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Common Fixed Points for Compatible Mappings of Type (A)

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Abstract. In this paper we give criteria for the existence of a common fixed point of a pair of mappings in 2-metric spaces and establish common fixed and coincidence point theorems for certain classes of contractive type mappings. The results presented here generalize, improve and unify a number of fixed point theorems given by Cho [1], Imdad *et al.* [5], Khan and Fisher [11], Kubiak [14], Murthy *et al.* [17], Rhoades [25], Singh *et al.* [34]and others.

1. Introduction

Gähler [4] introduced the concept of 2-metric spaces. A 2-metric space is a set X with a function $d : X \times X \times X \to [0, \infty)$ satisfying the following conditions:

(G1) for two distinct points $x, y \in X$, there exists a point $z \in X$ such that $d(x, y, z) \neq 0$,

(G2) $d(x, y, z) = 0$ if at least two of x, y, z are equal,

(G3) $d(x, y, z) = d(x, z, y) = d(y, z, x)$,

(G4) $d(x, y, z) \leq d(x, y, u) + d(x, u, z) + d(u, y, z)$ for all $x, y, z, u \in X$.

It has been shown by Gähler [4] that a 2-metric d is a continuous function of any one of its three arguments but it need not be continuous in two arguments. If it is continuous in two arguments, then it is continuous in all three arguments. A 2-metric d which is continuous in all of its arguments will be called continuous.

Iséki [7], for the first time, established a fixed point theorem in 2-metric spaces. Since then a quite number of authors ([1]-[3], [5]-[36]) have extended and generalized the result of Iséki and various other results involving contractive and expansive type mappings. Especially, Murthy *et al.* [17] introduced the concepts of compatible mappings and compatible mappings of type (A) in 2-metric spaces, derived some relations between these mappings and proved common fixed point theorems for compatible mappings of type (A) in 2-metric spaces.

On the other hand, Cho [1], Constantin [2], Khan and Fisher[11] and Kubiak [14] established some necessary and sufficient conditions which guarantee the existence of a common fixed point for a pair of continuous mappings in 2-metric spaces.

In this paper we establish criteria for the existence of a common fixed point of a pair of mappings in 2-metric spaces and establish common fixed and coincidence point theorems for certain classes of contractive type mappings. The results presented here generalize, improve and unify the corresponding results of Cho [1], Imdad *et al.* [5], Khan and Fisher [11], Kubiak [14], Murthy *et al.* [17], Rhoades [25], Singh *et al.* [34] and others.

2. Preliminaries

Throughout this paper, N and ω denote the sets of positive and nonnegative integers, respectively. Let $R^+ = [0, \infty)$ and

$$\Phi_1 = \left\{ \varphi : \varphi : \left(R^+\right)^5 \to R^+ \text{ satisfies conditions (a1) and (a2)} \right\},$$

$$\Phi_2 = \left\{ \varphi : \varphi : \left(R^+\right)^{11} \to R^+ \text{ satisfies conditions (a1) and (a3)} \right\},$$

where conditions (a1), (a2) and (a3) are as follows:

(a1) φ is upper semicontinuous, nondecreasing in each coordinate variable,

(a2) $b(t) = \max\left\{\varphi(t, 0, 0, t, t), \varphi(t, t, t, 2t, 0), \varphi(t, t, t, 0, 2t)\right\} < t$ for all $t > 0$,

(a3) $c(t) = \max\left\{\varphi(t, t, t, 0, 2t, t, 0, 2t, 0, 2t, 0), \varphi(t, 0, 0, t, t, 0, 0, 0, 0, 0, t), \varphi(t, t, t, 2t, 0, t, 2t, 0, 2t, 0, 0)\right\} < t$ for all $t > 0$.

Lemma 2.1. [37] *For every* $t > 0, c(t) < t$ *if and only if* $\lim_{n \to \infty} c^n(t) = 0$, *where* c^n *denotes the n-times composition of* c.

Definition 2.1. *A sequence* $\{x_n\}_{n \in N}$ *in a 2-metric space* (X, d) *is said to be convergent to a point* $x \in X$ *if* $\lim_{n \to \infty} d(x_n, x, a) = 0$ *for all* $a \in X$. *The point* x *is called the limit of the sequence* $\{x_n\}_{n \in N}$ *in* X.

Definition 2.2. *A sequence* $\{x_n\}_{n \in N}$ *in a 2-metric space* (X, d) *is said to be a Cauchy sequence if* $\lim_{m, n \to \infty} d(x_m, x_n, a) = 0$ *for all* $a \in X$.

Definition 2.3. *A 2-metric space* (X, d) *is said to be complete if every Cauchy sequence in* X *is convergent.*

Note that, in a 2-metric space (X, d), a convergent sequence need not be a Cauchy sequence, but every convergent sequence is a Cauchy sequence when the 2-metric d is continuous on X ([19]).

Definition 2.4. *Let f and g be mappings from a 2-metric spaces (X, d) into itself. f and g are said to be compatible if*

$$\lim_{n\to\infty} d\big(fgx_n, gfx_n, a\big) = 0$$

for all $a \in X$, whenever $\{x_n\}_{n\in N} \subset X$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$; f and g are said to be compatible of type (A) if

$$\lim_{n\to\infty} d\big(fgx_n, ggx_n, a\big) = \lim_{n\to\infty} d\big(gfx_n, ffx_n, a\big) = 0$$

for all $a \in X$, whenever $\{x_n\}_{n\in N} \subset X$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$.

Definition 2.5. *A mapping f from a 2-metric space (X, d) into itself is said to be continuous at $x \in X$ if for every sequence $\{x_n\}_{n\in N} \subset X$ such that $\lim_{n\to\infty} d(x_n, x, a) = 0$ for all $a \in X$, $\lim_{n\to\infty} d(fx_n, fx, a) = 0$. f is called continuous on X if it is so at all points of X.*

Lemma 2.2. [1] *Let f and g be compatible mappings of type (A) from a 2-metric spaces (X, d) into itself. If $ft = gt$ for some $t \in X$, then $fgt = ggt = gft = fft$.*

Lemma 2.3. [1] *Let f and g be compatible mappings of type (A) from a 2-metric spaces (X, d) into itself. If f is continuous at some $t \in X$ and if $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$, then $\lim_{n\to\infty} gfx_n = ft$.*

Lemma 2.4. [1] *Let f and g be compatible mappings from a 2-metric spaces (X, d) into itself. If f and g are continuous, then they are compatible of type (A).*

3. Characterizations of common fixed points

Our main results are as follows:

Theorem 3.1. *Let (X, d) be a complete 2-metric space with d continuous on X and let h and t be two mappings from X into itself. Then the following conditions are equivalent:*

(1) h *and* t *have a common fixed point;*

(2) there exist $r \in (0,1)$, $f : X \to t(X)$ *and* $g : X \to h(X)$ *such that*

(b1) the pairs f, h *and* g, t *are compatible of type (A),*

(b2) one of f, g, h *and* t *is continuous,*

(b3) $d(fx, gy, a) \le r\, max\{d(hx, ty, a), d(hx, fx, a), d(ty, gy, a),$
$\frac{1}{2}[d(hx, gy, a) + d(ty, fx, a)]\}$ *for all* $x, y, a \in X$;

(3) there exist $\varphi \in \Phi_1$, $f : X \to t(X)$ *and* $g : X \to h(X)$ *satisfying conditions (b1), (b2) and (b4):*

(b4) $d(fx, gy, a) \le \varphi(d(hx, ty, a), d(hx, fx, a), d(ty, gy, a), d(hx, gy, a), d(ty, fx, a))$
for all $x, y, a \in X$;

(4) there exist $\varphi \in \Phi_2$, $f : X \to t(X)$ *and* $g : X \to h(X)$ *satisfying conditions (b1), (b2) and (b5):*

(b5)
$$\begin{aligned} d^2(fx, gy, a) \le \varphi(& d^2(hx, ty, a), d(hx, ty, a)\, d(hx, fx, a), \\ & d(hx, ty, a)\, d(ty, gy, a), \\ & d(hx, ty, a) d(hx, gy, a), d(hx, ty, a)\, d(ty, fx, a), \\ & d(hx, fx, a) d(ty, gy, a), \\ & d(hx, fx, a)\, d(hx, gy, a), d(hx, fx, a) d(ty, fx, a), \\ & d(ty, gy, a)\, d(hx, gy, a), \\ & d(ty, gy, a)\, d(ty, fx, a), d(hx, gy, a) d(ty, fx, a)) \end{aligned}$$
for all $x, y, a \in X$;

Proof. $(1) \Rightarrow (2)$ and (4). Let z be a common fixed point of h and t. Define $f : X \to t(X)$ and $g : X \to h(X)$ by $fx = gx = z$ for all $x \in X$. Then (b1) and (b2) hold. For each $r \in (0,1)$ and $\varphi \in \Phi_2$, (b3) and (b5) also hold.

$(2) \Rightarrow (3)$ Take $\varphi(u, v, w, x, y) = r \max\{u, v, w, \frac{1}{2}(x + y)\}$ for all $u, v, w, x, y \in R^+$. Then $\varphi \in \Phi_1$ and (b3) implies (b4).

$(3) \Rightarrow (1)$ By using the method of Cho [1], we can similarly show that (3) implies (1).

$(4) \Rightarrow (1)$ Let x_0 be an arbitrary point in X. Since $f(X) \subset t(X)$ and $g(X) \subset h(X)$, there exist sequences $\{x_n\}_{n\in\omega}$ and $\{y_n\}_{n\in\omega}$ in X satisfying $y_{2n} = tx_{2n+1} = fx_{2n}$, $y_{2n+1} = hx_{2n+2} = gx_{2n+1}$ for all $n \in \omega$. Define

$d_n(a) = d(y_n, y_{n+1}, a)$ for all $a \in X$ and $n \in \omega$. We claim that for any $i, j, k \in \omega$

$$d(y_i, y_j, y_k) = 0. \tag{3.1}$$

Suppose that $d_{2n}(y_{2n+2}) > 0$. Using (b5), we have

$$\begin{aligned} d^2(fx_{2n+2}, gx_{2n+1}, y_{2n}) \le \varphi\big(&d^2(hx_{2n+2}, tx_{2n+1}, y_{2n}),\\ &d(hx_{2n+2}, tx_{2n+1}, y_{2n})d(hx_{2n+2}, fx_{2n+2}, y_{2n}),\\ &d(hx_{2n+2}, tx_{2n+1}, y_{2n})d(tx_{2n+1}, gx_{2n+1}, y_{2n}),\\ &d(hx_{2n+2}, tx_{2n+1}, y_{2n})d(hx_{2n+2}, gx_{2n+1}, y_{2n}),\\ &d(hx_{2n+2}, tx_{2n+1}, y_{2n})d(tx_{2n+1}, fx_{2n+2}, y_{2n}),\\ &d(hx_{2n+2}, fx_{2n+2}, y_{2n})d(tx_{2n+1}, gx_{2n+1}, y_{2n}),\\ &d(hx_{2n+2}, fx_{2n+2}, y_{2n})d(hx_{2n+2}, gx_{2n+1}, y_{2n}),\\ &d(hx_{2n+2}, fx_{2n+2}, y_{2n})d(tx_{2n+1}, fx_{2n+2}, y_{2n}),\\ &d(tx_{2n+1}, gx_{2n+1}, y_{2n})d(hx_{2n+2}, gx_{2n+1}, y_{2n}),\\ &d(tx_{2n+1}, gx_{2n+1}, y_{2n})d(tx_{2n+1}, fx_{2n+2}, y_{2n}),\\ &d(hx_{2n+2}, gx_{2n+1}, y_{2n})d(tx_{2n+1}, fx_{2n+2}, y_{2n})\big), \end{aligned}$$

which implies that

$$d_{2n}^{\,2}(y_{2n+2}) \le \varphi(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0) \le c\big(d_{2n}^{\,2}(y_{2n+2})\big) < d_{2n}^{\,2}(y_{2n+2}),$$

which is a contradiction. Hence $d_{2n}(y_{2n+2}) = 0$. Similarly, we have $d_{2n+1}(y_{2n+3}) = 0$. Consequently, $d_n(y_{n+2}) = 0$ for all $n \in \omega$. Note that

$$d(y_n, y_{n+2}, a) \le d_n(y_{n+2}) + d_n(a) + d_{n+1}(a) = d_n(a) + d_{n+1}(a). \tag{3.2}$$

By (b5) and (3.2) we have

$$
\begin{aligned}
d_{2n+1}{}^{2}(a) &= d^{2}\left(fx_{2n+2}, gx_{2n+1}, a\right) \\
&\le \varphi\Big(d^{2}\left(hx_{2n+2}, tx_{2n+1}, a\right), \\
&\qquad d\left(hx_{2n+2}, tx_{2n+1}, a\right) d\left(hx_{2n+2}, fx_{2n+2}, a\right), \\
&\qquad d\left(hx_{2n+2}, tx_{2n+1}, a\right) d\left(tx_{2n+1}, gx_{2n+1}, a\right), \\
&\qquad d\left(hx_{2n+2}, tx_{2n+1}, a\right) d\left(hx_{2n+2}, gx_{2n+1}, a\right), \\
&\qquad d\left(hx_{2n+2}, tx_{2n+1}, a\right) d\left(tx_{2n+1}, fx_{2n+2}, a\right), \\
&\qquad d\left(hx_{2n+2}, fx_{2n+2}, a\right) d\left(tx_{2n+1}, gx_{2n+1}, a\right), \\
&\qquad d\left(hx_{2n+2}, fx_{2n+2}, a\right) d\left(hx_{2n+2}, gx_{2n+1}, a\right), \\
&\qquad d\left(hx_{2n+2}, fx_{2n+2}, a\right) d\left(tx_{2n+1}, fx_{2n+2}, a\right), \\
&\qquad d\left(tx_{2n+1}, gx_{2n+1}, a\right) d\left(hx_{2n+2}, gx_{2n+1}, a\right), \\
&\qquad d\left(tx_{2n+1}, gx_{2n+1}, a\right) d\left(tx_{2n+1}, fx_{2n+2}, a\right), \\
&\qquad d\left(hx_{2n+2}, gx_{2n+1}, a\right) d\left(tx_{2n+1}, fx_{2n+2}, a\right)\Big) \\
&= \varphi\Big(d_{2n}{}^{2}(a), d_{2n}(a) d_{2n+1}(a), d_{2n}{}^{2}(a), \\
&\qquad 0, d_{2n}(a)\left[d_{2n}(a) + d_{2n+1}(a)\right], d_{2n+1}{}^{2}(a), 0, \\
&\qquad d_{2n+1}(a)\left[d_{2n}(a) + d_{2n+1}(a)\right], 0, \\
&\qquad d_{2n}(a)\left[d_{2n}(a) + d_{2n+1}(a)\right], 0\Big).
\end{aligned}
$$

Suppose that $d_{2n+1}(a) > d_{2n}(a)$. Then

$$
\begin{aligned}
d_{2n+1}{}^{2}(a) &\le \varphi\Big(d_{2n+1}{}^{2}(a), d_{2n+1}{}^{2}(a), d_{2n+1}{}^{2}(a), 0, 2d_{2n+1}{}^{2}(a), \\
&\qquad d_{2n+1}{}^{2}(a), 0, 2d_{2n+1}{}^{2}(a), 0, 2d_{2n+1}{}^{2}(a), 0\Big) \\
&\le c\left(d_{2n+1}{}^{2}(a)\right) < d_{2n+1}{}^{2}(a),
\end{aligned}
$$

which is a contradiction. Hence $d_{2n+1}(a) \le d_{2n}(a)$ and so $d_{2n+1}{}^{2}(a) \le c(d_{2n}{}^{2}(a))$. Similarly, we have $d_{2n}(a) \le d_{2n-1}(a)$ and $d_{2n}{}^{2}(a) \le c(d_{2n-1}{}^{2}(a))$. That is, for all $n \in N$

$$
d_{n+1}(a) \le d_{n}(a), \quad d_{n+1}{}^{2}(a) \le c\left(d_{n}{}^{2}(a)\right).
$$

Hence for all $n \in N$

$$
d_{n+1}{}^{2}(a) \le c\left(d_{n}{}^{2}(a)\right) \le c^{2}\left(d_{n-1}{}^{2}(a)\right) \le \cdots \le c^{n+1}\left(d_{0}{}^{2}(a)\right). \tag{3.3}
$$

It follows from (3.3) and Lemma 2.1 that

$$
d_{n+1}(a) \to 0 \quad \text{as} \quad n \to \infty. \tag{3.4}
$$

Let n, m be in ω. If $n \geq m$, then $0 = d_m(y_m) \geq d_n(y_m)$; if $n < m$, then

$$\begin{aligned} d_n(y_m) &\leq d_n(y_{m-1}) + d_{m-1}(y_n) + d_{m-1}(y_{n+1}) \\ &\leq d_n(y_{m-1}) + d_n(y_n) + d_n(y_{n+1}) \\ &\leq d_n(y_{m-1}) \leq d_n(y_{m-2}) \leq \cdots \leq d_n(y_{n+1}) = 0. \end{aligned}$$

Thus, for any $n, m \in \omega$

$$d_n(y_m) = 0. \tag{3.5}$$

For all $i, j, k \in \omega$, we may, without loss of generality, assume that $i < j$. It follows from (3.5) that

$$\begin{aligned} d(y_i, y_j, y_k) &\leq d_i(y_j) + d_i(y_k) + d(y_{i+1}, y_j, y_k) \\ &= d(y_{i+1}, y_j, y_k) \leq d(y_{i+2}, y_j, y_k) \leq \cdots \\ &\leq d(y_{j-1}, y_j, y_k) = d_{j-1}(y_k) = 0. \end{aligned}$$

Therefore (3.1) holds.

In order to show that $\{y_n\}_{n\in\omega}$ is a Cauchy sequence, by (3.4), it is sufficient to show that $\{y_{2n}\}_{n\in\omega}$ is a Cauchy sequence. Suppose that $\{y_{2n}\}_{n\in\omega}$ is not a Cauchy sequence. Then there exist $\varepsilon > 0$ and $a \in X$ such that for each even integer $2k$, there are even integers $2m(k)$ and $2n(k)$ with $2m(k) > 2n(k) > 2k$ and $d(y_{2m(k)}, y_{2n(k)}, a) \geq \varepsilon$.

For each even integer $2k$, let $2m(k)$ be the least even integer exceeding $2n(k)$ satisfying the above inequality, so that

$$d\left(y_{2m(k)-2}, y_{2n(k)}, a\right) \leq \varepsilon, \quad d\left(y_{2m(k)}, y_{2n(k)}, a\right) > \varepsilon. \tag{3.6}$$

For each even integer $2k$, by (3.1) and (3.6) we have

$$\begin{aligned} \varepsilon &< d\left(y_{2m(k)}, y_{2n(k)}, a\right) \\ &\leq d\left(y_{2m(k)-2}, y_{2n(k)}, a\right) + d\left(y_{2m(k)}, y_{2m(k)-2}, a\right) \\ &\quad + d\left(y_{2m(k)}, y_{2n(k)}, y_{2m(k)-2}\right) \\ &\leq \varepsilon + d\left(y_{2m(k)-2}, y_{2m(k)}, y_{2m(k)-1}\right) + d\left(y_{2m(k)-2}, y_{2m(k)-1}, a\right) \\ &\quad + d\left(y_{2m(k)-1}, y_{2m(k)}, a\right) \\ &= \varepsilon + d_{2m(k)-2}(a) + d_{2m(k)-1}(a) \end{aligned}$$

which implies that

$$\lim_{k\to\infty} d\left(y_{2m(k)}, y_{2n(k)}, a\right) = \varepsilon. \tag{3.7}$$

It follows from (3.7) that

$$\begin{aligned}0 < d\left(y_{2n(k)}, y_{2m(k)}, a\right) - d\left(y_{2n(k)}, y_{2m(k)-2}, a\right) \\ \le d\left(y_{2m(k)-2}, y_{2m(k)}, a\right) \\ \le d_{2m(k)-2}(a) + d_{2m(k)-1}(a).\end{aligned}$$

In view of (3.5) and (3.7) we immediately obtain that

$$\lim_{k\to\infty} d\left(y_{2n(k)}, y_{2m(k)-2}, a\right) = \varepsilon. \tag{3.8}$$

Note that

$$\left| d\left(y_{2n(k)}, y_{2m(k)-1}, a\right) - d\left(y_{2n(k)}, y_{2m(k)}, a\right)\right| \le d_{2m(k)-1}(a) + d_{2m(k)-1}\left(y_{2n(k)}\right),$$

$$\left| d\left(y_{2n(k)+1}, y_{2m(k)}, a\right) - d\left(y_{2n(k)}, y_{2m(k)}, a\right)\right| \le d_{2n(k)}(a) + d_{2n(k)}\left(y_{2m(k)}\right),$$

$$\left| d\left(y_{2n(k)+1}, y_{2m(k)-1}, a\right) - d\left(y_{2n(k)}, y_{2m(k)-1}, a\right)\right| \le d_{2n(k)}(a) + d_{2n(k)}\left(y_{2m(k)-1}\right).$$

It is easy to see that

$$\begin{aligned}\lim_{k\to\infty} d\left(y_{2n(k)}, y_{2m(k)-1}, a\right) &= \lim_{k\to\infty} d\left(y_{2n(k)+1}, y_{2m(k)}, a\right) \\ &= \lim_{k\to\infty} d\left(y_{2n(k)+1}, y_{2m(k)-1}, a\right) = \varepsilon. \end{aligned} \tag{3.9}$$

It follows from (b5) that

$$\begin{aligned}d\left(y_{2m(k)}, y_{2n(k)+1}, a\right) &= d\left(fx_{2m(k)}, gx_{2n(k)+1}, a\right) \\ &\le \varphi\Big(d^2\left(hx_{2m(k)}, tx_{2n(k)+1}, a\right), \\ &\qquad d\left(hx_{2m(k)}, tx_{2n(k)+1}, a\right)d\left(hx_{2m(k)}, fx_{2m(k)}, a\right), \\ &\qquad d\left(hx_{2m(k)}, tx_{2n(k)+1}, a\right)d\left(tx_{2n(k)+1}, gx_{2n(k)+1}, a\right), \\ &\qquad d\left(hx_{2m(k)}, tx_{2n(k)+1}, a\right)d\left(hx_{2m(k)}, gx_{2n(k)+1}, a\right), \\ &\qquad d\left(hx_{2m(k)}, tx_{2n(k)+1}, a\right)d\left(tx_{2n(k)+1}, fx_{2m(k)}, a\right), \\ &\qquad d\left(hx_{2m(k)}, fx_{2m(k)}, a\right)d\left(tx_{2n(k)+1}, gx_{2n(k)+1}, a\right), \\ &\qquad d\left(hx_{2m(k)}, fx_{2m(k)}, a\right)d\left(hx_{2m(k)}, gx_{2n(k)+1}, a\right), \\ &\qquad d\left(hx_{2m(k)}, fx_{2m(k)}, a\right)d\left(tx_{2n(k)+1}, fx_{2m(k)}, a\right),\end{aligned}$$

$$
\begin{aligned}
& d\big(tx_{2n(k)+1}, gx_{2n(k)+1}, a\big) d\big(hx_{2m(k)}, gx_{2n(k)+1}, a\big), \\
& d\big(tx_{2n(k)+1}, gx_{2n(k)+1}, a\big) d\big(tx_{2n(k)+1}, fx_{2m(k)}, a\big), \\
& d\big(hx_{2m(k)}, gx_{2n(k)+1}, a\big) d\big(tx_{2n(k)+1}, fx_{2m(k)}, a\big)\big) \\
= \varphi\big(& d^2\big(y_{2m(k)-1}, y_{2n(k)}, a\big), d\big(y_{2m(k)-1}, y_{2n(k)}, a\big)\, d_{2m(k)-1}(a), \\
& d\big(y_{2m(k)-1}, y_{2n(k)}, a\big) d_{2n(k)}(a), \\
& d\big(y_{2m(k)-1}, y_{2n(k)}, a\big) d\big(y_{2m(k)-1}, y_{2n(k)+1}, a\big), \\
& d\big(y_{2m(k)-1}, y_{2n(k)}, a\big) d\big(y_{2(k)}, y_{2m(k)}, a\big), \\
& d_{2m(k)-1}(a)\, d_{2n(k)}(a), \\
& d_{2m(k)-1}(a)\, d\big(y_{2m(k)-1}, y_{2n(k)+1}, a\big), \\
& d_{2m(k)-1}(a)\, d\big(y_{2n(k)}, y_{2m(k)}, a\big), \\
& d_{2n(k)}(a) d\big(y_{2m(k)-1}, y_{2n(k)+1}, a\big), \\
& d_{2n(k)}(a) d\big(y_{2n(k)}, y_{2m(k)}, a\big), d\big(y_{2m(k)-1}, y_{2n(k)+1}, a\big) \\
& d\big(y_{2n(k)}, y_{2m(k)}, a\big)\big),
\end{aligned}
$$

Letting $k \to \infty$, by (3.9), (3.7) and (3.5) we have

$$\varepsilon^2 \le \varphi\big(\varepsilon^2, 0, 0, \varepsilon^2, \varepsilon^2, 0, 0, 0, 0, 0, \varepsilon^2\big) \le c\big(\varepsilon^2\big) < \varepsilon^2,$$

which is a contradiction. Therefore $\{y_{2n}\}_{n\in\omega}$ is a Cauchy sequence in X. It follows from completeness of (X, d) that $\{y_n\}_{n\in\omega}$ converges to a point $u \in X$.

Now, suppose that t is continuous. Since g and t are compatible of type (A) and $\{gx_{2n+1}\}_{n\in\omega}$ and $\{tx_{2n+1}\}_{n\in\omega}$ converge to the point u, by Lemma 2.3 we get that gtx_{2n+1}, $ttx_{2n+1} \to tu$ as $n \to \infty$. In virtue of (b5), we have

$$
\begin{aligned}
d\big(fx_{2n}, gtx_{2n+1}, a\big) \le \varphi\big(& d^2\big(hx_{2n}, ttx_{2n+1}, a\big), d\big(hx_{2n}, ttx_{2n+1}, a\big) d\big(hx_{2n}, fx_{2n}, a\big), \\
& d\big(hx_{2n}, ttx_{2n+1}, a\big) d\big(ttx_{2n+1}, gtx_{2n+1}, a\big), \\
& d\big(hx_{2n}, ttx_{2n+1}, a\big) d\big(hx_{2n}, gtx_{2n+1}, a\big), \\
& d\big(hx_{2n}, ttx_{2n+1}, a\big) d\big(ttx_{2n+1}, fx_{2n}, a\big), \\
& d\big(hx_{2n}, fx_{2n}, a\big) d\big(ttx_{2n+1}, gtx_{2n+1}, a\big), \\
& d\big(hx_{2n}, fx_{2n}, a\big) d\big(hx_{2n}, gtx_{2n+1}, a\big), \\
& d\big(hx_{2n}, fx_{2n}, a\big) d\big(ttx_{2n+1}, fx_{2n}, a\big), \\
& d\big(ttx_{2n+1}, gtx_{2n+1}, a\big) d\big(hx_{2n}, gtx_{2n+1}, a\big), \\
& d\big(ttx_{2n+1}, gtx_{2n+1}, a\big) d\big(ttx_{2n+1}, fx_{2n}, a\big), \\
& d\big(hx_{2n}, gtx_{2n+1}, a\big) d\big(ttx_{2n+1}, fx_{2n}, a\big)\big).
\end{aligned}
$$

Letting $n \to \infty$, we have

$$\begin{aligned} d^2(u,tu,a) &\le \varphi\big(d^2(u,tu,a), 0, 0, d^2(u,tu,a), d^2(u,tu,a), 0, 0, 0, 0, 0, d^2(u,tu,a)\big) \\ &\le c\big(d^2(u,tu,a)\big), \end{aligned}$$

which implies that $u = tu$.

It follows from (b5) that

$$\begin{aligned} d^2(fx_{2n}, gu, a) \le \varphi\big(& d^2(hx_{2n}, tu, a), d(hx_{2n}, tu, a)d(hx_{2n}, fx_{2n}, a), \\ & d(hx_{2n}, tu, a)d(tu, gu, a), d(hx_{2n}, tu, a)d(hx_{2n}, gu, a), \\ & d(hx_{2n}, tu, a)d(tu, fx_{2n}, a), d(hx_{2n}, fx_{2n}, a)d(tu, gu, a), \\ & d(hx_{2n}, fx_{2n}, a)d(hx_{2n}, gu, a), d(hx_{2n}, fx_{2n}, a)d(tu, fx_{2n}, a), \\ & d(tu, gu, a)d(hx_{2n}, gu, a), d(tu, gu, a)d(tu, fx_{2n}, a), \\ & d(hx_{2n}, gu, a)d(tu, fx_{2n}, a)\big). \end{aligned}$$

As $n \to \infty$, we have

$$\begin{aligned} d^2(u,gu,a) &\le \varphi\big(0, 0, 0, 0, 0, 0, 0, 0, d^2(u,gu,a), 0, 0,\big) \\ &\le c\big(d^2(u,gu,a)\big). \end{aligned}$$

This gives that $u = gu$. It follows from $g(X) \subset h(X)$ that there exists $v \in X$ with $u = gu = hv$. From (b5) we get

$$\begin{aligned} d^2(fv,u,a) &= d^2(fv,gu,a) \\ &\le \varphi\big(d^2(hv,tu,a), d(hv,tu,a)d(hv,fv,a), d(hv,tu,a)d(tu,gu,a), \\ &\qquad d(hv,tu,a)d(hv,gu,a), d(hv,tu,a)d(tu,fv,a), \\ &\qquad d(hv,fv,a)d(tu,gu,a), d(hv,fv,a)d(hv,gu,a), \\ &\qquad d(hv,fv,a)d(tu,fv,a), d(tu,gu,a)d(hv,gu,a), \\ &\qquad d(tu,gu,a)d(tu,fv,a), d(hv,gu,a)d(tu,fv,a)\big) \\ &= \varphi\big(0, 0, 0, 0, 0, 0, 0, d^2(fv,u,a), 0, 0, 0\big) \\ &\le c\big(d^2(fv,u,a)\big). \end{aligned}$$

Therefore $u = fv$. Lemma 2.2 ensures that $fu = fhv = hfv = hu$. By (b5) we obtain again

$$\begin{aligned} d^2(fu, u, a) &= d^2(fu, gu, a) \\ &\le \varphi\big(d^2(hu, tu, a), d(hu, tu, a)d(hu, fu, a), d(hu, tu, a)d(tu, gu, a), \\ &\qquad d(hu, tu, a)d(hu, gu, a), d(hu, tu, a)d(tu, fu, a), \\ &\qquad d(hu, fu, a)d(tu, gu, a), d(hu, fu, a)d(hu, gu, a), \\ &\qquad d(hu, fu, a)d(tu, fu, a), d(tu, gu, a)d(hu, gu, a), \\ &\qquad d(tu, gu, a)d(tu, fu, a), d(hu, gu, a)d(tu, fu, a)\big) \\ &= \varphi\big(d^2(fu, u, a), 0, 0, d^2(fu, u, a), d^2(fu, u, a)\, 0, 0, 0, 0, 0, d^2(fu, u, a)\big) \\ &\le c\big(d^2(fu, u, a)\big). \end{aligned}$$

Hence $u = fu$. That is, u is a common fixed point of f, g, h and t.

Suppose that f is continuous. Since f and h are compatible of type (A) and $\{fx_{2n}\}_{n\in\omega}$ and $\{hx_{2n}\}_{n\in\omega}$ converge to the point u, by Lemma 2.3 we get that $fhx_{2n}, ffx_{2n} \to fu$ as $n \to \infty$. From (b5) we have

$$\begin{aligned} d^2(ffx_{2n}, gx_{2n+1}, a) \le \varphi\big(&d^2(hfx_{2n}, tx_{2n+1}, a), d(hfx_{2n}, tx_{2n+1}, a)d(hfx_{2n}, ffx_{2n}, a), \\ &d(hfx_{2n}, tx_{2n+1}, a)d(tx_{2n+1}, gx_{2n+1}, a), \\ &d(hfx_{2n}, tx_{2n+1}, a)d(hfx_{2n}, gx_{2n+1}, a), \\ &d(hfx_{2n}, tx_{2n+1}, a)d(tx_{2n+1}, ffx_{2n}, a), \\ &d(hfx_{2n}, ffx_{2n}, a)d(tx_{2n+1}, gx_{2n+1}, a), \\ &d(hfx_{2n}, ffx_{2n}, a)d(hfx_{2n}, gx_{2n+1}, a), \\ &d(hfx_{2n}, ffx_{2n}, a)d(tx_{2n+1}, ffx_{2n}, a), \\ &d(tx_{2n+1}, gx_{2n+1}, a)d(hfx_{2n}, gx_{2n+1}, a), \\ &d(tx_{2n+1}, gx_{2n+1}, a)d(tx_{2n+1}, ffx_{2n}, a), \\ &d(hfx_{2n}, gx_{2n+1}, a)d(tx_{2n+1}, ffx_{2n}, a)\big). \end{aligned}$$

Letting $n \to \infty$, we have

$$\begin{aligned} d^2(fu, u, a) &\le \varphi\big(d^2(fu, u, a), 0, 0, d^2(fu, u, a), d^2(fu, u, a), 0, 0, 0, 0, 0, d^2(fu, u, a)\big) \\ &\le c\big(d^2(fu, u, a)\big). \end{aligned}$$

This gives that $u = fu$. Note that $f(X) \subset t(X)$. Thus there exists a point $v \in X$ with $u = tv$. It follows from (b5) that

$$\begin{aligned} d^2(fx_{2n}, gv, a) \le \varphi\big(& d^2(hx_{2n}, tv, a), d(hx_{2n}, tv, a)d(hx_{2n}, fx_{2n}, a), \\ & d(hx_{2n}, tv, a)d(tv, gv, a), d(hx_{2n}, tv, a)d(hx_{2n}, gv, a), \\ & d(hx_{2n}, tv, a)d(tv, fx_{2n}, a), d(hx_{2n}, fx_{2n}, a)d(tv, gv, a), \\ & d(hx_{2n}, fx_{2n}, a)d(hx_{2n}, gv, a), d(hx_{2n}, fx_{2n}, a)d(tv, fx_{2n}, a), \\ & d(tv, gv, a)d(hx_{2n}, gv, a), d(tv, gv, a)d(tv, fx_{2n}, a), \\ & d(hx_{2n}, gv, a)d(tv, fx_{2n}, a)\big). \end{aligned}$$

Taking $n \to \infty$, we get that

$$\begin{aligned} d^2(u, gv, a) &\le \varphi\big(0, 0, 0, 0, 0, 0, 0, 0, d^2(u, gv, a), 0, 0\big) \\ &\le c\big(d^2(u, gv, a)\big), \end{aligned}$$

which implies that $u = gv = tv = fu$. It follows from Lemma 2.2 that $gu = gtv = tgv = tu$. Using again (b5), we have

$$\begin{aligned} d^2(fx_{2n}, gu, a) \le \varphi\big(& d^2(hx_{2n}, tu, a), d(hx_{2n}, tu, a)d(hx_{2n}, fx_{2n}, a), \\ & d(hx_{2n}, tu, a)d(tu, gu, a), d(hx_{2n}, tu, a)d(hx_{2n}, gu, a), \\ & d(hx_{2n}, tu, a)d(tu, fx_{2n}, a), d(hx_{2n}, fx_{2n}, a)d(tu, gu, a), \\ & d(hx_{2n}, fx_{2n}, a)d(hx_{2n}, gu, a), d(hx_{2n}, fx_{2n}, a)d(tu, fx_{2n}, a), \\ & d(tu, gu, a)d(hx_{2n}, gv, a), d(tu, gu, a)d(tu, fx_{2n}, a), \\ & d(hx_{2n}, gu, a)d(tu, fx_{2n}, a)\big). \end{aligned}$$

As $n \to \infty$, we have

$$\begin{aligned} d^2(u, gu, a) &\le \varphi\big(d^2(u, gu, a), 0, 0, d^2(u, gu, a), d^2(u, gu, a), 0, 0, 0, 0, 0, d^2(u, gu, a)\big) \\ &\le c\big(d^2(u, gu, a)\big), \end{aligned}$$

which means that $u = gu$. Since $g(X) \subset h(X)$, so there is $w \in X$ with $u = hw$. It follows from (b5) that

$$
\begin{aligned}
d^2(fw, gu, a) \le \varphi\big(&d^2(hw, tu, a), d(hw, tu, a)\,d(hw, fw, a),\\
&d(hw, tu, a)\,d(tu, gu, a), d(hw, tu, a)\,d(hw, gu, a),\\
&d(hw, tu, a)\,d(tu, fw), d(hw, fw, a)\,d(tu, gu, a),\\
&d(hw, fw, a)\,d(hw, gu, a), d(hw, fw, a)\,d(tu, fw),\\
&d(tu, gu, a)\,d(hw, gu, a), d(tu, gu, a)\,d(tu, fw),\\
&d(hw, gu, a)\,d(tu, fw)\big)\\
= \varphi\big(&0, 0, 0, 0, 0, 0, 0, d^2(u, fw, a), 0, 0, 0\big)\\
\le c\big(&d^2(u, fw, a)\big).
\end{aligned}
$$

This gives that $u = fw$. Therefore, $hw = fw$. Lemma 2.2 ensures that $u = fu = fhw = hfw = hu$. That is, u is a common fixed point of f, g, h and t. Similarly, we can complete the proof when g or h is continuous. This completes the proof.

Remark 3.1. Theorem 3.1 generalizes, improves and unifies Theorem 2 of Khan and Fisher [11] and Theorem 1 of Kubiak [14]. The following example reveals that Theorem 3.1 generalizes properly the results of Khan and Fisher [11] and Kubiak [14].

Example 3.1. Let $X = R^+ \times R^+$ and d be a 2-metric which expresses $d(x, y, a)$ as the area of the Euclidean triangle with vertices $x = (x_1, x_2)$, $y = (y_1, y_2)$, $a = (a_1, a_2)$. Define mappings $h, t : X \to X$ by

$$
h(x_1, x_2) = t(x_1, x_2) = \left(\frac{1}{3}x_1, 0\right) \text{ for } (x_1, x_2) \in [0, 1) \times R^+,
$$
$$
h(x_1, x_2) = t(x_1, x_2) = (1, 0) \text{ for } (x_1, x_2) \in [1, \infty) \times R^+.
$$

Then (X, d) is a complete 2-metric space with d continuous on X. Since

$$
\lim_{n\to\infty} d\left(\left(1 - \frac{1}{n}, 0\right), (1, 0), (a_1, a_2)\right) = \lim_{n\to\infty} \frac{1}{2n} a_2 = 0
$$

and

$$
\lim_{n\to\infty} d\left(h\left(1 - \frac{1}{n}, 0\right), h(1, 0), (a_1, a_2)\right) = \lim_{n\to\infty} \frac{1}{6}\left(2 + \frac{1}{n}\right)a_2 = \frac{1}{3}a_2,
$$

so h and t do not continuous at $(1, 0) \in X$. Thus we can not invoke Theorem 2 of Khan and Fisher [11] and Theorem 1 of Kubiak [14] to prove that h and t have a common fixed point in X.

Let f and g be two mappings from X into itself satisfying $fx = gx = (0, 0)$ for all $x \in X$. Note that

$$d(fx, fy, a) = d(fx, gy, a) = 0$$

for all $x, y, a \in X$. Hence f is continuous and (b3), (b4) and (b5) hold. Now we verify that f and h are compatible of type (A). Let $a = (a_1, a_2)$, $x_n = (x_{1n}, x_{2n}) \in X$ with $\lim_{n\to\infty} fx_n = (0, 0) = \lim_{n\to\infty} hx_n$. This means that

$$\lim_{n\to\infty} d(hx_n, (0, 0), a) = \lim_{n\to\infty} \frac{1}{6} x_{1n} a_2 = 0,$$

which implies that

$$\lim_{n\to\infty} d(fhx_n, hhx_n, a) = \lim_{n\to\infty} d\left((0, 0), \left(\frac{1}{9} x_{1n}, 0\right), a\right) = \lim_{n\to\infty} \frac{1}{18} x_{1n} a_2 = 0,$$

and

$$\lim_{n\to\infty} d(hfx_n, ffx_n, a) = \lim_{n\to\infty} d((0, 0), (0, 0), a) = 0.$$

That is, f and h are compatible of type (A). Using Theorem 3.1, we conclude that f, g, h and t have a common fixed point $(0, 0) \in X$. Thus $(0, 0)$ is also a common fixed point of h and t.

Remark 3.2. Note that $(0, 0)$ and $(1, 0)$ are common fixed points of h and t. This reveals that the common fixed points of h and t in Theorem 3.1 may not be unique.

Removing the assumption of continuity in Theorem 3.1, we have

Theorem 3.2. *Let (X, d) be a 2-metric space with d continuous on X and let h and t be two mappings from X into itself. Then (1) is equivalent to each of the following conditions (5), (6) and (7).*

(5) there exist $r \in (0, 1)$, $f, g : X \to X$ satisfying conditions (b1), (b3), (b6) and (b7):

(b6) $f(X) \cup g(X) \subset h(X) \cap t(X)$,

(b7) $h(X) \cap t(X)$ is a complete subspace of X;

(6) there exist $\varphi \in \Phi_1$, $f, g : X \to X$ satisfying conditions (b1), (b4), (b6) and (b7);

(7) there exist $\varphi \in \Phi_2$, $f, g : X \to X$ satisfying conditions (b1), (b5), (b6) and (b7).

Proof. As in the proof of Theorem 3.1, we conclude that (1) implies (7) and that (1), (5) and (6) are equivalent. We now prove that (7) implies (1). Let $x_0 \in X$. (b6) ensures that there exist two sequences $\{x_n\}_{n\in\omega} \subset X$ and $\{y_n\}_{n\in\omega} \subset h(X) \cap t(X)$ satisfying $y_{2n} = tx_{2n+1} = fx_{2n}$, $y_{2n+1} = hx_{2n+2} = gx_{2n+1}$ for all $n \in \omega$. Using the method of Theorem 3.1, we similarly conclude that $\{y_n\}_{n\in\omega}$ is a Cauchy sequence in $h(X) \cap t(X)$. Therefore $\{y_n\}_{n\in\omega}$ converges to a point $u \in h(X) \cap t(X)$ by (b7). Thus, there are two points $x, y \in X$ with $u = hx = ty$. By (b5), we have

$$\begin{aligned} d^2(fx, gx_{2n+1}, a) \le \varphi\big(& d^2(hx, tx_{2n+1}, a), d(hx, tx_{2n+1}, a)d(hx, fx, a), \\ & d(hx, tx_{2n+1}, a)d(tx_{2n+1}, gx_{2n+1}, a), \\ & d(hx, tx_{2n+1}, a)d(hx, gx_{2n+1}, a), \\ & d(hx, tx_{2n+1}, a)d(tx_{2n+1}, fx, a), \\ & d(hx, fx, a)d(tx_{2n+1}, gx_{2n+1}, a), \\ & d(hx, fx, a)d(hx, gx_{2n+1}, a), \\ & d(hx, fx, a)d(tx_{2n+1}, fx, a), \\ & d(tx_{2n+1}, gx_{2n+1}, a)d(hx, gx_{2n+1}, a), \\ & d(tx_{2n+1}, gx_{2n+1}, a)d(tx_{2n+1}, fx, a), \\ & d(hx, gx_{2n+1}, a)d(tx_{2n+1}, fx, a)\big) \end{aligned}$$

Letting $n \to \infty$, we have

$$d^2(fx, u, a) \le \varphi\big(0, 0, 0, 0, 0, 0, 0, d^2(fx, u, a), 0, 0, 0\big) \le c\big(d^2(fx, u, a)\big),$$

which implies that $u = fx = hx$. It follows from Lemma 2.2 that $fu = hu$. Similarly, we conclude that $gu = tu$. In view of (b5), we get that

$$\begin{aligned} d^2(fu, gu, a) &\le \varphi\big(d^2(fu, gu, a), 0, 0, d^2(fu, gu, a), d^2(fu, gu, a), 0, 0, 0, 0, 0, d^2(fu, gu, a)\big) \\ &\le c\big(d^2(fu, gu, a)\big). \end{aligned}$$

This gives that $fu = gu$. By virtue of (b5) again, we obtain that

$$\begin{aligned} d^2(fu, gx_{2n+1}, a) \le \varphi\big(& d^2(hu, tx_{2n+1}, a), d(hu, tx_{2n+1}, a)d(hu, fu, a), \\ & d(hu, tx_{2n+1}, a)d(tx_{2n+1}, gx_{2n+1}, a), \end{aligned}$$

$$d(hu, tx_{2n+1}, a)d(hu, gx_{2n+1}, a),$$
$$d(hu, tx_{2n+1}, a)d(tx_{2n+1}, fu, a),$$
$$d(hu, fu, a)d(tx_{2n+1}, gx_{2n+1}, a),$$
$$d(hu, fu, a)d(hu, gx_{2n+1}, a),$$
$$d(hu, fu, a)d(tx_{2n+1}, fu, a),$$
$$d(tx_{2n+1}, gx_{2n+1}, a)d(hu, gx_{2n+1}, a),$$
$$d(tx_{2n+1}, gx_{2n+1}, a)d(tx_{2n+1}, fu, a),$$
$$d(hu, gx_{2n+1}, a)d(tx_{2n+1}, fu, a)).$$

Taking $n \to \infty$, we have

$$d^2(fu, u, a) \le \varphi(d^2(fu, u, a), 0, 0, d^2(fu, u, a), d^2(fu, u, a), 0, 0, 0, 0, 0, d^2(fu, u, a)),$$

which means that $u = fu$. Consequently, $u = fu = gu = hu = tu$. This completes the proof.

From the proofs of Theorems 3.1 and 3.2, we immediately obtain the following results:

Theorem 3.3. *Let (X, d) be a complete 2-metric space with d continuous on X and let f, g, h and t be mappings from X into itself satisfying conditions (b1), (b2). If at least one of conditions (b3), (b4), (b5) holds, then f, g, h and t have a unique common fixed point in X.*

Theorem 3.4. *Let (X, d) be a 2-metric space with d continuous on X and let f, g, h and t be mappings from X into itself satisfying conditions (b1), (b6) and (b7). If at least one of conditions (b3), (b4), (b5) holds, then f, g, h and t have a unique common fixed point in $h(X) \cap t(X)$.*

Theorem 3.5. *Let (X, d) be a 2-metric space with d continuous on X and let f, g, h and t be mappings from X into itself satisfying conditions (b6) and (b7). If at least one of conditions (b3), (b4), (b5) holds, then*

(i) f *and* h *have a coincidence point in* X.
(ii) g *and* t *have a coincidence point in* X.

Remark 3.3. Theorem 4.1 of Cho [1] is obtained as a special case of Theorem 3.3 if (b4) holds. On the other hand, Theorem 3.3 extends Theorem 3.3 of Imdad *et al.* [5], Theorem 4 of Rhoades [25], Theorem 1 of Singh *et al.* [34]. The following examples show the greater generality of Theorem 3.3 over the results of Imdad *et al.* [5], Rhoades [25], Singh *et al.* [34].

Example 3.2. Let (X, d), f, h, t be as in Example 3.1. Define $A = f, S = T = h.$ By Theorem 3.3 and the proof of Example 3.1, we conclude that A, S, T have a unique common fixed point $(0, 0) \in X$. But Theorem 3.3 of Imdad *et al.* [5] is not applicable since S and T are not continuous.

Example 3.3. Let (X, d) be as in Example 3.1. Define mappings $f, g : X \to X$ by

$$f(x_1, x_2) = g(x_1, x_2) = (2x_1, 0) \text{ for all } (x_1, x_2) \in X.$$

It is easy to see that f, g do not satisfy the following inequality:

$$d(fx, gy, a) \le r \max\left\{ d(x, y, a), d(x, fx, a), d(y, gy, a), \frac{1}{2}[d(x, gy, a) + d(y, fx, a)]\right\}$$

for any $r \in (0, 1)$ and $a = (1, 1)$, $x = (2, 0)$, $y = (1, 0) \in X$. That is, Theorem 4 of Rhoades [25] is not applicable.

We next prove that f is continuous on X. Assume that $\{(x_{1n}, x_{2n})\}_{n \in N} \subseteq X$ and $(x_1, x_2) \in X$ such that $(x_{1n}, x_{2n}) \to (x_1, x_2)$ as $n \to \infty$. Then for any $(a_1, a_2) \in X$, we have

$$\begin{aligned} &\lim_{n\to\infty} d((x_{1n}, x_{2n}), (x_1, x_2), (a_1, a_2)) \\ &= \frac{1}{2} \lim_{n\to\infty} | x_1 a_2 - a_1 x_2 - x_{1n} a_2 + x_{2n} a_1 + x_{1n} x_2 - x_{2n} x_1 | = 0. \end{aligned} \tag{3.10}$$

Taking $(a_1, a_2) = (0, 0)$ in (3.10), we infer that

$$\lim_{n\to\infty} (x_{1n} x_2 - x_{2n} x_2) = 0. \tag{3.11}$$

Taking $a_1 = 0$ and $a_2 \ne 0$ in (3.10), by (3.11) we have

$$\lim_{n\to\infty} (x_1 - x_{1n}) = 0. \tag{3.12}$$

It follows from (3.12) that

$$\begin{aligned} \lim_{n\to\infty} d(f(x_{1n}, x_{2n}), f(x_1, x_2), (a_1, a_2)) &= \lim_{n\to\infty} d((2x_{1n}, 0), (2x_1, 0), (a_1, a_2)) \\ &= \lim_{n\to\infty} a_2 | x_1 - x_{1n} | = 0. \end{aligned}$$

for all $(a_1, a_2) \in X$. Hence f is continuous on X.

Put $h(x_1, x_2) = t(x_1, x_2) = (4x_1, 0)$ for all $(x_1, x_2) \in X$. Then $fX = X \times \{0\} = hX$ and $fh(x_1, x_2) = (8x_1, 0) = hf(x_1, x_2)$ for all $(x_1, x_2) \in X$. Note that commuting mappings are compatible. Lemma 2.4 ensures that f and h are compatible of type (A). Obviously, for any $r \in \left(\frac{1}{2}, 1\right)$ and all $(x_1, x_2), (y_1, y_2), (a_1, a_2) \in X$, we have

$$\begin{aligned} d(f(x_1, x_2), g(y_1, y_2), (a_1, a_2)) &= d((2x_1, 0), (2y_1, 0), (a_1, a_2)) \\ &= a_2|x_1 - y_1| \\ &\le rd((4x_1, 0), (4y_1, 0), (a_1, a_2)) \\ &= d(h(x_1, x_2), t(y_1, y_2), (a_1, a_2)) . \end{aligned}$$

Thus, by Theorem 3.3, we conclude that f, g, h, t have a unique common fixed point $(0, 0) \in X$. Therefore $(0, 0)$ is also a common fixed point of f and g.

Example 3.4. Let $X = \{1, 2, 3\} \subset R^+$. Define $d : X \times X \times X \to R^+$ by $d(x, y, z) = 1$ for all $x, y, z \in X$ with $x \ne y$, $y \ne z, z \ne x$ and $d(x, y, z) = 0$ for other. Then (X, d) is a compact 2-metric space and d is continuous. Let h, t, S, T be mappings from X into itself satisfying

$$h1 = 2, h2 = 1, h3 = 3, \text{ and } S = T = t = h.$$

Then Theorem 1 of Singh, Tiwari and Gupta [34] is not applicable since

$$d(Sx, Sy, a) \le r \max\left\{ d(Tx, Ty, a), d(Sx, Tx, a), d(Sy, Ty, a), \frac{1}{2}[d(Sx, Ty, a) + d(Sy, Tx, a)]\right\}$$

does not hold for any $r \in (0, 1)$ and $x = 1, y = 2, a = 3$. Take $fx = gx = 3$ for all $x \in X$. It is easy to verify that f is continuous on X. Since f, h are commuting, so they are compatible of type (A) by Lemma 2.4. Clearly,

$$d(fx, gy, a) \le r \max\left\{ d(hx, ty, a), d(hx, fx, a), d(ty, gy, a), \frac{1}{2}[d(hx, gy, a) + d(ty, fx, a)]\right\}$$

for any $x, y, a \in X$ and $r \in (0, 1)$. It follows from Theorem 3.3 that f, g, h, t have a unique common fixed point $3 \in X$. Hence S, T have a common fixed point $3 \in X$.

Remark 3.4. Theorems 4.1 and 3.1 of Murthy *et al.* [17] are special cases of our Theorems 3.4 and 3.5, respectively.

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