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Limit Theorems for the Process of Exceedances in Large Populations

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Abstract. We consider a discrete time branching process with n initial ancestors. Each of the ancestors generates a branching process with the same of spring distribution. Let $S(n,t)$ be the number of ancestors having more than $\theta(t)$ descendants at time t. We prove that $S([m(t)x],t)$ can, in a weak sense, be approximated as $m(t)\to\infty,\ t\to\infty$ by either a Poisson or a Binomial process depending on the criticality of the initial process.

1. Introduction

We consider a discrete time Markov branching stochastic process $X(t),\ t\in N_0=\{0,1,2,\cdots\}$. Let there be n ancestors at time zero. Each of these ancestors initiate a branching process. Let $\theta(t), t\in N_0$, be a positive valued function and $S(n,t)$ be the number of ancestors having more than $\theta(t)$ descendants at time t.

Branching processes started by the initial ancestors may be considered as population processes describing population growth in different regions of an area $\Re$. Then it is clear that $S(n,t)$ is the number of regions of $\Re$ whose populations at time t exceeds level $\theta(t)$. Process $S(n,t)$ can be associated with a problem on the number of vertexes of random rooted trees. In fact each realization of the scheme under the consideration can be interpreted as a forest containing n rooted trees. Consequently a realization of $S(n,t)$ is the number of trees in the forest having more that $\theta(t)$ vertexes of the level t.

It is clear that, if n is fixed and processes are critical or subcritical, then $S(n,t)$ in the long run equals to zero with probability one for any level function $\theta(t)$. The question which we are interested in is how the behaviour of $S(n,t)$ will change if the number of initial ancestors increase depending on the time of observation. More precisely, we consider family of stochastic processes $y(x,t)=S([m(t)x],t)$ where $x\in[0,\infty), m(t)\to\infty$ as $t\to\infty$.

We will try to approximate this process by some known processes with independent increments. Behaviour of the parameter $m(t)$ and the form of limit processes naturally depend on the criticality of the initial branching process. It turns out that, if the process is supercritical, then $y(x,t)$ may be approximated by a "binomial process" (process with independent and binomially distributed increments). If the process is subcritical or critical then the approximating process is Poisson process.

We use the following notations. Let $\{p_k, k \geq 0\}$ be the offspring distribution of the process. We denote

$$f(s) = \sum_{k=0}^{\infty} p_k s^k, \quad A = \sum_{k=1}^{\infty} kp_k, \quad \sigma^2 = \sum_{k=2}^{\infty} k(k-1)p_k,$$
$$Q(t) = P\{X(t) > 0\}.$$

Now, we give a rigorous definition of the process $S(n,t)$. Let $\mathsf{A}_t^{(i)}$ be the random population at time t generated by the ith initial ancestor, $i = 1, 2, \cdots, n.$ If $\theta(t)$ is a positive valued function, the functional $S(n,t) = S(n,t)[\theta]$ can be defined as the following:

$$S(n,t) = \#\{i : card\ \mathsf{A}_t^{(i)} > \theta(t)\}.$$

Interested readers can read more on the properties of exceedances of given levels by sequence of independent and identically distributed random variables in the literature (see [4], [9]) where these properties were discussed widely. The processes of exceedance were also considered in [5] and [6] where some limit theorems for the "index" of the first exceedance process were obtained.

The proofs of main results are based on direct analysis of Laplace transforms. One could obtain these theorems by using results on the theory of point processes (see [7], Prop. 3.2).

2. Approximating processes

First we consider the critical case, i.e. the case $A = 1, 0 < \sigma^2 < \infty$. In this case, we assume that there exists the following :

$$\lim_{t \to \infty} \frac{\theta(t)}{t} = \theta \in [0, \infty] \tag{1}$$

and consider $y(x,t) = S([tx], t)$, i.e. $m(t) = t.$

Theorem 1. *If* $A=1, 0<\sigma^2<\infty$ *and (1) is satisfied then* $y(x,t)\xrightarrow{D}y(x)$, *as* $t\to\infty$ *where D means convergence in the weak sense and* $y(x)$ *is the Poisson process with* $Ey(x)=\frac{2x}{\sigma^2}\exp\{-\frac{2\theta}{\sigma^2}\}$ *for* $\theta\in[0,\infty)$ *and it is a "zero process" (i.e.* $y(x)=0$ *with probability 1,* $x\in[0,\infty)$*) for* $\theta=\infty$.

Proof. Since the lives of individuals are independent, we obtain that

$$ES^{S(n,t)}=[1-(1-s)R(\theta(t),t)]^n \tag{2}$$

where $R(x,t)=P\{X(t)>x\}$ and $X(t)$ is the branching process generated by a single ancestor.

It is known that (see [2], p20), if $A=1, \sigma^2<\infty$ then

$$\lim_{t\to\infty}P\left\{\frac{X(t)}{t}>\theta\,\middle|\,X(t)>0\right\}=e^{\frac{-2\theta}{\sigma^2}},\quad \theta\ge 0.$$

Furthermore, it is known that if $A\le 1$, then $Q(t)\to 0$ as $t\to\infty$. It is also known (see [1], [3]) that if $A=1$ and $\sigma^2<\infty$ then $Q(t)\sim\frac{2}{\sigma^2 t}$ as $t\to\infty$.

It follows from the limit theorem for critical processes that under the condition (1)

$$R(\theta(t),t)\sim\frac{2}{\sigma^2 t}\exp\left\{\frac{-2\theta}{\sigma^2}\right\},\quad t\to\infty. \tag{3}$$

Therefore,

$$\begin{aligned}\lim_{t\to\infty}\ell n\, ES^{y(x,t)}&=-\lim_{t\to\infty}[tx]\,R(\theta(t),t)\,(1-s)\\&=-\frac{2x}{\sigma^2}\exp\left\{\frac{-2\theta}{\sigma^2}\right\}(1-s)\end{aligned} \tag{4}$$

for $\theta\in[0,\infty)$. Consequently the generating function of $y(x,t)$ tends to

$$\exp\left\{\frac{2x}{\sigma^2}e^{\frac{-2\theta}{\sigma^2}}(s-1)\right\}\text{ as } t\to\infty,$$

which is the generating function of the one dimensional distribution for Poisson process $y(x)$.

Now we consider

$$P\{y(x_i,t)=k_i, i=0,1,\cdots,r\},\ x_0<x_1<\cdots x_r<\infty .$$

First we prove that

$$\lim_{t\to\infty} ES^{y(x_2,t)-y(x_1,t)}=\exp\left\{\frac{2(x_2-x_1)}{\sigma^2}e^{\frac{-2\theta}{\sigma^2}}(s-1)\right\}. \tag{5}$$

Infact, since

$$y(x_2,t)-y(x_1,t)=\sum_{i=[tx_1]+1}^{[tx_2]}\varepsilon_i , \tag{6}$$

where

$$\varepsilon_i=\begin{cases}1 & if \quad X_i(t)>\theta(t)\\ 0 & if \quad X_i(t)\le\theta(t)\end{cases}$$

and $X_i(t)$ is the process generated by the *i*th ancestor, we have

$$\lim_{t\to\infty} \ell n\, ES^{y(x_2,t)-y(x_1,t)}=\lim_{t\to\infty}([tx_2]-[tx_1])\,R(\theta(t),t)\,(s-1).$$

Thus again due to the limit theorem for critical processes we obtain (5) from the last relation.

Let $r=2$. It is easy to see that

$$E\left[S_1^{\,y(x_1,t)},S_2^{\,y(x_2,t)}\right]=E(S_1,S_2)^{y(x_1,t)}ES_2^{\,y(x_2,t)-y(x_1,t)}. \tag{7}$$

Then by (4) and (5), it follows that as $t\to\infty$, the joint generating function from (7) tends to

$$\exp\left\{\frac{2x_1}{\sigma^2}e^{\frac{-2\theta}{\sigma^2}}(s_1s_2-1)+\frac{2(x_2-x_1)}{\sigma^2}e^{\frac{-2\theta}{\sigma^2}}(s_2-1)\right\}$$

which is the generating function of $(y(x_1),y(x_2))$. Convergence in distribution of $(y(x_1,t),\cdots,y(x_r,t))$ as $t\to\infty$ to $(y(x_1),\cdots,y(x_r))$ follows by induction. Thus the theorem is proved for $\theta\in[0,\infty)$. The proof for $\theta=\infty$ follows from the fact that in this case the limit on the right side of (4) is zero.

Now let $A<1$, i.e. the initial process is subcritical. In this case, we use the following limit theorem for subcritical processes (see [3] or [7], p29). If $A<1$, there exists

$$\lim_{t\to\infty} P\left\{ X(t)=j \mid X(t)>0 \right\} = P_j^{\ast}, \quad j\geq 1 \tag{8}$$

and the generating function $F^{\ast}(s)$ of $P_j^{\ast}$, $j\geq 1$ satisfies the equation

$$1-F^{\ast}(f(s)) = A(1-F^{\ast}(s)). \tag{9}$$

It is also known that, if $A\leq 1$, then $Q(t)=R(0,t)\to 0$ as $t\to\infty$. If $A<1$ and in addition, $EX(1)\,\ell n\,X(1)<\infty$, then we have the following asymptotics for $Q(t)$ (see [7], p56)

$$Q(t)\sim KA^{t},\ 0<K=\prod_{m=0}^{\infty} B(P\{X(m)=0\})<\infty \tag{10}$$

where $B(s)=(1-f(s))/(A(1-s))$,

$$\sum_{k=2}^{\infty} kp_k\,\ell n\,k<\infty \tag{11}$$

and $y(x,t)=S([A^{-t}x],t)$.

Theorem 2. *If* $A<1$, *(11) is satisfied, then* $y(x,t)\xrightarrow{D} y(x)$, *as* $t\to\infty$ *where, as before D means convergence in the weak sense and* $y(x)$ *is the Poisson process with* $Ey(x)=Kx\sum_{j>\theta} P_j^{\ast}$ *for* $\theta(t)=\theta\in N_0$ *and it is a zero process if* $\theta(t)\to\infty$.

Proof. We use again relation (2) with $n=[A^{-t}x]$. In this case, it follows from the above limit theorem for subcritical processes that under the condition (11).

$$R(\theta(t),t)\sim K\ A^{t}\sum_{j>\theta} P_j^{\ast}, \quad t\to\infty \tag{12}$$

for $\theta\in N_0$. Since

$$\lim_{t\to\infty} \ell n\, ES^{y(x,t)} = \lim_{t\to\infty} \left[A^{-t}x\right]\, R\left(\theta(t),t\right)(s-1),$$

we obtain that the limit of $ES^{y(x,t)}$ as $t\to\infty$ is $\exp\left\{Kx\sum_{j>\theta} P_j^{\ast}(s-1)\right\}$.

To prove convergence of finite dimensional distributions it is sufficient to show that

$$\lim_{t\to\infty} ES^{y(x_2,t)-y(x_1,t)} = \exp\left\{ K(x_2-x_1)\sum_{j>\theta} P_j^{\,*}(s-1)\right\} \tag{13}$$

for any $0<x_1<x_2<\infty$ and $0<s<1$. It follows from representation (6) and (2) that in this case,

$$\lim_{t\to\infty} ES^{y(x_2,t)-y(x_1,t)} = \lim_{t\to\infty} n(t)\,R\left(\theta(t),t)(s-1\right),$$

where $n(t)=[A^{-t}x_2]-[A^{-t}x_1]$. We obtain (13) from here taking into account relation (12). Thus convergence of finite dimensional distributions of $y(x,t)$ to ones of $y(x)$ follows from (13) by induction. The theorem is proved for $\theta\in N_0$. If $\theta=\infty$, again we have zero limit for $\ell n\, ES^{y(x,t)}$ since the limit distribution $\{P_j^{\,*}\},\ j\geq 1$ is proper.

Now we consider the case of supercritical processes. It is known (see [1]) that if $A>1,\ EX(1)\ \ell n\, X(1)<\infty$, then $X(t)A^{-t}$ converges with probability 1 to a random variable W and the Laplace transform $\phi(\lambda)$ of W satisfies the following equation

$$\phi(\lambda)=f\left(\phi\left(\frac{\lambda}{A}\right)\right).$$

It is also known that the distribution function $\Pi(x)$ of W is absolute continuous for $x>0$ and has an atom of the mass q at $x=0$. Here q is the extinction probability.
We assume that there exists

$$\lim_{t\to\infty}\theta(t)A^{-t}=\theta\in[0,\infty] \tag{14}$$

and consider "discrete time" process $S(n,t)$, $n=0,1,\cdots$ for $t\in N_0$.

Theorem 3. *If* $A>1$, *(11) and (14) are satisfied, then* $S(n,t)\xrightarrow{D}\xi(n)$, $n\in N_0$ *as* $t\to\infty$ *where, as before* D *means convergence in the weak sense and* $\xi(n)$ *is a stochastic process with independent and binomially distributed increments such that*

$$P\{\xi(n_i)-\xi(n_{i-1})=k\}=\binom{n_i-n_{i-1}}{k}[1-\Pi(\theta)]^k\ \Pi(\theta)^{n_i-n_{i-1}-k}$$

for any $0 \le n_{i-1} < n_i < \infty$, $n_i \in N_0$ *for* $\theta \in [0,\infty)$ *and it is a zero process for* $\theta = \infty$.

Proof. Let $n_0, n_1, \cdots, n_r$ be such numbers that $0 = n_0 < n_1 < \cdots < n_r < \infty$ and $n_i \in N_0$, $0 \le i \le r$. First we prove that for $1 \le i \le r$,

$$\lim_{t\to\infty} ES_i^{S(n_i,t)-S(n_{i-1},t)} = \left(\hat{\Pi}(\theta)S_i + \Pi(\theta)\right)^{n_i - n_{i-1}}, \tag{15}$$

where $\hat{\Pi}(\theta) = 1 - \Pi(\theta)$. It follows from representation (6) that

$$ES_i^{S(n_i,t)-S(n_{i-1},t)} = \left(R(\theta(t),t)\, S_i + 1 - R(\theta(t),t)\right)^{n_i - n_{i-1}}. \tag{16}$$

Now we consider the estimate

$$\left| P\{X(t) \le \theta(t)\} - \Pi(\theta) \right| \le \sup_x \left| P\{X(t)A^{-t} \le x\} - \Pi(x)\right| + \left| \Pi(\theta(t)A^{-t}) - \Pi(\theta)\right|. \tag{17}$$

First term on the right side of (17) tends to zero as $t \to \infty$ due to the limit theorem for supercritical processes. It follows from condition (14) and continuity of $\Pi(x)$ that the limit of the second term is also zero. Thus

$$R(\theta(t),t) \to 1 - \Pi(\theta) \tag{18}$$

as $t \to \infty$. From relations (16) and (18), we obtain relation (15).

Using independence of incfrements of $S(n,t)$ from relation (15) we have

$$\lim_{t\to\infty} E\left[\prod_{i=1}^{r} S_i^{S(n_i,t)-S(n_{i-1},t)} \right] = \prod_{i=1}^{r} \left\{ \hat{\Pi}(\theta)S_i + \Pi(\theta)\right\}^{n_i - n_{i-1}}$$

which proves the theorem for $\theta \in [0,\infty)$. In the case, if $\theta = \infty$, the limit on the right side of (15) equals 1 and the "limit process" $\xi(n)$ equals zero for all *n*. The theorem is proved.

Finally, we would like to make some remarks on the asymptotic behaviour of some moments of the process $S(n,t)$. In the critical case for which (1) holds, using the limit theorem for this case, then

$$E(S(n,t)) = Var(S(n,t)) \sim \frac{2x}{\sigma^2} e^{-\frac{2\theta}{\sigma^2}}, \quad t \to \infty.$$

For the subcritical case for which (11) holds, then

$$E(S(n,t)) = Var(S(n,t)) \sim Kx\sum_{j>\theta} P_j^* , \quad t \to \infty .$$

If the process is supercritical and (14) holds, using the limit theorem for the supercritical process, one can find that

$$E(S(n,t) \sim n(1-\Pi(\theta)), \qquad Var(S(n,t)) \sim n(1-\Pi(\theta))\,\Pi(\theta).$$

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