BULLETIN
of the
MALAYSIAN MATHEMATICAL SOCIETY

Bull. Malaysian Math. Soc. (Second Series) **22** (1999) 35-43

A Survey of Divisibility Tests with a Historical Perspective

MANGHO AHUJA AND JAMES BRUENING
Department of Mathematics, Southeast Missouri State University
One University Plaza, Cape Girardeau, MO 63701
e-mail: mahuja@semovm.semo.edu
e-mail: jbruening@semovm.semo.edu

1. Introduction

There is nothing simpler than the test for divisibility of a number by 2 or 5. A number N is divisible by 2 if its last digit is an even integer, and is divisible by 5 if the last digit is either 0 or 5. Equally simple is the reason why the tests work. But not so simple is the reason for the following test: A number N is divisible by 3 if the sum of its digits is divisible by 3. However, all three tests can be easily explained using modulo arithmetic. Since our Hindu Arabic numeral system has base 10, the test for 2 works because $10 \equiv 0 \pmod 2$, the test for 5 works because $10 \equiv 0 \pmod 5$, and the test for 3 works because $10 \equiv 1 \pmod 3$..

Can every divisibility test be explained by using the concept of Modulo Arithmetic? What other concepts form the basis for divisibility tests? These and other questions were the motivation for conducting a survey of divisibility tests. The purpose of the survey is to understand the concepts underlying the divisibility tests. To collect data, the authors used the ERIC system (Educational Resources Information Clearinghouse) on the computer, and located more than 70 articles on divisibility tests published in USA during the period 1965-97. Some of these are listed in the references. This paper is a summary of that survey. It classifies tests according to the mathematical concept used in the test. The authors hope that almost every mathematical concept used for a divisibility test is included here.

It is not surprising that every one of the tests in the survey has a historical origin. In other words, none of the tests in the survey are "new." Linking each test to its historical origin is an essential feature of this paper. The authors have intentionally chosen to write in a less rigid and informal style, sometimes using the first person "we" instead of the third person "the authors." We hope that this will make the paper more readable to undergraduates and teachers. Both authors are strongly committed to helping elementary and secondary school teachers of mathematics.*

2. Early Contributions from the Moslem mathematicians

Divisibility tests have been known since antiquity. The Babylonian Talmud contains a test for divisibility of a number by 7 [6]. Most of the early tests result from the genius of the Islamic mathematicians. Ibn Sina (980-1037 AD), known as Avicenna in the western world, is said to have discovered the method of "casting out 9" to check arithmetic operations. Al-Karkhi (c. 1015), who had studied Diophantus and is famous for his work *Fakhri* on Algebra, had a test for 9 and for 11. The "Father of Algebra," al-Khowarizimi (9th century), had a test for 9. The Arab mathematician al-Banna (1256-1321 AD) had tests for 7, 8, and 9. In the 15th century, another Arab mathematician, Sibt el-Maridini, checked addition by "casting out multiples of 7 or 8."

The Renaissance mathematicians were not far behind. Leonardo Fibonacci of Pisa, in his famous book *Liber Abaci* (1202 AD), had a proof of the test for 9, and indicated tests for 7 and 11. For a complete chronological record of the early tests the readers may refer to Volume I of Leonard Dickson's classic *History of the Theory of Numbers* [6]. For this paper we did not find it appropriate to classify the tests in chronological fashion. Instead, we have grouped the tests according to the mathematical concepts involved. Of course, any test could involve more than one concept. The tests within each group have a common thread, a common concept. We now begin with divisibility tests arranged in different groups.

I. A number N can be written as $N = 10t + u$. For example, if $N = 2536$, then $N = 10(253) + 6$, thus $t = 253$ and $u = 6$. The tests [7, 12, 13, 15, 16, 17] in this group *add a multiple of the unit digit* to the rest of the number (or to its multiple) and check divisibility of the number thus obtained. For example to check for divisibility by 7, we may proceed as follows:

$$N = 10t + u \equiv 3t - 6u \pmod 7 .$$

Since 3 is relatively prime to 7, we can factor out 3 and get $10t + u \equiv 0 \pmod 7$ iff $t - 2u \equiv 0 \pmod 7$. This gives us a test for 7.

Test Ia for 7. A number $N = 10t + u$ is divisible by 7 iff $t - 2u$ is divisible by 7.

Again, $N = 10t + u \equiv 10t + 8u \pmod 7$. We factor out 2 and conclude that $N = 10t + u \equiv 0 \pmod 7$ iff $5t + 4u \equiv 0 \pmod 7$.

Test Ib for 7. A number $N = 10t + u$ is divisible by 7 iff $5t + 4u$ is divisible by 7.

To design a test for 13, let $N = 10t + u$. Then $10t + u \equiv -3t - 12u \pmod{13}$. By factoring out -3, we find that $10t + u \equiv 0 \pmod{13}$ iff $t + 4u \equiv 0 \pmod{13}$.

Test Ic for 13. A number N=10t+u is divisible by 13 iff t +4u is divisible by 13.

Let us use this idea to develop a test for 97. Let $N = 10t + u$. Then $10t + u \equiv 10t + 98u \pmod{97}$. By factoring out 2, we find that $10t + u \equiv 0 \pmod{97}$ iff $5t + 49u \equiv 0 \pmod{97}$.

Test Id for 97. A number $N = 10t + u$ is divisible by 97 iff $5t + 49u$ is divisible by 97.

2. Comments and history

1. It is obvious that test Id is not very practical. It involves two multiplications and one addition, and even then the new number thus obtained is not much simpler than the original number *N*. Throughout this paper we will encounter tests that have very little practical use. Nonetheless such tests will be included for discussion in the paper.

2. Going back in history, the idea of these tests was known to A. Zbikowsi (1861 AD), V. Ziepel (1861 AD), and G. Dostor (1879 AD) [6]. Dostor generalized these tests in the following manner. Suppose we have to test if a prime number *p* divides a number *N*. The case when *p* is 2 or 5 is easy, so we assume that *p* is relative prime to 10. In that case we can find a multiple of *p* that is of the type $10k + 1$ (or $10k - 1$). Since *k* and *p* must be relatively prime, $N = 10t + u \equiv 0 \pmod{p}$ iff $10kt + ku \equiv 0 \pmod{p}$, but $10k \pm 1 \equiv 0 \pmod{p}$ implies $10k \equiv \pm 1 \pmod{p}$, so $10kt + ku \equiv 0 \pmod{p}$ iff $\mp t + ku \equiv 0 \pmod{p}$ iff $t \mp ku \equiv 0 \pmod{p}$. Thus a number $N = 10t + u$ is divisible by *p* iff $t - ku$ (or $t + ku$) is divisible by *p* [3, 4, 7, 9, 19, 20, 21]. Both Benjamin Bold [3] and Frank Smith [19] have shown this general version of the test. For each prime *p*, they obtained a suitable *k* and indicated if $t - ku$ or $t + ku$ should be used. Using this idea, we can find a test for 17. A multiple of 17 that ends in either 1 or 9 is 51. Because $17 \times 3 = 10(5) + 1$, the value of *k* is 5. The test will be:

Test Ie for 17. A number $N = 10t + u$ is divisible by 17 iff $t - 5u$ is divisible by 17.

To design a test for 19, we see that $19 \times 1 = 10(2) - 1$. Here $k = 2$.

Test If for 19. A number $N = 10t + u$ is divisible by 19 iff $t + 2u$ is divisible by 19.

3. It is obvious that when the number *N* is large, the test has to be used repeatedly. For example, suppose $N = 335986$ and we want to use test Ia to check if 7 is a divisor. The test says we have to look at another number $33598 - 2(6)$, or 33586. By repeating this test we get a succession of numbers each smaller than the previous one: 335986, 33586, then $3358 - 2(6) = 3346$, followed by $334 - 2(6) = 322$, and finally $32 - 2(2) = 28$. Since 28 is divisible by 7, then so is *N*. The practicality of the test lies in its being simple to execute and in its giving us the final decision, "*yes* or *no*," in a few short steps.

II. This is a group of tests where the digits of N are divided into two or three groups. To test divisibility of N, we look at a linear combination of these groups of digits. As we will see below, there are many different styles of tests in this group. We may think of a number N as $N = 100h + 10t + u$. If $N = 24539$, it can be written as $N = 100(245) + 10(3) + 9$, thus $h = 245$, $t = 3$, and $u = 9$. Here the digits of N are divided into three groups, 245, 3, and 9. Since $10 \equiv 3 \pmod 7$ and $100 \equiv 2 \pmod 7$, we can say $N = 100h + 10t + u \equiv 2h + 3t + u \pmod 7$. So the test will say:

Test IIa for 7. $N = 100h + 10t + u$ is divisible by 7 iff $2h + 3t + u$ is divisible by 7 [10].

We may also think of a number N as $N = 100h + x$. For example, if $N = 24539$, it can be written as $N = 100(245) + 39$, so that $h = 245$ and $x = 39$. Here the digits of N are divided into two groups, 245 and 39. The tests in this group add a multiple of h to x and test divisibility of the new number thus obtained [7, 15]. Now, since $100 \equiv 2 \pmod 7$ we have a test:

Test IIb for 7. A number $N = 100h + x$ is divisible by 7 iff $2h + x$ is divisible by 7.

Similar tests can be designed [22] when we consider $N = 1000a + b$, where b is the number consisting of the last three digits of N. Since $1000 \equiv -1 \pmod 7$, we have a test:

Test IIc for 7. A number $N = 1000a + b$ is divisible by 7 iff $a - b$ is divisible by 7.

3. Comments and history

1. Tests using groups of two were used by A. Loir (1898) [6]. He devised a test for 43 using the fact that $43 \times 7 = 301$, and $300 \equiv -1 \pmod{43}$. Let $N = 100a + b$, where b is a two-digit number. Then N is divisible by 43 iff $300a + 3b$ is. But $300a + 3b \equiv -a + 3b \pmod{43}$. This gives us the test:

Test IId for 43. A number $N = 100a + b$ is divisible by 43 iff $a - 3b$ is divisible by 43.

2. Similarly Castelvetri (1757) designed tests for 99 and 999 [6]. Since $100 \equiv 1 \pmod{99}$, he arranged the digits of the number in groups of two. So a number N with 6 digits a,b,c,d,e,f is $N = abcdef = (ab)(cd)(ef)$. Replacing each 100 as 1, he looked at the number $ef + cd + ab$ and tested it for divisibility by 99.

3. Rearranging the digits in groups of twos or threes has the following underlying idea. To divide a number N by a prime p, first we find the smallest number h such that $10^h \equiv 1 (\text{mod } p)$. The existence of such an h is guaranteed by Fermat's Little Theorem [7, *p*. 89]. We can then divide the digits of N into groups of h, adding zeros to the front of the number if necessary. Let us illustrate this method using $p = 37$. Since $10^3 \equiv 1 (\text{mod } 37)$, the value of h is 3, so we will arrange the digits of the number in groups of 3. Since $10^3 \equiv 1 (\text{mod } 37)$, we can replace each 1000 by 1. But this amounts to taking the sum of all three-digit numbers forming the groups. Here is an example. Let $N = 256234158277$. Breaking up the digits in groups of three, we get $N = 256, 234, 158, 277$. Replacing 1000 by 1, we get $277 + 158 + 234 + 256 = 925$, which is divisible by 37, hence so is N. If $N = 16, 158, 340, 078$, we treat it as $016, 158, 340, 078$, and then check if $078 + 340 + 158 + 016 = 592$ is divisible by 37.

Test IIe for 37. A number N whose digits are written in groups of three, like $\cdots, def, abc$ is divisible by 37 iff the sum $abc + def + \cdots$ is divisible by 37.

III. We may think of a number N as a polynomial in 10 (instead of x), and use the symbol $P(10)$. Thus $1967 = 1(10^3) + 9(10^2) + 6(10) + 7$. The tests in this group [1, 7] use the following idea:

Test III (general) for any divisor d. If $10 \equiv k (\text{mod } d)$, then a number $N = P(10)$ is divisible by d iff $P(k)$ is divisible by d.

Since $10 \equiv 3 (\text{mod } 7)$, we have

$$\begin{aligned} 1967 &\equiv 1(3^3) + 9(3^2) + 6(3) + 7 (\text{mod } 7) \equiv 1(27) + 9(9) + 6(3) + 7 (\text{mod } 7) \\ &\equiv 27 + 81 + 18 + 7 (\text{mod } 7) \equiv 133 (\text{mod } 7) \equiv 0 (\text{mod } 7) . \end{aligned}$$

This gives us a test for 7.

Test IIIa for 7. A number $N = P(10)$ is divisible by 7 iff $P(3)$ is divisible by 7.

The well-known tests for 3, 9 and 11 can be classified in this group [5]. Since $10 \equiv 1 (\text{mod } 3)$, $10 \equiv 1 (\text{mod } 9)$ and $10 \equiv -1 (\text{mod } 11)$, $P(1)$ amounts to the sum of the digits, whereas $P(-1)$ equals the difference of the two sums of the odd and even-numbered digits; thus we have the following tests for 9 and 11:

Test IIIb for 9. A number N is divisible by 9 iff the sum of its digits is divisible by 9.

Test IIIc for 11. A number N is divisible by 11 iff the difference of the two sums of the odd and even-numbered digits is divisible by 11.

4. Comments and history

Throughout history many mathematicians have used the idea of expressing a number as a polynomial in 10, or 100, or 1000, to devise different tests. Fontes (1893) considered a number N as a polynomial in 100 (not 10), whose coefficients were two-digit numbers [6]. He expressed the number $N = 10433$ as $1(100^2) + 4(100) + 33$ and devised a test for 19. Since $100 \equiv 5 (\mathrm{mod}\, 19)$ we can replace each 100 by 5. We have the following test:

Test IIIe for 19. A number $N = P(100)$, i.e., a polynomial in 100 with two-digit coefficients, is divisible by 19 iff $P(5)$ is divisible by 19.

We see that the test IIe for 37 could have been classified here. We first think of a number N as a polynomial in 10^3 with three-digit numbers as coefficients. For example, a number $N = 1,234,647,289$ can be expressed as a polynomial in 1000 as follows: $N = 1(1000^3) + 234(1000^2) + 647(1000) + 289$. Let us denote N by $P(1000)$. Since $1000 \equiv 1 (\mathrm{mod}\, 37)$, the test will say:

Test IIIf for 37. A number N expressed as $P(1000)$, a polynomial in 1000, is divisible by 37 iff $P(1)$ is divisible by 37.

A.L. Crelle (1844) used the fact that $1000 \equiv -1 (\mathrm{mod}\, 7)$. Here is an example. Suppose we want to test if 7 divides the number $N = 235,689,436,773$. Considering N as a polynomial in 1000, we may write

$$N = P(1000) = 235(1000^3) + 689(1000^2) + 436(1000) + 773.$$

Using $1000 \equiv -1 (\mathrm{mod}\, 7)$, we see that N is divisible by 7 if $N' = 773 - 436 + 689 - 235 = 791$ is. Since N' is divisible by 7, so is N.

IV. We may as well call this a group of Miscellaneous tests, because there is no central idea connecting them. Each test in this group uses a different concept.

Test IVa. In this test, to check the divisibility of a number $N = abcdef$ by a prime p, we add or subtract a suitable multiple of p to N so that the result ends in 0. This is possible if the prime p is relative prime to 10.

In [2], Bezuszka showed the divisibility of a number N by a prime p, say $p = 7$, as follows. We add a suitable multiple of 7 to N so that the sum ends in 0. Since 10 is relatively prime to 7, we can delete the 0 and test the new number N' thus obtained.

Here is an example of how the test works. To test 2366 for 7, we add to it a multiple of 7 that ends in 4. Since $2 \times 7 = 14$, adding 2366 and 14 gives us 2380. We drop the 0 and look at $N' = 238$. To repeat the test we need a multiple of 7 that ends in 2. Since $6 \times 7 = 42$, we add 42 to 238 which results in 280. Dropping the 0, we see that 28 is divisible by 7, hence so is *N*.

C.F. Moller and C. Holten (1875) used the same idea but subtracted (instead of adding) a multiple of 7 [6]. Here is how their test works. Let $N = 157892$. We subtract a multiple of 7 that ends in 2 to get $157892 - 42 = 157850$, Deleting 0 we now consider 15785. Subtracting again a multiple of 7 that ends in 5, we get $15785 - 35 = 15750$, and deleting 0 we look at 1575. Again subtracting 35 and deleting 0, we get 154. Subtracting 14 and deleting 0, we get 14. This is divisible by 7, hence so is *N*.

Test IVb. Let *N* be a number written as $N = abcdefg$. To check divisibility by *p*, we replace the number *ab* by $ab(\text{mod } p)$. Suppose that is *x*. Now we look at the new number $N' = xcdefg$.

Example. We will test divisibility of 2366 by 7. We replace 23 by 2 because $23 \equiv 2(\text{mod } 7)$. We look at the new number 266. Repeating the test, we replace 26 by $26(\text{mod } 7)$ which is 5. The new number is 56 which is divisible by 7, hence so is *N*.

We may use more than two digits to apply this test. In other words, if $N = abcdefg$, and we are testing divisibility by *p*, we may replace the number *abc* by a number $x = abc(\text{mod } p)$, and then look at the new number *xdefg*. Or we may replace *abcd* by $y = abcd(\text{mod } p)$, and look at the new number *yefg* [11].

Test IVc. There are various tests when the digits of a number *N* have a certain pattern. Our number *N* may be of the type *aabbcc*, or *ababab*, or *abcabc*. In each case we make use of the pattern and devise suitable tests.

Example. If $N = 234234$, then $N = 234(1001)$, and any divisor of 1001 or 234 will divide *N*.

Example. If $N = ababab$, then $N = ab(10101)$, and we look at the divisors of 10101 as well as *ab*.

These tests are not so useful because they lack generality.

5. Conclusion

We have seen that a multitude of tests are available to us, and in general, it is rather difficult to say if one test is better than another one. Of course, the ease of applying the test should be our major concern while selecting the test. Yet, while teaching divisibility tests, our focus should be less on the test itself and more on the inherent mathematics involved. As we have seen above, a fascinating interplay of interconnecting branches of mathematics, mainly Number Theory and Algebra, is displayed in these tests. A teacher must not only teach the test, but also use it to reinforce the underlying mathematical ideas. As is true in all mathematics, it is not enough to only teach how to obtain the final result. A good teacher will always try to display the inner beauty of mathematics. Like all journeys, it is not the final destination, but the excitement of the adventure and the experience along the way, which makes the journey so memorable.

References

1. L. Berenson, A divisibility test for amateur discoverers, *Arithmetic Teacher* **17** (1970), 39-41.
2. S.J. Bezuszka, A test for divisibility by primes, *Arithmetic Teacher* **33** (1985), 36-38.
3. B. Bold, A general test for divisibility by any prime (except 2 and 5), *The Mathematics Teacher* **58** (1965), 311-312.
4. D.L. Boley, A simple general criterion of divisibility, *The Mathematics Teacher* **61** (1968), 501-502.
5. D.M. Burton, *Elementary Number Theory,* 4th edition, The McGraw-Hill Companies, Inc., New York, 1998.
6. L.E. Dickson, *History of the Theory of Numbers*, Vol.1, Chelsea, New York, 1971.
7. R.L. Francis, Divisibility discoveries, *AMATYC Journal* **17** No. 2 (1996), 12-16.
8 F.J. Gardella, Divisibility - Another route, *Arithmetic Teacher* **3l** (1984), 55-56.
9. R.E. Kennedy, Divisibility by integers ending in 1, 2, 7, or 9, *The Mathematics Teacher* **64** (1971), 137-138.
10. C.T. Long, A simpler “7” divisibility rule, *The Mathematics Teacher* **64** (1971), 473-475.
11. E.R. Matthews, A simple 7-divisibility rule, *The Mathematics Teacher* **62** (1969), 461-464.
12. R.L. Morton, Divisibility by 7, 11, 13, and greater primes, *The Mathematics Teacher* **16** (1968), 370-373.
13. C. Oliver, Gus’s magic numbers: A key to the divisibility test for primes, *Arithmetic Teacher* **19** (1972), 183-189.
14. D. Papadopoulos, A divisibility test for large numbers, *Mathematical Spectrum* **24** (1991-1992), 114-115.
15. E. Parkerson, Patterns in divisibility, *Arithmetic Teacher* **25** (1978), 58.
16. F. Rogers, Divisibility rule for seven, *Arithmetic Teacher* **16** (1969), 63-64.
17. D.A. Rosen, An uncommon divisibility test, *Mathematics Teaching* **76** (1976), 32-33.
18. G. Schatzman, 252/7: A divisibility pattern, *The Mathematics Teacher* **79** (1986), 542-546.
19. F. Smith, Divisibility rules for the first fifteen primes, *Arithmetic Teacher* **18** (1971), 85-87.
20. L.T. Smith, A general test of divisibility, *The Mathematics Teacher* **71** (1978), 668-669.
21. B. Stastny, A test for divisibility, *The Mathematics Teacher* **53** (1960), 627-631.
22. W. Szetela, A general divisibility test for whole numbers, *The Mathematics Teacher* **73** (1980), 223-225.

* The first author spent the summer of 1997 visiting Malaysia, and enjoyed the hospitality offered by the faculty and administration of the USM (Universiti Sains Malaysia), Penang, Malaysia. He also learned that the problems of teachers in Malaysia are similar to those in the USA. To support and champion their cause, this paper is dedicated to the elementary and secondary school teachers of mathematics.