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A Note on Exact Sequences

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Abstract. In this paper a new notion of an exact sequence is introduced which is called *U*-exact sequence. Some interesting results concerning this concept are proved.

1. Introduction

Some basic definitions and theorems about exact sequences can be found in [1] and [2]. Suppose that we have the following exact sequence of *R*-modules and *R*-homomorphisms

$$\cdots \to M_{i-1} \xrightarrow{f_i} M_i \xrightarrow{f_{i+1}} M_{i+1} \to \cdots$$

then $\mathrm{Im}(f_i) = \ker(f_{i+1})$ or $\mathrm{Im}(f_i) = f_{i+1}^{-1}(\{0\})$. Our aim in this paper is to introduce a new notion exact sequence. It is a natural question to ask what does happen if we substitute a submodule U_{i+1} of M_{i+1} instead of the trivial submodule $\{0\}$ in the above definition.

In this paper we introduce the concept of *U*-exact sequences and answer the above question. We also obtain properties of the *U*-exact sequences, for example, a generalization of Five Lemma holds, and we obtain a relationship between *U*-exact sequences and ascending chain condition similar to the ordinary exact sequences. Finally from a *U*-exact sequence, we get an S^{-1} *U*-exact sequence of module of fractions.

Throughout this paper we let R to be a commutative ring and M_i, A, B, C be *R*-modules.

Definition 1. *A sequence of R-modules and R-homomorphisms*

$$\cdots \to M_{i-1} \xrightarrow{f_i} M_i \xrightarrow{f_{i+1}} M_{i+1} \to \cdots$$

is said to be U_{i+1}*-exact (where* U_{i+1} *is a submodule of* M_{i+1}*) at* M_i *if* $\mathrm{Im}(f_i) = f_{i+1}^{-1}(U_{i+1})$.

Definition 2. *Let* $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ *be* $\{0\}$*-exact at A, U-exact at B and* $\{0\}$*-exact at C, then to simplify, we say the sequence is U-exact.*

A sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is U-exact if and only if f is injective, g is surjective and $\mathrm{Im}(f) = g^{-1}(U)$.

Example 3. Let M be an R-module and U be a submodule of M, the following sequence is U-exact

$$0 \to U \xrightarrow{\subseteq} M \xrightarrow{i} M \to 0.$$

Example 4. Let U and V be two submodules of M such that $V \subseteq U \subseteq M$ then the sequence $0 \to U \xrightarrow{\subseteq} M \xrightarrow{\pi} M/V \to 0$ is U/V-exact where π is the natural homomorphism.

Corollary 5. *Let* $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ *be a U-exact sequence. Then the sequence is exact if and only if* $U = \{0\}$.

Also the dual notion of a U-exact sequence can be define as follows:

Definition 6. *A sequence* $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ *is V-coexact (V, a submodule of A), if f is injective, g is surjective and* $f(V) = \ker(g)$.

By the Snake Lemma, given a U-exact short sequence, there exists a natural map $A \to U$ with kernel V. Hence these two notions are equivalent.

2. Some results

Lemma 7. *(Generalization of Five Lemma). Let the following diagram be a commutative diagram of R-modules and R-homomorphisms such that the first row is U-exact and the second row is* U'*-exact. Then*

$$\begin{array}{ccccccccc} 0 & \to & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \to & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \to & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \to & 0 \end{array}$$

i) If α *and* γ *are monomorphisms, then* β *is a monomorphism,*
ii) If α *and* γ *are epimorphisms, then* β *is an epimorphisms,*
iii) If α *and* γ *are isomorphisms, then* β *is an isomorphism.*

Proof. The proof is straightforward and omitted.

Definition 8. *The U-exact sequence* $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ *is said to be isomorphic to the* U'*-exact sequence* $0 \to A' \xrightarrow{f'} B' \xrightarrow{g'} C' \to 0$ *if there exists a commutative diagram of R-homomorphisms*

$$\begin{array}{ccccccccc} 0 & \to & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \to & 0 \\ & & \downarrow\alpha & & \downarrow\beta & & \downarrow\gamma & & \\ 0 & \to & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \to & 0 \end{array}$$

such that α, β *and* γ *are isomorphisms.*

Corollary 9. *Isomorphism of U-exact sequences is an equivalence relation.*

Proposition 10. *If two U-exact and* U'*-exact sequences are isomorphic then* $U \cong U'$.

Proof. We consider the diagram defined in Definition 8. We have

$$\beta f = f'\alpha, \ \gamma g = g'\beta, \ \mathrm{Im}(f) = g^{-1}(U) \ \text{ and } \ \mathrm{Im}(f') = g'^{-1}(U').$$

It is enough to show that $\gamma(U) = U'$. Suppose that $x \in \gamma(U)$ then there exists $u \in U$ such that $x = \gamma(u)$. Since $u \in U$ so $g^{-1}(u) \subseteq g^{-1}(U) = \mathrm{Im}(f)$ and hence $gg^{-1}(u) \subseteq gf(A)$ which implies $u \in gf(A)$ then there exists $a_0 \in A$ such that $u = gf(a_0)$. Now, we have

$$x = \gamma(u) = \gamma(gf)(a_0) = (\gamma g)f(a_0) = (g'\beta)f(a_0)$$
$$= g'(\beta f)(a_0) = g'(f'\alpha)(a_0) = (g'f')(\alpha(\alpha_0)),$$

and so $x \in g'f'(A') \subseteq U'$. Therefore $\gamma(U) \subseteq U'$. Similarly, we get $\gamma^{-1}(U') \subseteq U$, and hence $U' \subseteq \gamma(U)'$.

Theorem 11. *Let* $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ *be a U-exact sequence. Then B satisfies the ascending (resp. descending) chain condition on submodules if and only if A and C do.*

Proof. If B satisfies the ascending chain condition then obviously A and C satisfy the ascending chain condition.

Suppose that *A* and *C* satisfy the ascending chain condition (*ACC*). So does *C/U*. By the definition of *U*-exactness, $0 \to A \to B \to C/U \to 0$ is a short exact sequence in the usual sence. Since both *A* and *C/U* satisfy the *ACC*, so does *B*.

Proposition 12. *Let S be a multiplicative subset of R and the sequence* $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ *be U-exact. Then the sequence*

$$0 \to S^{-1}A \xrightarrow{S^{-1}(f)} S^{-1}B \xrightarrow{S^{-1}(g)} S^{-1}C \to 0$$

is a *U*-exact sequence.

Proof. Obviously, $S^{-1}(f)$ is injective and $S^{-1}(g)$ is surjective. Suppose that $b/s \in \mathrm{Im}(S^{-1}(f))$ then there exists $a \in A$ such that $f(a)/s = b/s$. Since $f(a) \in \mathrm{Im}(f) = g^{-1}(U)$ there exists $u \in U$ such that $f(a) \in g^{-1}(u)$ and so $f(a)/s \in (S^{-1}(g))^{-1}(u/s)$ which implies $f(a)/s \in (S^{-1}(g))^{-1}(S^{-1}U)$.

Conversely, if $u/t \in (S^{-1}(g))^{-1}(S^{-1}U)$ then $u/t \in \{x/s \,|\, g(x) \in U, s \in S\}$. Therefore for some x_0 where $g(x_0) \in U$ we have $u/t = x_0/s$. From $g(x_0) \in U$ we get $x \in \mathrm{Im}(f)$ and so $x_0/s \in \mathrm{Im}(S^{-1}(f))$. Therefore $u/t \in \mathrm{Im}(S^{-1}(f))$.

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