(f[U]) \cap pcl_Y(f[V]) = \emptyset$ which assures that Y is pre-Urysohn.

Definition 2.5. A function $f : X \rightarrow Y$ is said to be quasi-pre irresolute (briefly *qpi*) [15] if for each $x \in X$ and for each $V \in PO(f(x))$ there exists $U \in PO(x)$ such that $f[U] \subset pcl_Y(V)$.

Theorem 2.4. If Y is pre-Urysohn and $f : X \rightarrow Y$ is *qpi* injection, then X is pre- T_2 .

Proof. Since f is injective, for any pair of distinct points $x_1, x_2 \in X$, $f(x_1) \neq f(x_2)$. The pre-Urysohn property of Y indicates that there exist $V_i \in PO(Y, f(x_i))$, $i = 1, 2$ such that $pcl_Y(V_1) \cap pcl_Y(V_2) = \emptyset$. Hence $f^{-1}[pcl_Y(V_1)] \cap f^{-1}[pcl_Y(V_2)] = \emptyset$. Since f is *qpi*, there exists $U_i \in PO(X, x_i)$, $i = 1, 2$ such that $f[U_i] \subset pcl_Y(V_i)$, $i = 1, 2$. It, then, follows that $U_i \subset f^{-1}[pcl_Y(V_i)]$, $i = 1, 2$. Hence $U_1 \cap U_2 \subset f^{-1}[pcl_Y(V_1)] \cap f^{-1}[pcl_Y(V_2)] = \emptyset$. This implies that X is pre- T_2 .

Remark 2.3. The Theorem 2.4 has been proved by Pal and Bhattacharyya [15]. Here we give an alternative proof of the same result utilising only the definition of *qpi*.

El-Deeb *et al.* [4] introduced in 1983, the following weak form of regularity called p -regularity.

Definition 2.6. A topological space X is said to be p -regular [4] if for each closed subset F of X and each point $x \notin F$ there exist $U, V \in PO(X)$ such that $x \in U$, $F \subset V$ and $U \cap V = \emptyset$.

The following lemma of El-Deeb *et al.* [4] is needed in the sequel.

Lemma 2.6. [4, Theorem 3.2(2)]. X is p -regular if and only if for each $x \in X$ and each $U \in \Sigma(x)$ there exists $V \in PO(X)$ such that $x \in V \subset pcl(V) \subset U$.

Remark 2.4. p -regular space is not, in general, T_2 as shown by

Example 2.3. Let $X = \{a, b, c\}$ be endowed with the topology $\tau = \{\emptyset, X, \{b\}, \{a, c\}\}$. Then X is p -regular. Since $\{c\}$ is not closed, (X, τ) is not a T_1 -space and hence is not a T_2 -space.

However, the following result holds.

Theorem 2.5. *A p-regular T_2 -space is pre-Urysohn.*

Proof. Let X be p-regular and T_2 . Since X is T_2 for any pair of points $x_1, x_2 \in X, x_1 \neq x_2$ there exist $U \in \Sigma(x_1), V \in \Sigma(x_2)$ with $U \cap V = \emptyset$. Now $X - Cl(U) \in \Sigma(x_2)$. The p-regularity of X gives the existence of a $W \in PO(X, x_2)$, by Lemma 2.6, such that $x_2 \in W \subset pcl(W) \subset X - Cl(U)$. This implies that $pcl(W) \cap Cl(U) = \emptyset$ which yields that $pcl(U) \cap pcl(W) = \emptyset$. Since every open set is p.o., the above relation assures that X is pre-Urysohn.

3. Product of pre-Urysohn spaces

In this section let $\{(X_\alpha, \tau_\alpha) : \alpha \in \Lambda\}$ be a family of topological spaces where Λ is non-empty. Let A_α be any non-empty subset of X_α for each $\alpha \in \Lambda$ and $(X, \tau) = \prod_{\alpha \in \Lambda} X_\alpha$, the product space and τ , the product topology, generated by τ_α 's.

Lemma 3.1. [4] *Let n be a positive integer and $A = \prod_{j=1}^n A_{\alpha_j} \times \prod_{\alpha \neq \alpha_j} X_\alpha$. Then $A \in PO(X)$ if and only if $A_{\alpha_j} \in PO(X_{\alpha_j})$ for each $j(1 \leq j \leq n)$.*

Lemma 3.2. [4] $pcl_X(\prod_{\alpha \in \Lambda} A_\alpha) \subset \prod_{\alpha \in \Lambda} pcl_{X_\alpha}(A_\alpha)$.

Theorem 3.1. *If X_α is pre-Urysohn for each $\alpha \in \Lambda$, then X is pre-Urysohn.*

Proof. Suppose each co-ordinate space X_α is pre-Urysohn. Let $x = \langle x_\alpha : \alpha \in \Lambda \rangle$ and $y = \langle y_\alpha : \alpha \in \Lambda \rangle$ be two distinct points of the product space X . Then $x_\beta \neq y_\beta$ for some $\beta \in \Lambda$. Now X_β being pre-Urysohn and $x_\beta, y_\beta (\neq x_\beta)$ are distinct points of X_β , there exist $U_\beta, V_\beta \in PO(X_\beta)$ such that $pcl_{X_\beta}(U_\beta) \cap pcl_{X_\beta}(V_\beta) = \emptyset$. Now let $U = (\prod_{\alpha \neq \beta} X_\alpha) \times U_\beta$ and $V = (\prod_{\alpha \neq \beta} X_\alpha) \times V_\beta$. Then, by Lemma 3.1, $U, V \in PO(X)$. Also, by Lemma 3.2,

$$pcl_X(U) \subset \left(\prod_{\alpha \neq \beta} X_\alpha \right) \times pcl_{X_\beta}(U_\beta) \text{ and } pcl_X(V) \subset \left(\prod_{\alpha \neq \beta} X_\alpha \right) \times pcl_{X_\beta}(V_\beta).$$

This gives $pcl_X(U) \times pcl_X(V) \subset (\prod_{\alpha \neq \beta} X_\alpha) \times (pcl_{X_\beta}(U_\beta) \cap pcl_{X_\beta}(V_\beta)) = \emptyset$ which assures that X is pre-Urysohn.

4. Interrelationship between pre-Urysohn spaces and mappings with some strongly closed graphs

Definition 4.1. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be any function. Then the subset $G(f) = \{(x, f(x)) : x \in X\}$ of the product space $(X \times Y, \tau \times \sigma)$ is called the graph of f [7]. $G(f)$ is said to be strongly pre-closed [16] if for each $(x, y) \in X \times Y - G(f)$ there exist $U \in PO(X, x)$, $V \in PO(Y, y)$ such that $[U \times pcl(V)] \cap G(f) = \emptyset$.

A useful characterization of functions with strongly preclosed graph is the following lemma given by Paul and Bhattacharyya [16].

Lemma 4.1. The function $f : X \rightarrow Y$ has a strongly preclosed graph if and only if for $(x, y) \in X \times Y - G(f)$ there exist $U \in PO(X, x)$, $V \in PO(Y, y)$ such that $f[U] \cap pcl_Y(V) = \emptyset$.

In our investigation for the properties of mappings with strongly preclosed graphs, we arrive at the following result which is independent of any separation axioms (hence holds for pre-Urysohn spaces) but is interesting in its own right.

Theorem 4.1. If $f : X \rightarrow Y$ is a mapping with strongly preclosed graph, then for each $x \in X$, $f(x) = \bigcap \{pcl_Y(f[U] : U \in PO(X, x))\}$.

Proof. Suppose the theorem is false. Then there exists a $y \neq f(x)$ such that $y \in \bigcap \{pcl_Y(f[U] : U \in PO(X, x))\}$. This implies that $y \in pcl_Y(f[U])$, $\forall U \in PO(X, x)$. So, $V \cap f[U] \neq \emptyset$, $\forall V \in PO(Y, y)$. This, in its turn, indicates that $pcl_Y(V) \cap f[U] \supset V \cap f[U] \neq \emptyset$ which contradicts the hypothesis that f is a mapping with strongly preclosed graph. Hence the theorem.

The next theorem provides us with conditions under which mappings have their graphs strongly pre-closed.

Theorem 4.2. If $f : X \rightarrow Y$ is *qpi* and Y is pre-Urysohn then $G(f)$ is strongly pre-closed.

Proof. Let $(x, y) \in X \times Y - G(f)$. Then $y \neq f(x)$. Since Y is pre-Urysohn there exist $V \in PO(Y, y)$, $W \in PO(Y, f(x))$ such that $pcl_Y(V) \cap pcl_Y(W) = \emptyset$. Since f is *qpi* there exists $U \in PO(X, x)$ such that $f[U] \subset pcl_Y(W)$. This, therefore, implies that $f[U] \cap pcl_Y(V) = \emptyset$. So, by Lemma 4.1, $G(f)$ is strongly preclosed.

Remark 4.1. If f is replaced by weaker notion qpc then the same result is arrived at provided “pre-Urysohn” is replaced by stronger notion ‘Urysohn’. This has been obtained in [16, Theorem 5.10].

Definition 4.2. Let X and Y be two spaces. A set $S \subset X \times Y$ is termed strongly p - θ -closed (resp. strongly pre- θ -closed) with respect to $X \times Y$ if for each $(x, y) \in S$, there exist $U \in PO(X, x)$ and $V \in PO(Y, y)$ such that $[Cl_X(U) \times Cl_Y(V)] \cap S$ (resp. $[pcl_X(U) \times pcl_Y(V)] \cap S$) = \emptyset .

Definition 4.3. A function $f : X \rightarrow Y$ is said to have a strongly p - θ -closed (resp. pre- θ -closed) graph if the graph $G(f)$ is strongly p - θ -closed (resp. pre- θ -closed [15]) in $X \times Y$.

A useful characterization of functions with strongly p - θ -closed (resp. pre- θ -closed) graph with respect to $X \times Y$ is the following lemma.

Lemma 4.2. The function $f : X \rightarrow Y$ has a strongly p - θ -closed (resp. pre- θ -closed) graph with respect to $X \times Y$ if and only if for each $(x, y) \notin G(f)$ there exist $U \in PO(X, x)$, $V \in PO(Y, y)$ such that $f[Cl_X(U)] \cap Cl_Y(V)$ (resp. $f[pcl_X(U)] \cap pcl_Y(V)$, [15]) = \emptyset .

Proof. It immediately follows from Definition 4.3.

Theorem 4.3. If a preclosed surjection $f : X \rightarrow Y$ is open and continuous with strongly p - θ -closed graph, then Y is pre-Urysohn.

Proof. Let $y_1, y_2 \in Y$, $y_1 \neq y_2$. The surjectivity of f gives the existence of a $x_1 \in X$ such that $f(x_1) = y_1$ which, then, implies that $(x_1, y_2) \notin G(f)$. Since $G(f)$ is strongly p - θ -closed, there exist $U \in PO(X, x_1)$, $V \in PO(Y, y_2)$ such that $f[Cl_X(U)] \cap Cl_Y(V) = \emptyset$. By [17, Theorem 9], the preclosedness of f gives $pcl_Y(f[U]) \subset f[Cl_X(U)]$. From this it follows now that $pcl_Y(f[U]) \cap Cl_Y(V) = \emptyset$. Again the openness and continuity of f together with the preopenness of U indicates by [9, Lemma 5], that $f[U] \in PO(Y, y_1 = f(x_1))$. Thus $f[U] \in PO(Y, y_1)$, $V \in PO(Y, y_2)$ such that $\emptyset = pcl_Y(f[U]) \cap Cl_Y(V) \supset pcl_Y(f[U]) \cap pcl_Y(V)$. Hence Y is pre-Urysohn.

Definition 4.4. [15] A function $f : X \rightarrow Y$ is termed to be strongly pre-irresolute (briefly spi) if for each $x \in X$ and for each $V \in PO(f(x))$ there exists $U \in PO(x)$ such that $f[pcl_X(U)] \subset V$.

Theorem 4.4. *If the injection $f : X \rightarrow Y$ is spi and has strongly p - θ -closed graph, then X is pre-Urysohn.*

Proof. Since f is injective, for any pair of distinct points $x_1, x_2, \in X$, $f(x_1) \neq f(x_2) = y_2$ (say). Since $G(f)$ is strongly p - θ -closed and $(x_1, y_2) \notin G(f)$ there exist $U \in PO(X, x_1)$, $W \in PO(Y, y_2)$ such that $f[Cl_X(U)] \cap Cl_Y(W) = \emptyset$. The strong pre-irresoluteness of f gives the existence of a $V \in PO(X, x_2)$ such that $f[pcl_X(V)] \subset W$ whence $f[pcl_X(V)] \subset Cl_Y(W)$. We observe from above, then, $f[Cl_X(U)] \cap f[pcl_X(V)] = \emptyset$ and the injectivity of f indicates $pcl_X(U) \cap pcl_X(V) = \emptyset$. Thus X is pre-Urysohn.

Theorem 4.5. *If the spi bijection $f : X \rightarrow Y$ is p -open and has strongly p - θ -closed graph, then both X and Y are pre-Urysohn.*

Proof. Since $f : X \rightarrow Y$ is a spi injection with strongly p - θ -closed graph so, by Theorem 4, X is pre-Urysohn. Again X being pre-Urysohn and $f : X \rightarrow Y$ being p -open and bijection, by Theorem 2.3, Y is pre-Urysohn.

As stated by Mashhour *et al.* [12], Katetov made some comments on the paper [11]. Of the remarks of Katetov the following one which is needed in our development deserves special mention. To find conditions under which the intersection of any two preopen sets is preopen.

Mashhour along with others offered an answer to this remark in the form of a theorem [12, Theorem 2.3]. We, in this paper, take this theorem as a property of certain space and introduce the following definition.

Definition 4.5. *A space (X, τ) will be said to have the property P if the closure is preserved under finite intersection or equivalently, if the closure of intersection of any two subsets equals the intersection of their closures.*

Remark 4.2. From the above definition it readily follows that if a space X has the property P , then the intersection of any two p.o. sets is p.o. As a consequence of this, $PO(X)$ is a topology for X and it is finer than τ .

The following two definitions and two lemmas will be utilised in the next theorem where relation between pre-Urysohn space and pre- θ -closed graph has been studied.

Definition 4.6. *X is said to be precompact [13] if every preopen cover of X admits a finite subcover (Mashhour *et al.* [13] defined this space as strongly compact).*

Definition 4.7. *A space X is termed pre-regular [15] if for each $F \in PO(X)$ and each $x \notin F$, there exist disjoint p.o. sets U and V such that $x \in U$ and $F \subset V$.*

Lemma 4.3. [15]. *X is pre-regular if and only if for each $x \in X$ and each $U \in PO(X)$ there exists $V \in PO(X)$ such that $x \in V \subset pcl(V) \subset U$.*

The proof is straightforward and is omitted.

Lemma 4.4. *In a space X enjoying the property P, $pcl(A \cup B) = pcl(A) \cup pcl(B)$ for $A, B \subset X$.*

Proof. By [1, Theorem 6(e)], $pcl(A) \cup pcl(B) \subset pcl(A \cup B)$. To prove the reverse inclusion, we observe that the preclosedness of $pcl(A)$, $pcl(B)$ and the fact that X enjoys the property P , indicate that $X - [pcl(A) \cup pcl(B)] = [X - pcl(A)] \cap [X - pcl(B)] =$ a p.o. set. So, $pcl(A) \cup pcl(B)$ is preclosed and $A \cup B \subset pcl(A) \cup pcl(B)$. By [1, Theorem 8(e)], $pcl(A \cup B)$ is the smallest preclosed set containing $A \cup B$. Therefore, $pcl(A \cup B) \subset pcl(A) \cup pcl(B)$. Hence the lemma.

Theorem 4.6. *Let X be a pre-Urysohn space, which has the property P and Y is pre-regular. Then any p-open bijection $f : X \rightarrow Y$ with pre-compact point inverses has strongly pre- θ -closed graph.*

Proof. Suppose $(x, y) \notin G(f)$. Then $y \neq f(x)$, whence $x \notin f^{-1}(y)$. By hypothesis, $f^{-1}(y)$ is precompact in X . Since X is pre-Urysohn for each $z \in f^{-1}(y)$ there exist $U_x \in PO(X, x)$ and $U_z \in PO(X, z)$ such that

$$pcl_X(U_x) \cap pcl_X(U_z) = \emptyset. \quad (1)$$

If we now allow z to run over $f^{-1}(y)$, we obtain a family of U_z 's whose union contains $f^{-1}(y)$. Hence the family $\{U_z : z \in f^{-1}(y)\}$ is a preopen cover of $f^{-1}(y)$. The precompactness of $f^{-1}(y)$ gives the existence of a finite number of points z_1, z_2, \dots, z_n in $f^{-1}(y)$ such that $f^{-1}(y) \subset \bigcup_{k=1}^n U_{z_k}$. Now consider the corresponding p.o. sets $U_{x_1}, U_{x_2}, \dots, U_{x_n}$ containing x . Let $G = \bigcap \{U_{x_k} : 1 \leq k \leq n\}$ and $H = \bigcup \{U_{z_i} : 1 \leq i \leq n\}$ where each pair (U_{x_k}, U_{z_k}) ($k = 1, 2, \dots, n$) satisfies the property (1). Again since X has the property P , by Lemma 4.4, $pcl_X(\bigcup_{i=1}^n U_{z_i}) = \bigcup_{i=1}^n pcl_X(U_{z_i})$. That is $pcl_X(H) = \bigcup_{i=1}^n pcl_X(U_{z_i})$. It is now easy to check that

$$pcl_X(G) \cap pcl_X(H) = \emptyset. \quad (2)$$

Also H is a p.o. set containing $f^{-1}(y)$. By the bijectivity and p-openness of f , $y \in f[H] \in PO(Y)$. Now $y \notin (f[H])^c \in PO(Y)$. Hence the pre-regularity of Y assures the existence of two sets $V, W \in PO(Y)$ such that $y \in V$, $(f[H])^c \subset W$ and $V \cap W = \emptyset$. The disjointness of V and W gives $V \subset W^c$ and this yields $pcl_Y(V) \subset W^c$ because of the preclosedness of W^c . So, by the injectivity of f ,

$$f^{-1}[pcl_Y(V)] \subset f^{-1}[W^c] \subset f^{-1}(f[H]) = H \subset pcl_X(H).$$

It now follows from (2), $pcl_X(G) \cap f^{-1}[pcl_Y(V)] = \emptyset$ which gives $f[pcl_X(G)] \cap pcl_Y(V) = \emptyset$. Therefore, f has strongly pre- θ -closed graph. This proves the theorem.

The study of pre-Urysohn spaces now raises the following pertinent and natural question: What additional condition is needed to make a pre-Urysohn space a Urysohn one? We conclude the present investigation after offering an answer to this question.

Bourbaki has introduced the following:

Definition 4.8. [3] *A topological space X is termed submaximal if each dense subset of X is open.*

Ganster, among others, established the following result.

Theorem 4.7. [6] *In any topological space (X, τ) . $A \in PO(X)$ if and only if A can be represented as $A = D \cap G$ where D is dense and $G \in \tau$.*

Theorem 4.8. *(X, τ) is submaximal if and only if $PO(X) = \tau$.*

Proof. Follows from Definition 4.8 and Theorem 4.7.

Theorem 4.9. *A pre-Urysohn space X is Urysohn if and only if it is submaximal.*

To prove the above theorem we need the following

Lemma 4.5. [4] *In a topological space X if $A \in SO(X)$ then $pcl(A) = Cl(A)$.*

Proof of the theorem. Necessity. Suppose X is a pre-Urysohn space which is submaximal. Since X is pre-Urysohn, for $x, y \in X$, $x \neq y$ there exist $U \in PO(X, x)$, $V \in PO(X, y)$ such that $pcl(U) \cap pcl(V) = \emptyset$. Since X is submaximal, by Theorem 4.8, $U, V \in \tau$. It then follows from Lemma 4.5, $Cl(U) \cap Cl(V) = \emptyset$. Thus X is Urysohn.

Sufficiency follows from the fact that every Urysohn space is pre-Urysohn.

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