On Sums of $k$-EP Matrices

A.R. MEENAKSHI AND S. KRISHNAMOORTHY
Department of Mathematics, Annamalai University,
26, North Second Cross, Mariyappa Nagar, Annamalai Nagar - 608 002, Tamil Nadu, South India

Abstract. Necessary and sufficient conditions for the sums of $k$-EP matrices to be $k$-EP are discussed. As an application it is shown that sum and parallel sum of parallel summable $k$-EP matrices are $k$-EP.

1. Introduction

Throughout we shall deal with $\mathbb{C}^{n \times n}$, the space of $n \times n$ complex matrices. Let $\mathbb{C}_n$ be the space of complex $n$-tuples. For $A \in \mathbb{C}_{n \times n}$, let $A^T$, $A^*$ denote the transpose, conjugate transpose of $A$, let $A^{-}$ be a generalized inverse $(AA^{-}A = A)$ and $A$ be the Moore-Penrose inverse of $A[5]$. A matrix $A$ is called $EP_r$ if $\rho(A) = r$ and $N(A) = N(A^*)$ or $R(A) = R(A^*)$ where $\rho(A)$ denotes the rank of $A$; $N(A)$ and $R(A)$ denote the null space and range space of $A$ respectively. Throughout let `$k$' be a fixed product of disjoint transpositions in $\{1, 2, \ldots, n\}$ and $K$ be the associated permutation matrix. A matrix $A = (a_{ij}) \in \mathbb{C}_{n \times n}$ is $k$-hermitian if $a_{ij} = \overline{a}_{k(i), k(j)}$ for $i, j = 1, \ldots, n$. A theory for $k$-hermitian matrices is developed in [1]. For $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{C}_n$, let us define the function $k(x) = (x_{k(1)}, x_{k(2)}, \ldots, x_{k(n)})^T \in \mathbb{C}_n$. A matrix $A \in \mathbb{C}_{n \times n}$, is said to be $k$-EP if it satisfies the condition $Ax = 0 \Leftrightarrow A^*k(x) = 0$ or equivalently $N(A) = N(A^*K)$. In addition to that, $A$ is $k$-EP $\Leftrightarrow KA$ is $EP$ or $AK$ is $EP$ and $A$ is $k$-EP $\Leftrightarrow A^*$ is $k$-EP. Moreover, $A$ is said to be $k$-EP$_r$ if $A$ is $k$-EP and of rank $r$. For further properties of $k$-EP matrix one may refer [4]. In this paper we give necessary and sufficient conditions for sums of $k$-EP matrix to be $k$-EP. As an application it is shown that sum and parallel sum of parallel summable $k$-EP matrices are $k$-EP.

2. Sums of $k$-EP matrices

Lemma 2.1. Let $A_1, A_2, \ldots, A_m \in \mathbb{C}_{n \times n}$ and let $A = \sum_{i=1}^{m} A_i$. Consider the following conditions:
(a) $N(A) \subseteq N(A_i)$ for $i = 1, \cdots, m$;

(b) $N(A) = \bigcap_{i=1}^{m} N(A_i)$;

(c) $\rho(A) = \rho \begin{pmatrix} A_1 \\ \vdots \\ A_m \end{pmatrix}$;

(d) $\sum_{i=1}^{m} \sum_{j=1}^{m} A_i^* A_j = 0$;

(e) $\rho(A) = \sum_{i=1}^{m} \rho(A_i)$.

Then the following statements hold:

(i) Conditions (a), (b) and (c) are equivalent.

(ii) Condition (d) implies (a), but condition (a) does not implies (d).

(iii) Condition (e) implies (a), but condition (a) does not implies (e).

Proof.

(i) $(a) \iff (b) \iff (c)$: $N(A) \subseteq N(A_i)$ for each $i \Rightarrow N(A) \subseteq \cap N(A_i)$.

Since $N(A) = N(\sum A_i) \supseteq N(A_1) \cap N(A_2) \cdots \cap N(A_m)$, it follows that $N(A) \supseteq \cap N(A_i)$.

Always $\bigcap_{i=1}^{m} N(A_i) \subseteq N(A)$. Hence $N(A) = \bigcap_{i=1}^{m} N(A_i)$. Thus (b) holds.

Now,

$$N(A) = \bigcap_{i=1}^{m} N(A_i) = N \begin{pmatrix} A_1 \\ \vdots \\ A_m \end{pmatrix}$$

Therefore,

$$\rho(A) = \rho \begin{pmatrix} A_1 \\ \vdots \\ A_m \end{pmatrix}$$

and (c) holds.
Conversely, Since \( \rho \left( \begin{bmatrix} A_1 \\ \vdots \\ A_m \end{bmatrix} \right) = \rho(A) \) and

\[
N \left( \begin{bmatrix} A_1 \\ \vdots \\ A_m \end{bmatrix} \right) = \bigcap_{i=1}^{m} N(A_i) \subseteq N(A) \Rightarrow N(A) = \bigcap_{i=1}^{m} N(A_i)
\]

and (b) holds.

Hence, \( N(A) \subseteq N(A_i) \) for each \( i \) and (a) holds.

(ii) \( (d) \Rightarrow (a) \):

Since \( \sum_{i \neq j} A_i^* A_j = 0 \),

\[
A^* A = (\Sigma A_i)^* (\Sigma A_i)
\]

\[
= (\Sigma A_i^*) (\Sigma A_i)
\]

\[
= \Sigma A_i^* A_i
\]

\[
N(A) = N(A^* A) = N(\Sigma A_i^* A_i)
\]

\[
= N \left( \begin{bmatrix} A_1^* \\ \vdots \\ A_m^* \end{bmatrix} \right) \begin{bmatrix} A_1 \\ \vdots \\ A_m \end{bmatrix}
\]

\[
= N \left( \begin{bmatrix} A_1 \\ \vdots \\ A_m \end{bmatrix} \right)
\]

\[
= N(A_1) \cap N(A_2) \cdots \cap N(A_m)
\]

\[
= \bigcap_{i=1}^{m} N(A_i).
\]

Hence \( N(A) \subset N(A_i) \) for each \( i \) and (a) holds.
(a) ⇒ (d): Let us consider the following example.

Let \( A_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \)
and \( A_2 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \). \( A_1 + A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \).

Clearly, \( N(A_1 + A_2) \subseteq N(A_1) \). Also \( N(A_1 + A_2) \subseteq N(A_2) \). But \( A_1^* A_2 + A_2^* A_1 \neq 0 \).

(iii) (e) ⇒ (a):
If rank is additive, that is \( \rho(A) = \sum \rho(A_i) \), then by \([3]\), \( R(A_i) \cap R(A_j) = \{0\}, \ i \neq j \Rightarrow N(A) \subseteq N(A_i) \) for each \( i \) and (a) holds.

(a) ⇒ (e): Consider the example,

Let \( A_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \) and \( A_2 = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} \).

\( A_1 + A_2 = \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix} \).

Here, \( N(A_1 + A_2) \subseteq N(A_1) \) and \( N(A_1 + A_2) \subseteq N(A_2) \).
But \( \rho(A_1 + A_2) \neq \rho(A_1) + \rho(A_2) \).

**Theorem 2.2.** Let \( A_1, A_2, \cdots, A_m \in \mathbb{C}_{n \times n} \) be k-EP matrices. If any one of the conditions (a) to (e) of Lemma 2.1 holds, then

\[ A = \sum_{i=1}^{m} A_i \] is k-EP.

**Proof.** Since each \( A_i \) is k-EP, \( N(A_i) = N(A_i^* K) \) for each \( i \).

Now, \( N(A) \subseteq N(A_i) \) for each \( i \)

\[ \Rightarrow N(A) \subseteq \bigcap_{i=1}^{m} N(A_i) = \bigcap_{i=1}^{m} N(A_i^* K) \subseteq N(A^* K) \]
and \( \rho(A) = \rho(A^* K) \). Hence \( N(A) = N(A^* K) \). Thus \( A \) is \( k\)-EP. Hence the Theorem.

**Remark 2.3.** In particular, if \( A \) is non-singular the conditions automatically hold and \( A \) is \( k\)-EP. Theorem 2.2 fails if we relax the conditions on the \( A_i \)’s.

**Example 2.4.** Consider \( A_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \) and \( A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \). Let \( k = (1, 2) \), then the associated permutation matrix

\[
K = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad KA_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ is } EP.
\]

Therefore, \( A_1 \) is \( k\)-EP.

\[
KA_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ is not } EP. \quad \text{Therefore } A_2 \text{ is not } k\text{-EP}.
\]

\[
A_1 + A_2 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \text{ and } K(A_1 + A_2) = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}
\]

which is not \( EP \). Therefore \( (A_1 + A_2) \) is not \( k\)-EP. However,

\[
N(A_1 + A_2) \subseteq N(A_1^* K) \subseteq N(A_1) \quad \text{and} \quad N(A_1 + A_2) \subseteq N(A_2^* K) \subseteq N(A_2).
\]

Moreover, \( \rho\left( \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \right) = \rho(A_1 + A_2) \).

**Remark 2.5.** Theorem 2.2 fails if we relax the condition that \( A_i \)'s are \( k\)-EP. For, let

\[
A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}
\]

and let the associated permutation matrix be \( K = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \).
Therefore \( A_1 \) is not \( k\)-EP.

\[
KA_1 = \begin{bmatrix}
0 & 0 & 0 \\
0 & -1 & 0 \\
1 & 0 & 0
\end{bmatrix}
\]

is not EP.  Therefore \( A_1 \) is not \( k\)-EP.

Therefore \( A_2 \) is not \( k\)-EP.

\[
KA_2 = \begin{bmatrix}
0 & -1 & 0 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix}
\]

is not EP.  Therefore, \((A_1 + A_2)\) is not \( k\)-EP.  But \( A_1^* A_2 + A_2^* A_1 = 0 \).

**Remark 2.6.** The conditions given in Theorem 2.2 are only sufficient for the sum of \( k\)-EP matrices to be \( k\)-EP, but not necessary is illustrated in the following example.

**Example 2.7.** Let \( A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \) and \( A_2 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \).  For \( K = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \), \( A_1 \) and \( A_2 \) are \( k\)-EP.  The conditions in Theorem 2.2 does not hold. However \( (A_1 + A_2) \) is \( k\)-EP.

**Remark 2.8.** If \( A_1 \) and \( A_2 \) are \( k\)-EP matrices, then by Theorem 2.4(p.221,[4]), \( A_1^* = H_1 K A_1 K \) and \( A_2^* = H_2 K A_2 K \) where \( H_1 \) and \( H_2 \) are non-singular \( n \times n \) matrices.

If \( H_1 = H_2 \), then \( A_1^* + A_2^* = H_1 K (A_1 + A_2) K \)

\[
(A_1 + A_2)^* = H_1 K (A_1 + A_2) K \Rightarrow (A_1 + A_2) \text{ is } k\text{-EP.}
\]

If \((H_1 - H_2)\) is non-singular, then the above conditions are also necessary for the sum of \( k\)-EP matrices to be \( k\)-EP is given in the following Theorem.

**Theorem 2.9.** Let \( K \) be the permutation matrix associated with the fixed transposition \( 'k' \).  Let \( A_1^* = H_1 K A_1 K \) and \( A_2^* = H_2 K A_2 K \) such that \((H_1 - H_2)\) is non-singular.  Then \((A_1 + A_2)\) is \( k\)-EP if and only if \( N(A_1 + A_2) \subseteq N(A_i) \) for some (and hence both) \( i \in \{1, 2\} \).
Proof. Since \( A_1^* = H_1 K A_1 K \) and \( A_2^* = H_2 K A_2 K \), by Remark 2.8, \( A_1 \) and \( A_2 \) are \( k\)-EP matrices. Since, \( N(A_1 + A_2) \subseteq N(A_2) \) by Theorem 2.2, \( (A_1 + A_2) \) is \( k\)-EP. Conversely, let us assume that \( (A_1 + A_2) \) is \( k\)-EP. By Remark 2.8, there exists a non-singular matrix \( G \) such that

\[
(A_1 + A_2)^* = GK(A_1 + A_2)K
\]

\[
\Rightarrow A_1^* + A_2^* = GK(A_1 + A_2)K
\]

\[
\Rightarrow H_1 K A_1 K + H_2 K A_2 K = GK(A_1 + A_2)K
\]

\[
\Rightarrow (H_1 K A_1 + H_2 K A_2)K = GK(A_1 + A_2)K
\]

\[
\Rightarrow H_1 K A_1 + H_2 K A_2 = G K A_1 + G K A_2
\]

\[
\Rightarrow (H_1 K - G K) A_1 = (G K - H_2 K) A_2
\]

\[
\Rightarrow (H_1 - G) K A_1 = (G - H_2) K A_2
\]

\[
L K A_1 = MK A_2 \text{ where } \\
L = H_1 - G \text{ and } \\
M = G - H_2
\]

Now \( (L + M)(K A_1) = L K A_1 + M K A_1 = M K A_2 + M K A_1 = M K (A_1 + A_2) \)

and \( (L + M)(K A_2) = L K (A_1 + A_2) \)

By hypothesis, \( L + M = H_1 - G + G - H_2 = H_1 - H_2 \) is non-singular. Therefore,

\[
N(A_1 + A_2) \subseteq N(MK(A_1 + A_2)) = N((L + M)K A_1) = N(K A_1) = N(A_1).
\]

Therefore, \( N(A_1 + A_2) \subseteq N(A_1) \). Similarly \( N(A_1 + A_2) \subseteq N(A_2) \). Hence the Theorem.

Remark 2.10. The condition \( (H_1 - H_2) \) to be non-singular is essential in Theorem 2.9 is illustrated in the following example.

Example 2.11. Let \( A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \) and \( A_2 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \) are both \( k\)-EP matrices for \( K = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \). Further \( A_1^* = A_1 = K A_1 K \) and \( A_2^* = A_2 = K A_2 K \Rightarrow H_1 = H_2 = I. \)
(A_1 + A_2) = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \text{ is also } k\text{-EP. But } N(A_1 + A_2) \nsubseteq N(A_1) \text{ (or) } N(A_1 + A_2) \nsubseteq N(A_2). \text{ Thus Theorem 2.9 fails.}

3. Parallel summable k-EP matrices

In this section we shall show that sum and parallel sum of parallel summable k-EP matrices are k-EP. First we shall give the definition and some properties of parallel summable matrices as in (p.188, [5]).

**Definition 3.1.** \( A_1 \) and \( A_2 \) are said to be parallel summable (p.s.) if 
\[ N(A_1 + A_2) \subseteq N(A_2) \text{ and } N(A_1 + A_2)^\ast \subseteq N(A_2^\ast) \text{ (or) equivalently } N(A_1 + A_2) \subseteq N(A_1) \text{ and } N(A_1 + A_2)^\ast \subseteq N(A_1^\ast). \]

**Definition 3.2.** If \( A_1 \) and \( A_2 \) are parallel summable then parallel sum of \( A_1 \) and \( A_2 \) denoted by \( A_1 \underline{\mp} A_2 \) is defined as \( A_1 \underline{\mp} A_2 = A_1(A_1 + A_2)^{-1} A_2 \). The product \((A_1 + A_2)^- A_2\) is invariant for all choices of generalized inverse \((A_1 + A_2)^-\) of \((A_1 + A_2)\) under the conditions that \( A_1 \) and \( A_2 \) are parallel summable (p.188, [5]).

**Properties 3.3.** Let \( A_1 \) and \( A_2 \) be a pair of parallel summable (p.s.) matrices. Then the following hold:

\[ P.1 \quad A_1 \underline{\mp} A_2 = A_2 \underline{\mp} A_1 \]
\[ P.2 \quad A_1^* \text{ and } A_2^* \text{ are p.s. and } (A_1 \underline{\mp} A_2)^* = A_1^* \underline{\mp} A_2^* \]
\[ P.3 \quad \text{If } U \text{ is non-singular then } UA_1 \text{ and } UA_2 \text{ are p.s. and } (UA_1 \underline{\mp} UA_2) = U(A_1 \underline{\mp} A_2) \]
\[ P.4 \quad R(A_1 \underline{\mp} A_2) = R(A_1) \cap R(A_2) \]
\[ N(A_1 \underline{\mp} A_2) = N(A_1) + N(A_2) \]
\[ P.5 \quad (A_1 \underline{\mp} A_2) \underline{\mp} A_3 = A_1 \underline{\mp} (A_2 \underline{\mp} A_3) \]
if all the parallel sum operations involved are defined.

**Lemma 3.4.** Let \( A_1 \) and \( A_2 \) be k-EP matrices. Then \( A_1 \) and \( A_2 \) are p.s if and only if 
\[ N(A_1 + A_2) \subseteq N(A_i) \text{ for some (and hence both) } i \in \{1, 2\}. \]

**Proof.** \( A_1 \) and \( A_2 \) are p.s. \( \Rightarrow N(A_1 + A_2) \subseteq N(A_1) \) follows from the Definition 3.1. Conversely, if 
\[ N(A_1 + A_2) \subseteq N(A_1), \text{ then } N(KA_1 + KA_2) \subseteq N(KA_1). \text{ Also } N(KA_1 + KA_2) \subseteq N(A_2). \text{ Since } A_1 \text{ and } A_2 \text{ are } k\text{-EP matrices, } KA_1 \text{ and } KA_2 \text{ are.
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$EP$ matrices. $N(KA_1 + KA_2) \subseteq N(KA_1)$ and $N(KA_1 + KA_2) \subseteq N(KA_2)$, therefore $(KA_1 + KA_2)$ is $EP$.

Hence

$$N(KA_1 + KA_2)^* = N(KA_1 + KA_2) = N(KA_1) \cap N(KA_2) = N(KA_1)^* \cap N(KA_2)^*.$$ 

Therefore, $N(KA_1 + KA_2)^* \subseteq N(KA_1)^*$, $N(KA_1 + KA_2)^* \subseteq N(KA_2)^*$.

Also, $N(KA_1 + KA_2)^* \subseteq N(KA)$ by hypothesis. Hence, by Definition 3.1, $KA_1$ and $KA_2$ are p.s. $N(KA_1 + KA_2) \subseteq N(KA_1)$ by hypothesis. Hence, by Definition 3.1, $KA_1$ and $KA_2$ are p.s. $N(KA_1)$ by hypothesis. Hence, by Definition 3.1, $KA_1$ and $KA_2$ are p.s. $N(KA_1) \Rightarrow N(KA_1 + A_2) \subseteq N(A_1)$. Similarly, $N(A_1 + A_2)^* \subseteq N(A_1^*)$. Therefore, $A_1$ and $A_2$ are p.s. Hence the Theorem.

**Remark 3.5.** Lemma 3.4 fails if we relax the condition that $A_1$ and $A_2$ are $k$-EP.

Let $A_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ and $A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

Let the associated permutation matrix be

$$K = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$ 

$A_1$ is $k$-EP. $A_2$ is not $k$-EP. $N(A_1 + A_2) \subseteq N(A_1)$ and $N(A_1 + A_2) \subseteq N(A_2)$, but $N(A_1 + A_2)^* \nsubseteq N(A_1^*)$; $N(A_1 + A_2)^* \nsubseteq N(A_2^*)$. Hence $A_1$ and $A_2$ are not parallel summable.

**Theorem 3.6.** Let $A_1$ and $A_2$ be p.s. $k$-EP matrices. Then $(A_1 \pm A_2)$ and $(A_1 + A_2)$ are $k$-EP.

**Proof.** Since $A_1$ and $A_2$ are p.s. $k$-EP matrices, by Lemma 3.4,

$$N(A_1 + A_2) \subseteq N(A_1) \quad \text{and} \quad N(A_1 + A_2) \subseteq N(A_2).$$

$$N(K(A_1 + A_2)) \subseteq N(KA_1) \quad \text{and} \quad N(K(A_1 + A_2)) \subseteq N(KA_2).$$

$$N(KA_1 + KA_2) \subseteq N(KA_1) \quad \text{and} \quad N(KA_1 + KA_2) \subseteq N(KA_2).$$

Therefore, $KA_1 + KA_2 = K(A_1 + A_2)$ is $EP$. Then $(A_1 + A_2)$ is $k$-EP. Since $A_1$ and $A_2$ are p.s. $k$-EP matrices, $KA_1$ and $KA_2$ are p.s. $EP$ matrices. Therefore,
\[ R(KA_1)^* = R(KA_2) \quad \text{and} \quad R(KA_2)^* = R(KA_2) \]
\[ R(KA_1 \mp KA_2)^* = R\left((KA_1)^* \mp (KA_2)^*\right) \quad \text{[By P.2]} \]
\[ = R\left((KA_1)^*\right) \cap R\left((KA_2)^*\right) \quad \text{[By P.4]} \]
\[ = R(KA_1) \cap R(KA_2) \quad \text{[Since } KA_1 \text{ and } KA_2 \text{ are EP]} \]
\[ = R(KA_1 \mp KA_2). \]

Thus, \( KA_1 \mp KA_2 \) is EP \( \Rightarrow \) \( K(A_1 \mp A_2) \) is EP \( \Rightarrow \) \( (A_1 \mp A_2) \) is k-EP. Thus \( (A_1 \mp A_2) \) is k-EP whenever \( A_1 \) and \( A_2 \) are k-EP. Hence the Theorem.

**Corollary 3.7.** Let \( A_1 \) and \( A_2 \) be k-EP matrices such that \( N(A_1 + A_2) \subset N(A_2) \).
If \( A_3 \) is k-EP commuting with both \( A_1 \) and \( A_2 \), then \( A_3(A_1 + A_2) \) and \( A_3(A_1 \mp A_2) = (A_3A_1 \mp A_3A_2) \) are k-EP.

**Proof.** \( A_1 \) and \( A_2 \) are k-EP with \( N(A_1 + A_2) \subset N(A_2) \). By Theorem 2.2, \( (A_1 + A_2) \) is k-EP. Now \( KA_1, KA_2 \) and \( K(A_1 + A_2) \) are EP. Since \( A_3 \) commutes with \( A_1, A_2 \) and \( (A_1 + A_2) \), \( KA_3 \) commutes with \( KA_1, KA_2 \) and \( K(A_1 + A_2) \) and by Theorem (1.3) of [2], \( K(A_3A_1) \), \( K(A_3A_2) \) and \( K(A_3(A_1 + A_2)) \) are EP. Therefore, \( (A_3A_1, A_3A_2, A_3(A_1 + A_2)) \) are k-EP. Now by Theorem 3.6 \( (A_3A_1 \mp A_3A_2) \) is k-EP. By P.3 (Properties 3.3),
\[ K(A_3(A_1 \mp A_2)) = K(A_3A_1 \mp A_3A_2). \]

Since \( A_3A_1 \mp A_3A_2 \) is k-EP, \( K(A_3A_1 \mp A_3A_2) \) is EP \( \Rightarrow K(A_3(A_1 \mp A_2)) \) is EP \( \Rightarrow A_3(A_1 \mp A_2) \) is k-EP. Hence the corollary.

**References**