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On P_k and P_k' Near-Rings

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Abstract. In page 297 of Pilz[4] a right near-ring N is called a $C_1(C_2)$ near-ring if $xN = xNx(Nx = xNx)$ for all x in N. Szasz, Frence, in [6] calls a ring N, with the property $xN = xNx$ for all x in N, a P_1-ring. We shall, in this paper, refer to a near-ring N with the property $xN = xNx(Nx = xNx)$ for all x in N, a $P_1(P_1')$ near-ring. Motivated by these concepts we introduce P_k and P_k' near-rings (Definition 2.1). We further generalize these concepts by introducing $P_k(r,m)$ and $P_k'(r,m)$ near-rings (Definition 3.1). We discuss the properties of all these newly introduced structures in detail. We also obtain complete charcterisations and structure theorems for such near rings.

1. Introduction

Near-rings are generalized rings. If in a ring $(N,+,\cdot)$ we do not stipulate (i) the commutativity of '+' and (ii) the left distributive law of '.' over '+' then $(N,+,\cdot)$ becomes a right near-ring. Throughout this paper N stands for a right near-ring $(N,+,\cdot)$ with at least two elements, "xy" stands for "$x.y$" for all x, y in N and 0 denotes the identity of the group $(N,+)$.

A subgroup M of N is called an N-subgroup if $NM \subset M$. An ideal I of N is called

(i) a prime ideal if for all ideals J, K of N, $JK \subset I \Rightarrow J \subset I$ or $K \subset I$,
(ii) a completely prime ideal if for all $a, b \in N$, $ab \in I \Rightarrow a \in I$ or $b \in I$ and
(iii) a completely semiprime ideal if for $a \in N, a^2 \in I \Rightarrow a \in I$.

If for x, y in N, $xy = 0 \Rightarrow xny = 0$ for all n in N, we say that N has *IFP* (i.e. "Insertion of Factors property").

A map 'm' from N into N is called a mate function for N if $x = xm(x)x$ for all x in N. $m(x)$ is called a mate of x. This concept has been introduced in [5] to handle the regularity structure in a near-ring with considerable ease and also to discuss the properties of "mates" in detail.

All the near-fields in this paper are zero-symmetric. Basic concepts and terms used but left undefined in this paper can be found in Pilz [4].

1.1. Notations

(i) E denotes the set of all idempotents of N.

(ii) L is the set of all nilpotent elements of N.

(iii) $N_d = \{ n \in N / n(x+y) = nx + ny$ for all x, y in $N\}$ the set of all distributive elements of N.

(iv) $N_0 = \{ n \in N / n0 = 0\}$ - the zero-symmetric part of N. (N is called zero-symmetric if $N = N_0$).

(v) If S is a non-empty subset of N, $C(S) = \{n \in N / nx = xn$ for all $x \in S\}$ and for $C(\{x\})$, we write $C(x)$ for x in N.

(vi) If N has *IFP* and if $xy = 0 \Rightarrow yx = 0$ (for x, y in N) then we say that N has (*, *IFP*).

(vii) As in [1], N is a $P(r, m)$ near-ring if $x^r N = Nx^m$ for all x in N where r, m are positive integers.

1.2. Preliminary results

We freely make use of the following results from [1], [2], [3], [4] and [5] and designate them as $K(1)$, $K(2)$ etc. (K for "Known Result").

$K(1)$: When N admits mate functions the following are equivalent
(i) N is $P(1,2)$
(ii) $E \subset C(N)$
(iii) N is $P(2,1)$ (Theorem 2.20 of [1]).

$K(2)$: When N admits mate funtions, N is a $P(r, m)$ near-ring (for all positive integers r, m) if and only if N is a $P(1, 2)$ near-ring (Theorem 2.22 of [1]).

$K(3)$: If N is a $P(r, m)$ near-ring with a mate function, we have from $K(2)$, the following results:
(a) The concepts of N-subgroups, left ideals, right ideals and ideals are equivalent in N (Remark 2.26 (a) of [1]).
(b) $NaNb = Na \cap Nb = Nab$ for all a, b in N (Proposition 2.32 of [1]).
(c) N is subdirectly irreduible if and only if it is a near-field (Theorem 3.1 of [1]).
(d) N is isomorphic to a subdirect product of near-fields (Theorem 3.3 of [1]).

$K(4)$: A near-ring N has no non-zero nilpotent elements if and only if $x^2 = 0 \Rightarrow x = 0$ for all x in N. (This result in prob 14, P.9 of [3] in respect of rings is valid for N as well).

$K(5)$: A zero-symmetric near-ring N has *IFP* if and only if (0: S) is an ideal where S is any non-empty subset of N (9.3, p.289 of [4]).

$K(6)$: N is subdirectly irreducible if and only if the intersection of any family of non-zero ideals of N is again non-zero (1.60(c), p.25 of [4] and [2]).

$K(7)$: If N admits mate functions and is subdirectly irreducible then it has no non-zero idempotent zero-divisors (vide stage (2) of the proof of Theorem 3.1 of [1]).

$K(8)$: If N admits a mate function m, then $xm(x), m(x)x \in E$ and $Nx = Nm(x)x$ and $xN = xm(x)N$ for all x in N (Lemma 3.2 of [5]).

$K(9)$: Let $a^2 = ba$ and $b^2 = ab$ for a, b in N. Let $u_1 = a - b$, $u_2 = au_1$ and $u_3 = bu_1$. If there exist x_i's in N such that $u_i = x_i u_i^2$ $(i = 1, 2, 3)$ then $a = b$ (Lemma 2.5 of [5]).

2. P_k and P_k' near-rings

Definition 2.1. *A near-ring N is called a P_k near-ring (P_k' near-ring) if there exists a positive integer k such that* $x^k N = xNx (Nx^k = xNx)$ *for all x in N.*

Remark 2.2. Obviously any P_k' near-ring is zero-symmetric.

Examples 2.3.

(a) The direct product of any two near-fields is a P_k as well as a P_k' near-ring.

(b) The near-ring $(N,+,\cdot)$ where $(N,+)$ is the Klein's four group with $N = \{0, a, b, c\}$ and '.' satisfies the following table (scheme 12, p.408 of Pilz[4]).

.	0	a	b	c
0	0	0	0	0
a	0	a	0	a
b	0	0	0	0
c	0	a	0	a

is a P_k as well as a P_k' near-ring.

(c) Suppose in the example (b) above we define '.' (as per scheme 8, p. 408 of Pilz[4]) as follows

.	0	a	b	c
0	0	0	0	0
a	0	0	0	a
b	0	a	b	b
c	0	a	b	c

Then $(N,+,\cdot)$ is a P_k' near ring but not a P_k near ring for any $k>1$. It is neither a P_1 near-ring nor a P_1' near-ring.

(d) Consider the near-ring $(Z_4,+,\cdot)$ where $(Z_4,+)$ is the group of integers modulo 4 and '.' is defined as per scheme 2, p.407 of Pilz [4].

.	0	1	2	3
0	0	0	0	0
1	0	1	0	0
2	0	2	0	0
3	0	3	0	0

This is a P_k near-ring for $k>1$. It is not a P_k' near-ring for any positive integer *k*.

(e) Any constant near-ring (i.e. $ab=a$ for all $a,b\in N$) is a P_k near-ring.
It is easy to verify the following:

Proposition 2.4. *Any homomorphic image of a $P_k(P_k')$ near-ring is a $P_k(P_k')$ near-ring.*

As an immediate consequence of Proposition 2.4 we have the following:

Theorem 2.5. *Every $P_k(P_k')$ near-ring N is isomorphic to a subdiret product of subdirectly irreducible $P_k(P_k')$ near-rings.*

Proof. By 1.62, p.26 of Pilz[4], *N* is isomorphic to a subdirect product of subdirectly irreducible near-rings N_i\`s, say, and each N_i is a homomorphic image of *N* under the usual projection map π_i. The desired result now follows from Proposition 2.4.

Before proceeding further we have the following:

Definition 2.6. We say that a near-ring *N* is an $S_r(S_r')$ near-ring if $x\in Nx^r\ (x\in x^rN)$ for all *x* in *N*. When $r=1$ we write "$S(S')$ near-ring" instead of "$S_1(S_1')$ near-ring".

Examples 2.7. Let *r* be any positive integer.

(i) The near-ring of example 2.3(a) is an S_r as well as an S_r' near-ring.
(ii) Trivially any Boolean near-ring is an S_r as well as an S_r' near-ring.
(iii) The near-ring of example 2.3(e) is an S_r' near-ring (but it is not an S_r near-ring).

(iv) Let N be an arbitrary near-ring and let I be the ideal generated by $\{x - nx^r / n, x \in N\}$. Then $\overline{N} = N / I$ is an S_r near-ring.

(v) If in example (iv), I is the ideal generated by $\{\{x - x^r n / x, n \in N\}$ then $\overline{N} = N / I$ is an $S_r{}'$ near-ring.

Remark 2.8. Examples 2.7(iv) and 2.7(v) provide devices for manufacturing S_r and $S_r{}'$ near-rings from an arbitrary near-ring.

Proposition 2.9. *Every* $S_r(S_r{}')$ *near-ring is an* $S(S')$ *near-ring.*

Proof. Let N be an $S_r(S_r{}')$ near-ring with $r \geq 2$. Clearly then for all $x \in N, x \in Nx^r = (Nx^{r-1})x \subset Nx(x \in x^r N = x(x^{r-1}N = x(x^{r-1}N) \subset xN)$ $\therefore$ N is an $S(S')$ near-ring.

Remark 2.10. The converse of Proposition 2.9 is not valid. Obviously the near-ring of example 2.3 (c) is an *S* as well as an *S′* near-ring. But it is neither an S_r near-ring nor an $S_r{}'$ near-ring for $r > 1$.

Also the (near -) ring of integers $(Z, +, \cdot)$ is an *S* as well as an S' near-ring. But it is neither an S_r near-ring nor an $S_r{}'$ near-ring for $r > 1$. Thus even in the case of rings, the converse of Proposition 2.9 is not valid, in general.

Before discussing the proporties of $P_k(P_k{}')$ near-rings we have the following:

Lemma 2.11. *Let N be a* P_1 *near-ring - ie* $xN = xNx$ *for all x in N - with a mate function. Then N admits a mate function f such that* $f(x) \in C(x)$.

Proof. Let *m* be a mate function for *N*. For $x, m(x)$ in *N*, we can find a '*t*' in *N* such that $xm(x) = xtx$ (as $xN = xNx$) i.e. $x = xm(x)x = xtx^2 = x' x^2$ with $x' = xt$. We shall define $f : N \to N$ such that $f(x) = x'$. Clearly then if we set $y = xf(x)x$, it follows that $xy = y^2$ and $yx = x^2$ and *K*(9) guarantees $x = y$. Thus $x = xf(x)x$ and therefore *f* is a mate function for *N*.

Again let us set $a = xf(x)$, $b = f(x)x$, $a - b = w_1$, $xw_1 = w_2$ and $aw_1 = w_3$. By a slight modification of the proof of the Lemma 2.5 (*K*(9)) of [5], we get a = b and the desired result follows.

Proposition 2.12. *Let N be a* P_k *or a* $P_k{}'$ *near-ring. If N admits mate functions then* $L = \{0\}$.

Proof. Suppose *N* is a P_k near-ring with a mate function '*m*'.

Case (i). $k=1$ i.e. $xN = xNx$ for all *x* in *N*. We appeal to Lemma 2.11 and observe that *N* admits a mate function f such that $f(x) \in C(x)$ for all *x* in *N*. Therefore for all *x* in *N*, we have $x = xf(x)x = x(xf(x)) = x^2 f(x)$ and this guarantees "$x^2 = 0 \Rightarrow x = 0$". From *K*(4) we get $L = \{0\}$.

Case 2. Let $k > 1$. Now $x^k N = xNx$ for all *x* in *N*. Since $x = xm(x)x \in xNx(= x^k N)$ we get $x = x^k n$ for some *n* in *N*. If $k = 2$ then $x = x^2 n$ and therefore "$x^2 = 0 \Rightarrow x = 0$". If $k > 2$ we have $x = x^2(x^{k-2}n)$ and again we get "$x^2 = 0 \Rightarrow x = 0$". This yields $L = \{0\}$ for all $k > 1$.

In view of Remark 2.2 we can prove the above result when *N* is a P_k' near-ring, with $k > 1$, in a similar fashion. When $k = 1$ we observe that for $x \in N$, $xNx = Nx$. Consequently for all *x* in *N*, $x = xm(x)x \in xNx = x(Nx) = x(xNx) = x^2 Nx$ i.e. $x \in x^2 Nx$. Hence "$x^2 = 0 \Rightarrow x = 0$" and the proof is complete.

Remark 2.13. The converse of Proposition 2.12 is not valid. Examples are plentiful to justify this. To cite two such we have the following:

(i) Consider the near-ring constructed on the Klein's four group as per scheme (21) p. 408 of Pilz[4].

.	0	*a*	*b*	*c*
0	0	0	0	0
a	*a*	*a*	*a*	*a*
b	0	0	*b*	0
c	*a*	*a*	*c*	*a*

It is a $P_k (k \geq 2)$ near-ring without nilpotent elements. But it has no mate function.

(ii) Even in the case of rings, the converse does not hold. The ring $(Z,+,\cdot)$ of integers which is a P_2 as well as a P_2' near-ring comes in handy to justify this. The following result will be made use of throughout this paper.

Proposition 2.14. *A $P_k (P_k')$ near-ring N has a mate function if and only if N is an $S_k'(S_k)$ near-ring.*

Proof. Let *N* be a P_k near-ring with a mate function '*m*'. Therefore $x^k N = xNx$ for all *x* in *N*. Now $x = xm(x)x \in xNx(= x^k N)$ and this implies $x \in x^k N$ i.e. *N* is an S_k' near-ring.

Conversely let N be an $S_k'-P_k$ near-ring. $\therefore$ $x \in x^k N(= xNx)$ for all x in N. This implies $x = xnx$ for some n in N. Therefore $x = xm(x)x$ where we set $m(x) = n$. Hence 'm' is a mate function for N.

The proof in respect of P_k' near-rings is similar.

Lemma 2.15. *Let N be a zero-symmetric near-ring with a mate function `m'. Then*

(i) *N has (*, IFP) if and only if* $L = \{0\}$.
(ii) $ene = en$ *for all* $e \in E$ *and* $n \in N$.

Proof.

(i) Suppose N has (*, *IFP*). If $a^2 = 0$ for any a in N then by *IFP*, $am(a)a = 0$. i.e. $a = 0$. Hence $L = \{0\}$ (by K(4)).

Conversely if $L = \{0\}$, then $xy = 0 \Rightarrow$ (1) $(yx)^2 = (yx)(yx) = y(xy)x = y0x = y0 = 0 \Rightarrow yx = 0$ and (2) $(xny)^2 = (xny)(xny) = xn(yx)ny = xn0 = 0 \Rightarrow xny = 0$ for all n in N. Thus N has (*, *IFP*).

(ii) For $e \in E$ and $n \in N$ we have $(ene - en)e = 0$ and by (*, *IFP*), we get $e(ene - en) = 0,$ $en(ene - en) = 0$ and $ene(ene - en) = 0.$ These demand $(ene - en)^2 = 0$ and therefore $ene = en$ since $L = \{0\}$.

As an immediate consequence of Lemma 2.15 - read with Propositions 2.12 and 2.14 - we have the following:

Corollary 2.16.

(i) *If N is an* $S_k - P_k'$ *near-ring then N has (*, IFP).*
(ii) *If* $N = (= N_0)$ *is an* $S_k' - P_k$ *near-ring then N has (*, IFP).*

Theorem 2.17. *Let N be a near-ring with a mate function f. Then the following statements are equivalent:*

(i) *N is* P_k' *for any positive integer k.*
(ii) $E \subset C(N)$.
(iii) *N is* $P(r, m)$ *for all positive integers* r, m.

Proof.

(i) $\Rightarrow$ (ii): Since $Nx^k = xNx$ for all x in N, $Ne = eNe$ for all e in E. $\therefore$ For any $n \in N$, there exists $u \in N$ such that $ne = eue$. This implies $ene = (eue =)ne$. By Lemma 2.15 and Corollary 2.16(i) we have $en = ene$. Thus we get $en = (ene =)ne$ and (ii) follows.

(ii) $\Rightarrow$ (i): **Case 1 :** Let $k = 1$. For all x in N, $Nx = Nxf(x)x = (xf(x)N)x = xNx$ (using K(8)) $\therefore$ N is a P_1' near-ring.

Case 2. Let $k > 1$. For all n, x in N, $nx^k = (nx)x^{k-1} = n(xf(x)x)x^{k-1}$ $= xf(x)(nx)x^{k-1} = (xf(x)nx^{k-1})x \in xNx$. Therefore $Nx^k \subset xNx$. Also $xnx = (xf(x)x)nx = xnf(x)x^2 = xnf(x)x^2 = (xf(x)x)\,nf(x)x^2 = xn(f(x))^2\,x^3$. Repeating this process we ultimately obtain $xnx = (xn(f(x))^{k-1})x^k \in Nx^k$ for all positive integers k. Therefore $xNx \subset Nx^k$. Thus $xNx \subset Nx^k$ for all x in N and (i) follows.

"(ii) $\Leftrightarrow$ (iii)" follows from $K(1)$ and $K(2)$.

Theorem 2.18. *Any N-subgroup of an* $S_k - P_k'$ *near-ring N is also an* $S_k - P_k'$ *near-ring in its own right.*

Proof. Let N be an $S_k - P_k'$ near-ring. Proposition 2.14 guarantees the existence of a mate function '*f*' for N. Let M be any N-subgroup of N. From Theorem 2.17 we see that N is a $P(r, m)$ near-ring also. $K(3)$(a) demands that M is a right ideal of N. Therefore $f(x)xf(x) \in NMN \subset M$ for all x in M. Thus we can define a map g: $M \to M$ such that $g(x) = f(x)xf(x)$ for all $x \subset M$. Obviously g serves as a mate function for M - as $xg(x)x = x$.

Now let $x, s \in M$. Since $Nx^k = xNx$ there exists $n \in N$ such that $sx^k = xnx = x(nxg(x))x \in x(NM)x \subset xMx$. $\therefore Mx^k \subset xMx$. Also since $xsx \in xNx (= Nx^k)$ there exists $n_1 \in N$ such that $xsx = n_1x^k$. Again $xsx = xg(x)(xsx) = xg(x)n_1x^k = s_1x^k$ where $s_1 = xg(x)n_1 \in MN \subset M$. $\therefore$ $xMx \subset Mx^k$. Thus $Mx^k = xMx$ for all $x \in M$ i.e. M is a P_k' near-ring. Since M has a mate function '*g*', M is an S_k near-ring as well (from Proposition 2.14).

Corollary 2.19. *Any N-homomorphic image of an* $S_k - P_k'$ *near-ring N is an* $S_k - P_k'$ *near-ring.*

Proof. Let $h: N \to N$ be an N-homomorphism. Obviously $h(N)$ is a subgroup of N. For n, $n' \in N$, $nh(n') = h(nn') \in h(N) \Rightarrow Nh(N) \subset h(N) \Rightarrow h(N)$ is an N-subgroup of N. The result now follows from Theorem 2.18.

Theorem 2.20. *Let N be a near-ring with a mate function 'm'. If $E \subset C(N)$ then N is a P_k near-ring for all positve integers k.*

Proof. Suppose $E \subset C(N)$. If $x \in N$ then $xN = x(m(x)xN) = x(Nm(x)x) = xNx$ (using $K(8)$). This proves the theorem for $k = 1$. For $k > 1$ and for any $x \in N, x^k N = x(x^{k-1}N) \subset xN = xNx$ (using the result for $k = 1$) i.e. $x^k N \subset xNx$. Also $xNx = xNxm(x)x = x(Nxm(x))x = x(xm(x)N)x = x(xN)x$ (using $K(8)$) $= x^2 Nx = x(xNx) = x(x^2 Nx) = x^3 Nx$ and continuing in the same vein we ultimately get $xNx = x^k Nx \subset x^k N$ where k is any positive integer. Hence $xNx = x^k N$ for all $x \in N$ and the result follows.

Remark 2.21. The converse of Theorem 2.20 is not valid. For example let us consider the near-ring $(N, +, \cdot)$ where $(N, +)$ is the Klein's four group with $N = \{0, a, b, c\}$ and '.' satisfies the following table (defined as per scheme 1, p.408 of Pilz [4]).

.	0	a	b	c
0	0	0	0	0
a	0	a	a	a
b	0	b	b	b
c	0	c	c	c

This is a P_k near-ring for all positive integers k. In fact it is a Boolean near-ring and identity function serves as a mate function. But $E(= N) \not\subset C(N)$.

Corollary 2.22. *Let N be a near-ring admitting mate functions. Then N is a P_k' near-ring for a fixed positive integer k if and only if it is a P_r as well as a P_r' near-ring for all positive integers r.*

Proof. For the 'only if ' part we proceed as follows: We appeal to Theorems 2.17 and 2.20 and observe that N is $P_k' \Rightarrow E \subset C(N) \Rightarrow N$ is P_r as well as P_r' for all positive integers r.

Proof of 'if part' is obvious.

Remark 2.23. It is worth-noting that a P_k near-ring with a mate function need not necessarily be a $P_k{}'$ near-ring. The example cited under Remark 2.21 comes in handy to justify this. It is a P_k near-ring, for all positive integers *k*, and it admits mate functions. But it is not a $P_k{}'$ near-ring for any *k*.

We shall now discuss the conditons under which a P_k near ring becomes a $P_k{}'$ near-ring.

Theorem 2.24. *Let N be a zero-symmetric near-ring admitting mate functions. Then N is a P_k near-ring with $E \subset N_d$ if and only if N is a $P_k{}'$ near-ring.*

Proof. For the 'only if' part we note that as $N = N_o$ and *N* has a mate funciton, it has $(*, IFP)$ and $L = \{0\}$ (by Propositon 2.14 and Corollary 2.16(ii)). From Lemma 2.15 we get $ene = en$ for all $e \in E$ and $n \in N$. Since $E \subset N_d$, $e(ne - ene) = 0$. This ultimately yields $(ne - ene)^2 = 0$ and consequently $ne = ene.$ Thus we get $E \subset C(N)$. Theorem 2.17 now guarantees that *N* is a $P_k{}'$ near-ring.

For the 'if part' we observe that *N* is $P_k{}' \Rightarrow E \subset C(N)$ (from Theorem 2.17) $\Rightarrow N$ is a P_k near-ring (Theorem 2.20). Again $E \subset C(N) \Rightarrow E \subset N_d$ and the result follows.

As an immediate consequence of Theorems 2.17 and 2.24 we get:

Corollary 2.25. *The following statements are equivalent in a near-ring N with mate functions.*

(i), (ii), (iii) of Theorem 2.17 and
(iv) N is a zero-symmetric P_k near-ring with $E \subset N_d$.

From Corollaries 2.22 and 2.25 we obtain the following:

Proposition 2.26. *Let N admit mate functions. Then N is a zerosymmetric P_k near-ring (for a fixed k) with $E \subset N_d$ if and only if N is a P_r as well as a $P_r{}'$ near-ring for all positive integers r.*

Theorem 2.27. *Let N be a P_k near-ring with a mate function. If N is zero-symmetric and has a left identity 1 then N is a $P_r{}'$ near-ring for any positive integer r.*

Proof. Since *N* is a zero-symmetric P_k near-ring with a mate function it has $(*, IFP)$ and $L = \{0\}$ (from proposition 2.14, Lemma 2.15(i) and Corollary 2.16(ii)). Again by Lemma 2.15 (ii), $ene = en$ for all $e \in E$ and $n \in N$. Also since $(1-e)e = 0$ we get $(1-e)ne = 0$ for all $n \in N$. $\therefore$ $ne = ene.$ Thus $en = ene = ne$ for all $n \in N$ and for all $e \in E$. Hence $E \subset C(N)$ and Theorem 2.17 guarantees that *N* is a $P_r{}'$ near-ring for all positive integers *r*.

Proposition 2.28. *A left identity of a P_k near-ring is also a right identity.*

Proof. Let e be a left identity of N. $\therefore$ $x = ex$ for all $x \in N$. Now $e^k N = eNe \Rightarrow eN = eNe.$ Then there exists $n \in N$ such that $x = ex = ene$ and this implies $x = ne.$ Hence $xe(= ne^2) = ne = x.$ Thus 'e' is a right identity as well.

Remark 2.29. A right identity of a P_k near-ring need not be a left identity. In the example given under Remark 2.21 a, b, c are right identities but none is a left identity. This very same example together with Remark 2.23 asserts that Theorem 2.27 fails if N has only a right identity.

Proposition 2.30. Every ideal of an $S_k{'}-P_k$ near-ring is a completely semiprime ideal.

Proof. Case (i) Let $k = 1.$ Then N has a mate function and $xN = xNx$ for all $x \in N.$ From Lemma 2.11 we observe that N has a mate function f such that $xf(x) = f(x)x$ for all x in $N.$ Let I be any ideal of $N.$ If $a^2 \in I$ then $a = af(a)a = a(af(a)) = a^2 f(a) \in IN \subset I.$ i.e. $a \in I$ and the result follows.

Case (ii) Let $k > 1.$ Since N is an $S_k{'} - P_k$ near-ring Proposition 2.14 guarantees that N has a mate function say 'm'. For $a \in N,$ $a = am(a)a \in aNa(= a^k N)$ and consequently there exists $n \in N$ such that $a = a^k n.$. When $k = 2,$ $a^2 \in I \Rightarrow a = a^2 n \in IN \subset I.$ i.e. $a \in I.$ When $k > 2,$ $a^2 \in I$ $\Rightarrow$ $a = a^2 (a^{k-2} n) \in IN \subset I$ i.e. $a \in I$ and the desired result follows.

Propositions 2.31. *Any prime ideal of an $S_k - P_k{'}$ near-ring is a completely prime ideal.*

Proof. Let N be an $S_k - P_k{'}$ near-ring. Then N has a mate function, say f (Proposition 2.14). Let P be a prime ideal of N and let $ab \in P.$ $\therefore$ $Nab \subset NP \subset P$ (from Theorem 2.17 and $K(3)(a)$). Clearly then $NaNb(= Nab) \subset P$ (from $K(3)$(b)). Since Na and Nb are ideals in N and since P is prime, $Na \subset P$ or $Nb \subset P.$ Therefore $(a =)af(a)a \in P$ or $(b =)bf(b)b \in P$ and the desired result follows.

Proposition 2.32. *Any prime ideal of an $S_k - P_k{'}$ near-ring N is a maximal ideal.*

Proof. Let I be a prime ideal of N. Let J be an ideal of N such that $J \neq 1$ and that $I \subset J \subset N.$ We need only to show that $J = N.$ Let $x \in J - I.$ From proposition 2.14 we see that N has a mate function, say, 'm'. For $x \in N, x = xm(x)x = m(x)x^2$

($\because$ $E \subset C(N)$ from Theorem 2.17). Thus for all $n \in N$, $nx = nm(x)x^2$ and this implies $(n - nm(x)x)x = 0$. Since N has *IFP* (Corollary 2.16(i)) we get $z(n - nm(x)x)zx = 0$ for all $z \in N$. Consequently $N(n - nm(x)x)Nx = \{0\}$. If we let $y = n - nm(x)x$ then $NyNx = \{0\} \subset I$. Since I is prime and Nx, Ny are ideals in N (Theorem 2.17 and $K(3)(a)$) we get $Ny \subset I$ or $Nx \subset I$. If $Nx \subset I$ then $x(= xm(x)x) \in I$ which is clearly a contradiction. If $Ny \subset I$ then $Ny \subset J$ and this demands $y(= ym(y)y) \in J$. i.e. $n - nm(x)x \in J$. From Theorem 2.17 and K(3)(a) we get $nm(x)x \in J$ and therefore $n \in J$ forcing $J = N$. The desired result now follows.

Propositon 2.33. *If N is an $S_k - P_k{}'$ near-ring then for any $e \in E$, N is the direct sum of ideals Ne and $(0 : Ne)$.*

Proof. By Peirce decomposition $N = Ne \oplus (0:e)$ ($\because$ every $n \in N$ can be uniquely written as $n = ne + (-ne + n)$). Since N is an $S_k - P_k{}'$ near-ring Theorem 2.17 and K(3)(a) guarantee that Ne is an ideal of N. We shall show that $(0:e) = (0:Ne)$. If $y \in (0:e)$ then for all $n \in N$, $yne = yen$ ($\because$ $E \subset C(N)$ from Theorem 2.17 as N is an $S_k - P_k{}'$ near-ring) = 0. $\therefore$ $y \in (0:Ne)$ and hence $(0:e) \subset (0:Ne)$. Now $x \in (0:Ne) \Rightarrow xne = 0$ for all n in N. In particular, $xee = xe = 0$. Thus $xe = 0$ or $x \in (0:e)$. Therefore $(0:Ne) \subset (0:e)$. Thus $(0:e) = (0:Ne)$. Since N has *IFP* (Corollary 2.16(i)) K(5) demands that $(0:Ne)$ is an ideal. Obviously $(0:Ne) \cap Ne = \{0\}$ and the desired result follows.

Theorem 2.17 and K(3)(d) guarantee the following structure theorem for an $S_k - P_k{}'$ near-ring :

Theorem 2.34. *An $S_k - P_k{}'$ near-ring is isomorphic to a subdirect product of near-fields.*

We shall make use of the following result for obtaining a structure theorem for an $S_k{}' - P_k$ near-ring.

Theorem 2.35. *Let N be a zero-symmetric $S_k{}' - P_k$ near-ring. Then N is subdirectly irreducible if and only if N is simple.*

Proof. Since N is an $S_k{}' - P_k$ near-ring it admits mate functions (Proposition 2.14). Suppose N is subdirectly irreducible. K(7) guarantees that no non-zero idempotent element of N is a zero-divisor and this demands $Nx = N$ for all non-zero x in N. Hence N has no non-trivial N-subgroups. Since $N = N_o$ as well, all ideals are N-subgroups. Consequently N is simple.

The converse is obvious from K(6).

As an immediate consequence of Theorem 2.35 we have the following:

Theorem 2.36. *If N is a zero-symmetric* $S_k' - P_k$ *near-ring then N is isomorphic to a subdirect product of simple near-rings.*

Proof. From Theorem 2.5, *N* is isomorphic to a subdirect product of subdirectly irreducible P_k near-rings, N_i's say. Since *N* has a mate function (Proposition 2.14) we see that each N_i, being a homomorphic image of *N*, has a mate function. Theorem 2.35 takes care of the rest of the proof.

Corollary 2.37. *If N is a zero-symmetric* $S_k' - P_k$ *near-ring and if every homomorphic image of N has at least one non-zero distributive element then N is isomorphic to a subdirect product of near-fields.*

Proof. This result is an immediate consequence of 8.3 of Pilz[4] and Theorem 2.36.

Corollary 2.38. *Let N be a zero-symmetric* $S_k' - P_k$ *near-ring with at least one non-zero distributive element. Then N is subdirectly irreducible if and only if N is a near-field.*

Proof. The desired result follows from 8.3 of Pilz[4] and Theorem 2.35.

Corollary 2.39. *If N is a zero-symmetric* $S_k' - P_k$ *near-ring with* $E \subset N_d$ *then N is isomorphic to a subdirect product of near fields.*

Proof. Corollary 2.25 and Theorem 2.34 guarantee the desired result.

Theorem 2.40. *If a* P_k *near-ring N fulfills the left cancellation law then N is a near-integral domain.*

Proof. Let *S* be any subsemigroup of *N* and let $n \in N$. Since $s^k N = sNs$ for $s \in S$ there exists $n' \in N$ such that $s^k n = sn's$. i.e. $s_1 n = n_1 s$ where $s_1 = s^k \in S$ and $n_1 = sn' \in N$. i.e. *S* satisfies the left Ore condition. 9.60 of Pilz[4] guarantees that *N* is a near-integral domain.

Corollary 2.41. *If N, satisfying the conditions of Theorem 2.40, has DCCN then N is a near-field.*

Proof. This is an immediate consequence of 9.62(c) of Pilz [4] and Theorem 2.40.

3. $P_k(r,m)$ and $P_k'(r,m)$ Near-Rings

In this section we shall introduce $P_k(r,m)$ and $P_k'(r,m)$ near-rings by way of generalizing the concepts of P_k and P_k' nearrings.

Definition 3.1. *A near-ring N is called a $P_k(r,m)$ ($P_k'(r,m)$) near-ring if there exist positive integers k, r, m such that for all x in N, $x^k N = x^r N x^m$ ($Nx^k = x^r N x^m$).*

Remark 3.2. Obviously a $P_k'(r,m)$ near-ring is zero-symmetric. A $P_k(1,1)$ ($P_k'(1,1)$) near-ring is nothing but a P_k (P_k') near-ring.

Examples 3.3. Let k, r, m be any three positive integers.

(i) The direct product of any two near-fields is a $P_k(r,m)$ as well as a $P_k'(r,m)$ near-ring.
(ii) A constant near-ring is a $P_k(r,m)$ near-ring. It is not a $P_k'(r,m)$ near-ring.
(iii) Near-ring of example 2.3(b) is a $P_k(r,m)$ as well as a $P_k'(r,m)$ near-ring.
(iv) The near-ring given in example 2.3(c) is a $P_k'(r,m)$ but not a $P_k(r,m)$ near-ring for $k > 1$. It is not $P_1(r,m)$ nor $P_1'(r,m)$.

Lemma 3.4. *Let f be a mate function for N. If $E \subset C(N)$ then $(xf(x) =) (xf(x))^r = x^r (f(x))^r$ for all x in N and for all positive integers r.*

Proof. As $E \subset C(N)$ we have, $(xf(x) =)(xf(x))^2 = xf(x)(xf(x)) = x(xf(x))f(x) = x^2(f(x))^2$. Continuing in the same vein we get $(xf(x) =)(xf(x))^r = x^r(f(x))^r$ for all positive integers *r*.

Theorem 3.5. *Let N admit a mate function 'f'. Then N is a $P_k'(1,1)$ near-ring for a fixed k if and only if N is a $P_t'(r,m)$ and a $P_t(r,m)$ near-ring for all positive integers t, r, m.*

Proof. Suppose *N* is a $P_k'(1,1)$ near-ring for a fixed positive integer *k*. Theorem 2.17 demands that $E \subset C(N)$. Let t, r, m be any three positive integers. For $x \subset N$,

$$x^r N x^m \subset xNx = Nx^k \ (\because N \text{ is } P_k'(1,1)). \quad \therefore x^r N x^m \subset Nx^k. \qquad 1$$

Now let $z \in Nx^k (= xNx)$. Then there exists $y \in N$ such that $z = xyx$ $= (xf(x))(xyx(f(x)x) = (xf(x))^r (xyx(f(x)x)^m$ as $xf(x)$, $f(x)x \in E$. Since $E \subset C(N)$

we can make use of Lemma 3.4 and get $z = x^r((f(x))^r xyx(f(x))^m)x^m \in x^r Nx^m$. Thus $Nx^k \subset x^r Nx^m$ 2

Hence $x^r Nx^m = Nx^k (= xNx)$. From Corollary 2.22 we observe that N is a $P_t(1,1)$ as well as a $P_t'(1,1)$ near-ring for all positive integers t. Hence $x^t N = Nx^t = xNx = Nx^k = x^r Nx^m$ for all x in N. i.e. N is a $P_t(r,m)$ as well as a $P_t'(r,m)$ near-ring for all positive integers t, r, m.

Converse is obvious.

Theorem 3.6. *Let N be a near-ring with a mate function. Then N is a zero-symmetric* $P_k(1,1)$ *near-ring with* $E \subset N_d$ *if and only if N is a* $P_t(r,m)$ *as well as a* $P_t'(r,m)$ *near-ring for all positive integers* t, r, m.

Proof. N is zero-symmetric, $P_k(1,1)$ with $E \subset N_d \Leftrightarrow N$ is a $P_k'(1,1)$ near-ring for any positive integer k(Corollary 2.25) $\Leftrightarrow N$ is a $P_t(r,m)$ as well as a $P_t'(r,m)$ near-ring for all positive integers t, r, m (Theorem 3.5).

Collecting the pieces of results from Theorem 2.17, Corollaries 2.22, 2.25, Proposition 2.26 and Theorems 3.5 and 3.6 we obtain:

Theorem 3.7. *The following statements are equivalent in a near-ring N with mate functions.*

(1) *N is a* $P_k'(= P_k'(1,1))$ *near-ring for a fixed positive integer k.*

(2) *N is a zero-symmetric* $P_k(= P_k(1,1))$ *near-ring for a fixed positive integer k with* $E \subseteq N_d$.

(3) *N is a* $P(r,m)$ *near-ring for all positive integers r, m.*

(4) *N is a* $P_t'(u,v)$ *near-ring for all positive integer t, u, v.*

(5) *N is a zero-symmetric* $P_t(u,v)$ *near-ring for all positive integers t, u, v with* $E \subseteq Nd$.

We conclude our discussion with the following:

Remark 3.8. Theorem 3.7 - when read with K(3)(d) - guarantees that if N admits mate functions and is either a $P_t(u,v)$ near-ring with $E \subset N_d$ or simply a $P_t'(u,v)$ near-ring - where t, u, v are any three positive integers - then N is isomorphic to a subdirect product of near-fields.

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