BULLETIN
of the
MALAYSIAN MATHEMATICAL SOCIETY

Bull. Malaysian Math. Soc. (Second Series) **22** (1999) 57-65

A Cusp-Like Free-Surface Flow Caused by a Source/Sink in a Channel of Finite Depth

LEO HARI WIRYANTO
Department of Mathematics, Institut Teknologi Bandung, Jl. Ganesha 10 Bandung, Indonesia
e-mail: leo@dns.math.itb.ac.id

Abstract. A flow caused by a line sink/source is considered in a channel of finite depth. The sink is placed at the bottom of the channel, and produces a free surface with a cusp pointing to the sink. Numerical solutions of this free-surface flow are computed by an integral equation method. The relationship between the nondimensional parameter Froude number F, based on the downstream flow, and the distance y_a of the separation point of the cusp to the sink is presented as a plot from our computations. When $F \to \infty$, we obtain the limiting cusp solution with $y_a = 0.363$.

1. Introduction

Since the 1980s much progress has been achieved in numerical calculations of free-surface flow caused by a source/sink. The fluid surface above the sink may be drawn down such that it is cusp-like. Tuck and Vanden-Broeck [3] computed such a problem for fluid of infinite depth. When the wall below the sink is sloped with angle β from the vertical and the sink is located at the corner, Hocking [1] obtained numerical solutions for a sequence of angles ranging from 0 to $\pi/2$. In each case, there was just one cusp solution. Vanden-Broeck and Keller [4] then recomputed these 2-D flows, and found horizontal-bottom solutions for all values of a nondimensional Froude number F greater than some particular value. The results treated by Tuck and Vanden-Broeck, and Hocking were also confirmed in Vanden-Broeck and Keller [4].

In this paper, we recompute solutions of Vanden-Broeck and Keller for a sink at the corner of the horizontal bottom by an integral equation method, and we determine the limiting solution as $F \to \infty$. For finite Froude numbers, we agree with the results in [4], i.e., that there is a relationship between F and the nondimensional distance y_a of the separation point to the bottom with $F > 1$. When we increase the Froude number, we found that y_a also increases but it does not exceed a certain value obtained by solving the problem with neglecting the effect of gravity. Our computations show that the output parameter is in an interval $0.244 < y_a < 0.363$.

In solving the cusp flow, we first formulate the problem as an integral equation for a hodograph variable. The derivation of this equation follows that of Wiryanto [5] for free-surface flow emerging from a tunnel. Wiryanto assumed that the flow is uniform in the tunnel and far downstream. The upper bound of the flow domain is a streamline which changes the condition from solid to free surface. This problem is similar to the cusp flow here. The flow in the tunnel can be described as the source, and the upper streamline, in Wiryanto [5], corresponds to the streamline along the vertical wall and the free surface of the cusp flow. In order to solve the boundary value problem in the physical plane, Wiryanto mapped the flow domain into a lower half-plane which is called an artificial plane, and formulated the boundary value problem into an integral equation presenting the relationship between the hodograph variable and the artificial one. For the cusp flow, the derivation of the integral equation is presented in Section 2. A numerical procedure is then developed for solving the integral equation. We present this procedure in Section 3. In the case of the flow where the effect of gravity is neglected, the integral equation is derived in Section 4. The solution of this zero-gravity case is then compared with the results for finite Froude numbers. We describe these results in Section 5.

2. Problem formulation

The steady, irrotational motion of an inviscid, incompressible fluid in the presence of gravity is to be examined. The fluid is of finite depth and has a free surface above a line source *S* placed at the corner of a semi-infinite channel (see Figure 1). The flow caused by this source produces a uniform stream far downstream with depth *H* and velocity *U*. Therefore, we can define the Froude number as

$$F = \frac{U}{\sqrt{gH}},$$

where g is the acceleration due to gravity. On the other side, the free surface, just above the source, separates smoothly from the vertical wall at point *A* with level H_a.

Now we introduce the system of coordinates for the flow domain as shown in Figure 1. We choose the Cartesian coordinate with the origin at the source *S*. Therefore, the flow domain can be written as a complex plane $z = x + iy$. Another complex plane for the flow domain can be defined as $f = \phi + i\psi$, where ϕ and ψ are the velocity potential and stream function respectively. The flow domain in the *f*-plane is an infinite strip (see Figure 2(a)). Without loss of generality we choose the origin corresponding to the separation point *A*.

Mathematically, our problem is to determine the complex potential $f(z)$ which satisfies Laplace's equation $(\nabla^2 f = 0)$ within the flow domain, conditions of no flow across the solid boundaries and the free surface, and the condition of constant pressure on the free surface provided by Bernoulli's equation

$$\frac{1}{2}(\phi_x^2 + \phi_y^2) + gy = \frac{1}{2}U^2 + gH, \tag{1}$$

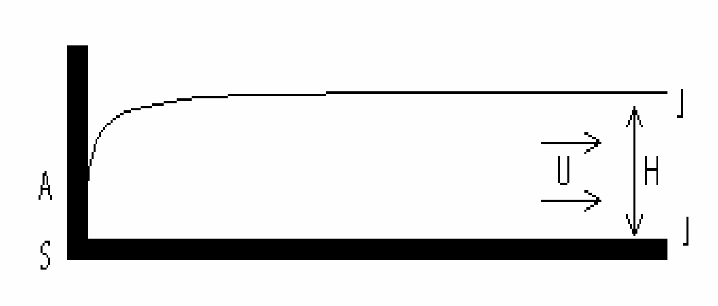

Figure 1. Sketch of flow in the physical z-plane.

where y is the elevation of the free surface. The constant value on the right of (1) represents the condition of free surface far downstream. If we nondimensionalize the problem with respect to the length H and velocity U, then equation (1) becomes

$$\frac{1}{2}F^2(\phi_x^2 + \phi_y^2) + y = \frac{1}{2}F^2 + 1, \tag{2}$$

and the flow separates the vertical wall at level y_a as the nondimensional quantity corresponding to H_a. In terms of these variables, the flow domain in the f-plane is then an infinite strip of height 1.

To derive the integral equation, we follow Wiryanto [5]. The transformation

$$f = \frac{1}{\pi}\log\zeta, \tag{3}$$

where $\zeta = \xi + i\eta$, maps the flow domain in the f-plane into the lower half ζ–plane (see Figure 2). We derived the transformation (3) by Schwarz and Christoffel's Theorem (see Milne-Thomson [2]). The hodograph variable $\Omega = \tau - i\theta$ is then introduced such that it has a relation to the velocity

$$\frac{df}{dz} = e^{\Omega}. \tag{4}$$

Therefore, the components τ and θ of Ω are interpreted respectively as the logarithm of the magnitude and the angle of the velocity on a streamline. In term of θ, the kinematics conditions on the solid boundaries are

$$\theta(\xi) = 0 \quad \text{for} \quad \xi < 0 \tag{5}$$

and

$$\theta(\xi) = \frac{\pi}{2} \quad \text{for} \quad 0 < \xi < 1. \tag{6}$$

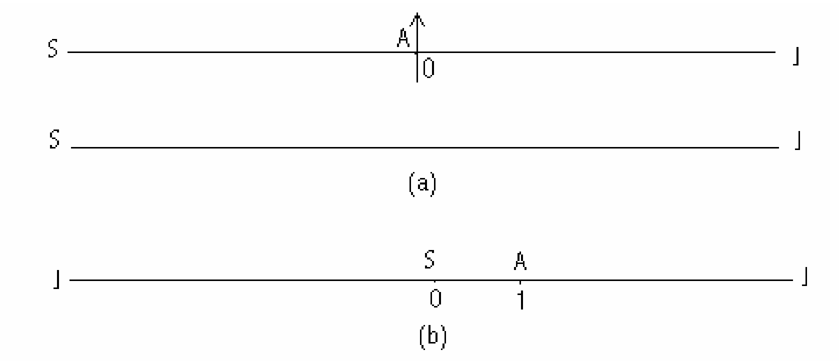

Figure 2. The flow domain in the f-plane (a), and in the ζ-plane (b).

On the other hand, the dynamic condition (2) becomes

$$F^2 e^{2\tau} + 2y = F^2 + 2 \tag{7}$$

on the free surface $\xi > 1$. Equation (7) is the integral equation for θ after we substitute the values of y and τ evaluated from

$$\frac{dy}{d\xi} = \frac{e^{-\tau}\sin\theta}{\pi\xi} \tag{8}$$

and

$$\tau(\xi) = \frac{1}{2}\log\left|\frac{1-\xi}{\xi}\right| + \frac{1}{\pi} PV \int_1^\infty \frac{\theta}{s-\xi}\, ds. \tag{9}$$

Note that the integral in (9) is a Cauchy principal value denoted by PV, and we use s as the dummy variable.

The relation (8) is obtained from the imaginary part of $dz/d\zeta$, where this form is expressed as

$$\frac{dz}{d\zeta} = \frac{e^{-\Omega}}{\pi\zeta} \tag{10}$$

from (3) and (4). Meanwhile, equation (9) is the result of applying Cauchy's Theorem to Ω on a part consisting of the real ζ-axis, a semi-circle at $|\zeta| = \infty$ in the lower half-plane, and a circle of vanishing radius about the point ζ. Hence, for $\mathrm{Im}(\zeta) < 0$ we have

$$\Omega(\zeta) = -\frac{1}{2\pi i} PV \int_{-\infty}^{\infty} \frac{\Omega(s)}{s-\zeta}\, ds,$$

since $\Omega \to 0$ as $|\zeta| \to \infty$. If we let $\mathrm{Im}(\zeta) \to 0^-$, we obtain

$$\tau(\xi) = -\frac{1}{\pi} PV \int_{-\infty}^{\infty} \frac{\theta(s)}{s-\xi} ds \tag{11}$$

and

$$\theta(\xi) = \frac{1}{\pi} PV \int_{-\infty}^{\infty} \frac{\tau(s)}{s-\xi} ds .$$

Substituting the known values of θ, equations (5) and (6), into (11) gives the relation (9).

3. Numerical procedure

To solve equation (7), we first truncate the integration domain in (9) up to $s = T$ with sufficiently large T representing uniform flow far downstream, and we approximate the integral into a summation over N mesh points. We can apply a numerical method such as the Trapezoidal Rule for this integral approximation. The N mesh points ξ_j are suggested to be distributed along the interval with the same distance of ϕ between two points to get better accuracy with less number of points. If we denote $\phi_T = \log T/\pi$, the mesh points can be defined as

$$\xi_j = e^{(j-1)\phi_T/(N-1)} \quad \text{for} \quad j = 1, 2, \cdots, N.$$

The value of θ for each ξ_j is the unknowns that we have to determine, except at $\xi = 1$ which is defined as $\theta = \pi/2$ indicating that the flow separates smoothly the vertical wall.

The next step is to construct a system of nonlinear algebraic equations from (7). This can be done by choosing $N-1$ collocation points ξ_j^* which are defined as the midpoints between ξ_j and ξ_{j-1}. The values of θ corresponding to these collocation points are denoted by θ_j^*, and are also defined as the midpoints between θ_j and θ_{j-1}. For each ξ_j^*, a nonlinear algebraic equation is obtained from (7), after substituting $\tau(\xi_j^*)$ and $y(\xi_j^*)$. The first value is evaluated from discretizing (9)

$$\tau(\xi_j^*) \approx \frac{1}{2}\log\left|\frac{1-\xi_j^*}{\xi_j^*}\right| + \frac{1}{2\pi}\sum_{l=1}^{N-1}\left(\frac{\theta_{j+1}}{\xi_{l+1}-\xi_j^*} - \frac{\theta_j}{\xi_l - \xi_j^*}\right)\left(\xi_{l+1}-\xi_l\right). \tag{12}$$

Meanwhile, $y(\xi_j^*)$ is evaluated by integrating (8) and discretizing the integral, similar to (12) using the Trapezoidal Rule.

The $N-1$ equations from the collocation points are used to determine the N unknowns $\theta_2, \theta_3, \cdots, \theta_N$ and y_a for a given Froude number F. Therefore, an extra equation is required to obtain a closed system. This equation is constructed from the level of the separation point A by integrating (8), i.e.,

$$1 - y_a \approx \frac{1}{\pi} \int_1^T \frac{e^{-\tau} \sin\theta}{\xi} d\xi.$$

This closed system is then solved by Newton's method.

As the method requires an initial guess for θ, we first use $\theta_j = \pi(1 - j/N)/2$ for $j = 2, \cdots, N$. We run the computer program with this initial guess for small values of N. Once the iteration converges, we double N by adding one point between ξ_j and ξ_{j-1}, and linearizing between θ_j and θ_{j-1} for the initial value of θ corresponding to the addition point. This can be repeated for another doubling of N. Therefore, we can observe the accuracy of our procedure. Using $N = 200$, the procedure is able to compute solutions up to three-figure of accuracy for y_a. This number is then used to perform most of our results. The free-surface profile is the output of our procedure which is computed in the post process. The coordinates of the free surface corresponding to each collocation point ξ_j^* are evaluated by integrating (8) for the y values, and integrating

$$\frac{dx}{d\xi} = \frac{e^{-\tau} \cos\theta}{\pi\xi}$$

for the x values. This equation is the real part of (10). We then perform the plot of these coordinates as the free surface.

4. Infinite Froude number

Very large Froude number corresponds to low effective importance of gravity relative to others forces such as inertia. When gravity is absent, the free-surface boundary condition is that of unit velocity magnitude, or $\tau = 0$. This can be obtained from (7). Therefore, our integral equation for flow formulated in Section 2 is reduced to determining θ satisfying

$$\frac{1}{2} \log\left|\frac{1-\xi}{\xi}\right| + \frac{1}{\pi} PV \int_1^\infty \frac{\theta}{s-\xi} ds = 0. \tag{13}$$

Basically, equation (13) can be solved numerically, similar to the procedure described in Section 3. The integral is truncated up to $s = T$ and approximated by summation over N discrete points in the truncated integration domain. For each collocation point ξ_j^* between those two discrete points, it is substituted to the approximate equation of (13) giving

$$\pi \log\left|\frac{1-\xi_j^*}{\xi_j^*}\right| + \sum_{l=1}^{N-1}\left(\frac{\theta_{l+1}}{\xi_{l+1}-\xi_j^*} - \frac{\theta_l}{\xi_l - \xi_j^*}\right)\left(\xi_{l+1}-\xi_l\right) = 0.$$

Here we denote $\theta_l = \theta(\xi_l)$, and the integral is approximated by the Trapezoidal Rule. The system of $N-1$ equations is therefore obtained for determining $\theta_2, \cdots, \theta_N$. In our calculation, we include determining the level y_a of the separation point A satisfying

$$1 - y_a \approx \frac{1}{\pi}\int_1^T \frac{\sin\theta}{\xi}\, d\xi.$$

When we apply Newton's method to solve the system of equations, iterations converge to values of θ_l decreasing monotonically from $\pi/2$ to 0, and $y_a = 0.363$. We perform the plots of θ versus ξ and the free surface in the next section, together with results for finite Froude numbers.

5. Numerical results

The numerical procedure described in Section 3 was used to compute solutions for various values of F. But the parameters N and T were first observed related to the accuracy of our procedure. We found that our numerical results reached three-figure accuracy using $N = 200$ and $T = 3{\cdot}10^6$. This was also tested to the procedure for infinite Froude number. The resulting values of y_a for $N = 50, 100$ and 200 were 0.3585, 0.3621 and 0.3630. These values converged to $y_a = 0.3633$ as $N \to \infty$. This process can be repeated for other values of T, but we found that the results are different in the fourth decimal place.

When the iteration of Newton's method converges, we obtain values of $\theta_l = \theta(\xi_l)$, for $l = 2, \cdots, N$, and y_a. The plot of θ versus ξ is shown in Figure 3 with the horizontal axis in $\phi = \log \xi/\pi$. We show three curves for different Froude numbers, namely $F = \infty$, 2.0 and 1.0 (from bottom to top). Our procedure is able to compute solutions less than $F = 1$, but the values of θ corresponding to large ξ start oscillating with increasing amplitude on decreasing F. Cusp solutions with a train of waves are a possibility for the free-surface configuration. Vanden-Broeck and Keller [4] also obtained cusp solutions without waves for $F > 1$.

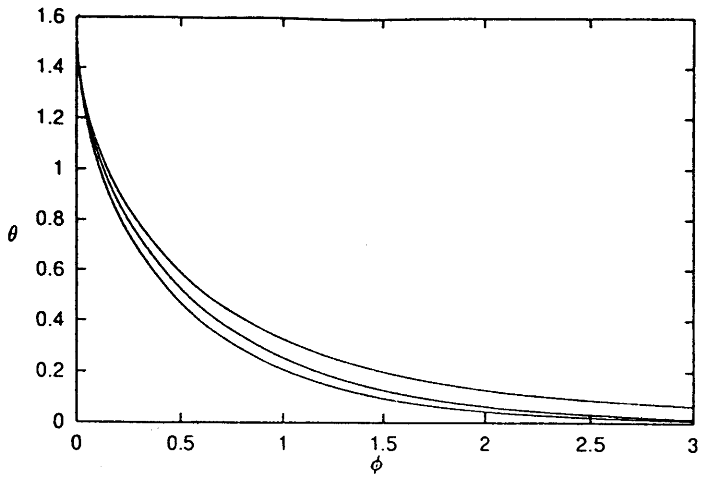

Figure 3. Plot of θ versus $\phi(= \log \zeta / \pi)$ for $F = 1, 2$ and ∞, from top to bottom

In Figure 4 we show a plot of two free surfaces corresponding to $F = \infty$ and 1.0. The resulting values of y_a for both Froude numbers are 0.363 and 0.245 respectively. For other values of the Froude number, the free-surface solution is characterized by the value of y_a. The plot of y_a versus F is shown in Figure 5, where y_a is less than the value corresponding to $F = \infty$.

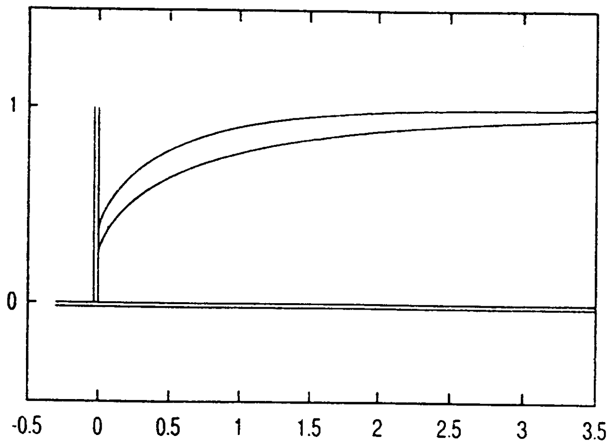

Figure 4. Plot of two free surfaces corresponding to $F = 1$ (lower) and $F = \infty$ (upper)

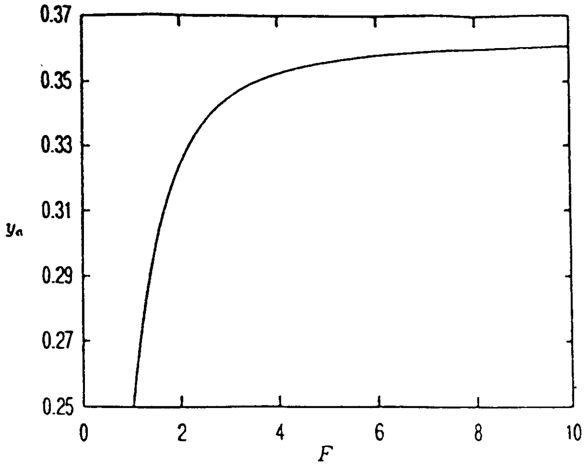

Figure 5. Plot of y_a versus F

Conclusions

We have performed numerical calculations for flow caused by a source/sink in a channel of finite depth. Free-surface solutions with a cusp are obtained for uniform stream far downstream with Froude number greater than one. From our calculations, we found that the separation point on the vertical wall tends to the level computed by zero-gravity case as we increase the Froude number.

Acknowledgement. This research was partly supported by Center for Reseach on Application and Advancement of Mathematics-Institut Teknologi Bandung, and this is gratefully acknowledged.

References

1. G.C. Hocking, Cusp-like free-surface flows due to a submerged source or sink in the presence of a flat or sloping bottom, *J. Austral. Math. Soc. Ser. B* **26** (1985), 470-486.
2. L.M. Milne-Thomson, *Theoretical Hydrodynamics*, 5th Ed., Dover, New York (1968), 276-280.
3. E.O. Tuck and J.-M. Vanden-Broeck, A cusp-like free-surface flow due to a submerged source or sink, *J. Austral. Math. Soc. Ser. B* **25** (1984), 443-450.
4. J.-M. Vanden-Broeck and J.B. Keller, Free surface flow due to a sink, *J. Fluid Mech.* **175** (1987), 109-117.
5. L.H. Wiryanto, A 2-D flow emerging from a tunnel, *Proceedings of ISASTI'98* (1998), 391-394.

Keywords: source/sink, integral equation, Froude number.