BULLETIN
of the
MALAYSIAN MATHEMATICAL SOCIETY

Bull. Malaysian Math. Soc. (Second Series) **22** (1999) 127-133

Mertens Theorem and Closed Orbits Of Ergodic Toral Automorphisms

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Abstract: In this paper we consider closed orbits of an ergodic (not necessarily hyperbolic) toral automorphism and prove an analogue of Mertens theorem of analytic number theory for these closed orbits.

1. Introduction

Mertens theorem of analytic number theory gives us the following asymptotic formula for primes p:

$$\prod_{p \le x} (1 - \frac{1}{p}) \sim \frac{e^{-\gamma}}{\log x}$$

where γ is Euler's constant and $f(t) \sim g(t)$ means $f(t)/g(t) \to 1$ when $t \to \infty$ (see [1]).

A dynamical version of this theorem, in the context of closed orbits of axiom A flows and diffeomorphisms, was established by Sharp [4]. In particular, the asymptotic formula derived by Sharp also hold for a special class of axiom A diffeomorphisms namely automorphisms of the finite dimensional torus of the hyperbolic type.

The purpose of this paper is to further extend Sharp's work to the general setting of an ergodic (not necessarily hyperbolic) toral automorphism A. Recall that if we denote the n-torus by T^n then this mean $A : T^n \to T^n$ where the associated $n \times n$ matrix which we shall also denote by A have integer entries, $\det (A) = \pm 1$ and no eigenvalues of A are roots of unity. Let τ denotes a closed orbit of A and $\lambda(\tau)$ be its (least) period (i.e $\tau = \{x, A(x), \cdots, A^{m-1}(x)\}$ for some $x \in T^n$ with $A^m(x) = x$ and $\lambda(\tau) = m$ (m least)). Then the main result of this paper is as follows:

Theorem 1. *Let h be the topological entropy of A and for each closed orbit τ of A, let $\lambda(\tau)$ be its period. Also let $\zeta(z)$ be the zeta function of A. Then*

$$\prod_{\lambda(\tau)\le x}\left(1-\frac{1}{e^{h\lambda(\tau)}}\right) \sim \frac{e^{-m\gamma}}{x^m}\nu\,, \quad as\ x\to\infty$$

where $m = 2^{d/2}$, d is the number of eigenvalues of A of modulus one, γ is Euler's constant and ν is the value of the non-zero and analytic function $\zeta(z)(1-e^h z)^m$ at $z = e^{-h}$.

The proof of this result is modelled along the line of Sharp's paper which in turn was motivated by the analogoues number theoretic proof (see [1]). One should note that, unlike the case of an axiom *A* diffeomorphism which relies heavily on the associated symbolic dynamics and zeta functions, the result for toral automorphisms (hyperbolic or not) can be derived directly without appealing to the symbolic model. This is so since in this case the corresponding dynamical zeta function whose analytic properties is relied upon is easily found. Observe that this is just as well since Markov partitions, the necessary tools for symbolic modelling can never exists in the case of ergodic toral automorphisms (see [3]).

2. The proof

As before let $A : T^n \to T^n$ be an ergodic automorphism on the *n*-torus and let $\mathrm{Fix}_n(A)$ denotes the number of fixed points of A^n. The Artin-Mazur zeta function for A is defined by

$$\zeta(z) = \exp\sum_{n=1}^{\infty} \mathrm{Fix}_n(A).$$

It is well-known that $\zeta(z)$ has radius of convergence e^{-h} where h is the topological entropy of A (see [2]). In fact if λ denotes an arbitrary eigenvalue of A then $\mathrm{Fix}_n(A) = |\prod_\lambda (1-\lambda^n)|$ and $h = \sum_{|\lambda|>1}\log|\lambda|$ (see [6]).

The analytic properties of $\zeta(z)$ beyond $|z| = e^{-h}$ is given by the following theorem which is due to Waddington [5].

Proposition 1. *There exists a finite set U whose elements ρ lies on the unit circle S^1 and integers $K(\rho)$, for each $\rho\in U$, such that*

$$\zeta(z) \;=\; B(z)\prod_{\rho\in U}\frac{1}{(1-e^h\rho z)^{K(\rho)}}$$

where $B(z)$ is analytic and non-zero in the region $|z| < Re^{-h}$ for some $R>1$.

Remarks

1. The set U arises from the expansion $\prod_{|\lambda|=1}(1-\lambda)=\sum_{\rho\in U}K(\rho)\rho$ where the product is taken over all eigenvalues λ of A of modulus one. Clearly $K(\rho)$ are just integers and $1\in U$.
2. Let $m=2^{d/2}$ where d is the number of eigenvalues of A of modulus one. Then it is not hard to see that $K(1)=m$.
3. If we know the eigenvalues of A explicitly then a straight-forward calculation will enable us to derive a closed form for $B(z)$ and hence for $\zeta(z)$. It is clear that this closed form will only involve the eigenvalues of A .

The following example illustrates the above proposition and the accompanying remarks.

Example. The following matrix induces an ergodic but non-hyperbolic automorphism on T^4 (see [7]):

$$A=\begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 8 \end{pmatrix}$$

The matrix A has 2 positive eigenvalues α_1 and α_2 with $0<\alpha_2<1<\alpha_1$, and 2 complex eigenvalues β and $\overline{\beta}$ on the unit circle. Note that the topological entropy h of A is $\log\ \alpha_1$. It is straight-forward to check that $\mathrm{Fix}_n(A)=-(1-\alpha_1^n)(1-\alpha_2^n)(1-\beta)(1-\overline{\beta})$. Hence by using the log series, the zeta function of A is given by the following formula:

$$\zeta(z)=\frac{(1-z)^4(1-\alpha_1\beta z)(1-\alpha_1\overline{\beta}z)(1-\alpha_2\beta z)(1-\alpha_2\overline{\beta}z)}{(1-\alpha_1 z)^2(1-\alpha_2 z)^2(1-\beta z)^2(1-\overline{\beta}z)^2}$$

and this expression is valid for $|z|<e^{-h}$. Also in this case, U is the set $\{1,\beta,\overline{\beta}\}$ with $K(1)=2$ and $K(\beta)=-1=K(\overline{\beta})$. We take the function $B(z)$ (in the notation of Proposition 1) as

$$B(z) = \frac{(1-z)^4(1-\alpha_2\beta z)(1-\alpha_2\overline{\beta}z)}{(1-\beta z)^2(1-\overline{\beta}z)^2(1-\alpha_2 z)^2}$$

Since $\alpha_2 < 1 < \alpha_1$, it is then clear that $B(z)$ is non-zero and analytic in the region $|z| < Re^{-h}$, for some $R > 1$.

For an integer x, let $\pi(x) = \mathrm{Card}\{\tau : \lambda(\tau) \le x\}$. Then using the above proposition, Waddington [5] showed that

Proposition 2. [Prime Orbit Theorem]

$$\pi(x) \sim \frac{e^{h(x+1)}}{x} \sum_{\rho \in U} K(\rho) \frac{\rho^{x+1}}{\rho e^h - 1} \quad \text{as} \quad x \to \infty .$$

Waddington also showed that the sum $\sum_{\rho \in U} K(\rho) \dfrac{\rho^{x+1}}{\rho e^h - 1}$ is bounded away from zero and infinity. Thus we have the following corollory to the Prime Orbit theorem which we shall call upon for later use:

Corrolary 1. *There exists positive constants C_1 and C_2 such that*

$$C_1 \frac{e^{hx}}{x} \le \pi(x) \le C_2 \frac{e^{hx}}{x}.$$

We shall need the following result.

Proposition 3. *The function $\zeta(z)(1-e^h z)^m$ is analytic and non-zero in a neighbourhood of $z = e^{-h}$. Moreover*

$$\zeta(z)(1-e^h z)^m = \exp \sum_{n=1}^{\infty} \frac{1}{n} \left(z^n \ \mathrm{Fix}_n(A) - m(e^h z)^n \right).$$

Proof. The first statement of the proposition follows from proposition 1 and the accompanying remarks. Now write

$$\zeta(z)(1-e^h z)^m = \exp\left(\sum_{n=1}^{\infty} \frac{z^n}{n} \mathrm{Fix}_n(A) \right) \exp\left(-m \log\left(1-e^h z\right)^{-1} \right).$$

The rest follows from the logarithmic expansion.

Corollary 2. *Let* $\nu(z) = \zeta(z)(1-e^h z)^m$. *Then*

$$\log \nu(e^{-h}) = \sum_{n=1}^{x} \frac{1}{n}\left(e^{-nh}\,\mathrm{Fix}_n(A) - m\right) + o(1) \quad \textit{as } x \to \infty .$$

Observe that by remark (3) above, for a given A the function $\nu(z)$ can be made explicit.

Proposition 4. *Let* $K(x) = \sum_{n=1}^{x} \frac{e^{-hn}}{n}\,\mathrm{Fix}_n(A)$. *Then*

$$K(x) = m\ \log\ x + \log\ \nu(e^{-h}) + m\gamma + o(1)$$

where γ *is Euler's constant.*

Proof.

$$\begin{aligned} K(x) - \sum_{n=1}^{x} \frac{m}{n} &= \sum_{n=1}^{x} \frac{e^{-hn}}{n}\,\mathrm{Fix}_n(A) - \sum_{n=1}^{x} \frac{m}{n} \\ &= \sum_{n=1}^{x} \frac{1}{n}\left[\frac{e^{-hn}}{n}\,\mathrm{Fix}_n(A) - m\right] \\ &= \log \nu(e^{-h}) + o(1) \end{aligned}$$

From number theory we have $\sum_{n=1}^{x} \frac{1}{n} = \log\ x + \gamma + o(1)$.

Thus $K(x) = \log \nu(e^{-h}) + m\ \log x + m\gamma + o(1)$.

Proposition 5.

$$\sum_{\lambda(\tau) \le x} \log \frac{1}{1 - e^{-h\lambda(\tau)}} = K(x) + o(1)$$

Proof. From the Euler product form of $\zeta(z)$, one can show that

$$\begin{aligned} K(x) &= \sum_{n=1}^{x} \frac{e^{-hn}}{n}\,\mathrm{Fix}_n(A) \\ &= \sum_{\lambda(\tau)\le x} \sum_{k=1}^{\left[\frac{x}{\lambda(\tau)}\right]} \frac{e^{-h\lambda(\tau)k}}{k} \end{aligned}$$

where [y] denotes the fractional part of y.

Now

$$\begin{aligned}
0 &< \sum_{\lambda(\tau)\le x} \log \frac{1}{1-e^{-h\lambda(\tau)}} - K(x) \\
&= \sum_{\lambda(\tau)\le x} \sum_{k=\left[\frac{x}{\lambda(\tau)}\right]+1}^{\infty} \frac{e^{-h\lambda(\tau)k}}{k} \\
&\le \sum_{\lambda(\tau)\le x} \frac{1}{e^{hx}} \sum_{\ell=2}^{\infty} \frac{e^{-h\lambda(\tau)(\ell-1)}}{\ell} \\
&\le \sum_{\lambda(\tau)\le x} \frac{e^{h\lambda(\tau)}}{e^{hx}} \frac{1}{2e^{h\lambda(\tau)}(e^{h\lambda(\tau)}-1)} \\
&= \frac{1}{2e^{hx}} \sum_{\lambda(\tau)\le x} \frac{1}{e^{h\lambda(\tau)}-1}
\end{aligned}$$

By Stieltjes integrating the last term in the above inequality with respect to $\pi(x)$, we have

$$\sum_{\lambda(\tau)\le x} \frac{1}{e^{h\lambda(\tau)}-1} = \frac{\pi(x)}{e^{hx}-1} + h\int_1^x \frac{\pi(t)}{e^{ht}-1}\,dt.$$

Now apply corollary 1 to deduce that the above sum converge to zero as $x \to \infty$ and hence completing the proof.

Combining Propositions 4 and 5 then gives us our main result where $\nu = \nu(e^{-h})$. When A is also hyperbolic (i.e $d = 0$ so that $m = 1$) then we can retrieve Sharp's result as mentioned in the introduction.

Corollary 3. *Let A be a hyperbolic toral automorphism with topological entropy h. Then*

$$\prod_{\lambda(\tau)\le x} \left(1 - \frac{1}{e^{h\lambda(\tau)}}\right) \sim \frac{e^{-\gamma}}{x}\nu \quad \text{as} \quad x \to \infty,$$

where γ is Euler's constant and some constant ν.

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