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Exploiting Symmetry in Electromagnetic Imaging Problems by using Group Representation Theory

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Abstract. Electromagnetic Imaging problems involve development of an algorithm for estimation of complex permittivities of a *N*-cell body from the knowledge of scattered electric fields at *N* receiver locations, incident electric fields at *N*-cell centroid locations, cell sizes, cell locations and receiver locations. In this algorithm, it is necessary to invert a scattering matrix, which relates polarization current inside the body to scattered electric fields outside the body. To allow for a large number of cells, it is necessary to reduce matrix formation and inversion time. This is achieved by block diagonalization of the scattering matrix using standard point symmetry groups D_{2h}, C_{2v} and C_{1h}. For three planes of symmetry, two planes of symmetry and one plane of symmetry, it is necessary to use groups D_{2h}, C_{2v} and C_{1h}, respectively. Group multiplication and character tables for these groups will be discussed. Due to $n\,(n = 1$ or 2 or $3)$ planes of symmetry which depend on the configuration of the imaging problem regarding cell locations, cell sizes, receiver locations and transmitter location, it is possible to block diagonalize the scattering matrix. In this paper, block diagonalized matrix is derived for one plane of symmetry which can be generalized to two or three planes of symmetry. Because of block diagonalization, it is observed that it is necessary to consider 2^n matrices of $3N/2^n \times 3N/2^n$ instead of one $3N \times 3N$ scattering matrix. Hence, it is observed that the matrix formation time and storage requirements of the scattering matrix are reduced by a factor 2^n and inversion time is reduced by a factor of 2^{2n}.

1. Introduction

Recently, the applications of electromagnetic imaging in biomedical areas have received much attention. This is due to the relative safety of non ionizing radiation as compared to *X*-rays or radioactive isotopes. Other applications of electromagnetic imaging include nondestructive testing and geophysical explorations.

Electromagnetic Imaging problems [1] involve development of an algorithm for estimation of complex permittivities of a *N*-cell body from the knowledge of scattered electric fields at *N* receiver locations, incident electric fields at *N*-cell centroid locations, cell sizes, cell locations and receiver locations. The estimation of complex permittivities (ε*s) requires the inversion of the scattering matrix $\vec{B}_1$. The scattering matrix relates polarization current $\vec{j}^{\,p}$ inside the body to scattered electric fields $\vec{E}^{s}$ outside the body, by the relationship

$$\vec{E}^{s} = \vec{\vec{B}}_1 \cdot \vec{j}^{\,p} \tag{1}$$

where the Green's function matrix $\vec{\vec{B}}_1$ is a $3N \times 3N$ matrix corresponding to the three components of the electric field for each of the N locations. The time required to compute the inverse of $\vec{\vec{B}}_1$ is proportional to $(3N)^3$, while the storage requirements and matrix formation time are proportional to $(3N)^2$. The reduction in the matrix formation and inversion time can be achieved by block diagonalization [2] of the scattering matrix $\vec{\vec{B}}_1$ using standard point symmetry groups.

Vertical planes of symmetry for solution of forward scattering problems have been used [3-5], whereby the scattering matrix is required to have physical and electrical (with respect to ε*s) symmetry about the planes of symmetry. In this paper, we will derive unitary transformation matrices for one, two, and three planes of symmetry for a parallelepiped shaped body, which is surrounded by symmetrical locations of receiving dipoles, using group representation theory. The symmetries between polarization currents and scattered electric fields in the presence of planes of symmetry will be discussed. In addition, block diagonalized scattering matrix is derived for one plane of symmetry.

2. Mathematical formulation

Let us consider an arbitrarily shaped biological body, which is made physically symmetric by extending it with incorporation of the surrounding medium. The ε*s of the peripheral cells are determined by volume averaging of the ε*s of the biological body and surrounding medium. Use of saline or water as the surrounding medium for biological bodies justifies volume averaging of peripheral cells and extension for a physically symmetric body. It is preferable to have the extended body in the shape of a parallelepiped, which will have one, two or three planes of symmetry depending on cell sizes, cell locations, and receiving dipole locations.

For block diagonalization of scattering matrix in equation (1), we define a unitary transformation matrix $\vec{\vec{V}}$ with either 1 or –1 or 0 as its elements, such as

$$\vec{\vec{B}}_{1b} \cdot \vec{j}_b^{\,p} = \vec{E}_b^{S} \tag{2}$$

$$\vec{\vec{B}}_{1b} = \vec{\vec{V}} \cdot \vec{\vec{B}}_1 \cdot \vec{\vec{V}}^{T} \tag{3}$$

$$\vec{j}_b^{\,p} = \vec{\vec{V}} \cdot \vec{j}^{\,p} \tag{4}$$

$$\vec{\vec{V}} \cdot \vec{\vec{V}}^{T} = \vec{\vec{I}} \tag{5}$$

$$\vec{E}_b^{S} = \vec{\vec{V}} \cdot \vec{E}^{S} \tag{6}$$

where $\vec{\vec{B}}_{1b}$ is the block diagonalized matrix, $\vec{\vec{V}}^T$ is the transpose of $\vec{\vec{V}}$ and $\vec{\vec{I}}$ is an identity matrix.

3. Symmetries in polarization currents and scattered fields

We will show symmetries between polarization currents and scattered electric fields for one plane of symmetry using even and odd mode excitation of polarization currents [1,2]. Besides that, the invariance Green's function matrices for even and odd modes will be established which form the basis for block diagonalization of the scattering matrix $\vec{\vec{B}}_1$.

Consider a cell ℓ with cell centroid at $\vec{r}_\ell$ and a field point $\vec{r}_m$ in the positive Y region of the three-dimensional space. Assume a cell ℓ' with cell centroid at $\vec{r}_\ell^{\,'}$ and a field point at $\vec{r}_m^{\,'}$ are the mirror images of cell ℓ and field point at $\vec{r}_m$, respectively, with respect to xz plane. The cell sizes of cells ℓ and ℓ' are assumed to be equal. The Green's function $\vec{\vec{G}}(\vec{a},\vec{b})$ denotes a matrix of order 3 which relates three orthogonal components (x,y, and z) of scattered electric fields at a point $\vec{a}$ to three orthogonal components of polarization currents of a cell with cell centroid at $\vec{b}$.

We define matrices $\vec{\vec{T}}$, $\vec{\vec{G}}^e$ and $\vec{\vec{G}}^o$ as follows:

$$\vec{\vec{T}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\vec{\vec{G}}^e = \vec{\vec{G}}(\vec{r}_m, \vec{r}_\ell) + \vec{\vec{G}}(\vec{r}_m^{\,'}, \vec{r}_\ell^{\,'}) + \vec{\vec{G}}(\vec{r}_m^{\,'}, \vec{r}_\ell) + \vec{\vec{G}}(\vec{r}_m, \vec{r}_\ell^{\,'})$$

$$\vec{\vec{G}}^o = \vec{\vec{G}}(\vec{r}_m, \vec{r}_\ell) - \vec{\vec{G}}(\vec{r}_m^{\,'}, \vec{r}_\ell^{\,'}) + \vec{\vec{G}}(\vec{r}_m^{\,'}, \vec{r}_\ell) - \vec{\vec{G}}(\vec{r}_m, \vec{r}_\ell^{\,'})$$

Since the locations of cells and field points are symmetrical, it is observed that

$$\vec{\vec{G}}(\vec{r}_m^{\,'}, \vec{r}_\ell^{\,'}) = \vec{\vec{T}} \cdot \vec{\vec{G}}(\vec{r}_m, \vec{r}_\ell) \cdot \vec{\vec{T}}$$

$$\vec{\vec{G}}(\vec{r}_m, \vec{r}_\ell^{\,'}) = \vec{\vec{T}} \cdot \vec{\vec{G}}(\vec{r}_m^{\,'}, \vec{r}_\ell) \cdot \vec{\vec{T}}$$

and it can be shown that

$$\vec{\vec{G}}^e \cdot \vec{\vec{T}} = \vec{\vec{T}} \cdot \vec{\vec{G}}^e \tag{7}$$

$$\vec{\vec{G}}^o \cdot \vec{\vec{T}} = -\vec{\vec{T}} \cdot \vec{\vec{G}}^o \tag{8}$$

Equations (7) and (8) show invariance of $\vec{\vec{G}}^e$ and $\vec{\vec{G}}^o$ under rotation described by $\vec{\vec{T}}$. This type of invariance is the basis for block diagonalization of the scattering matrix [6]. If the polarization currents in the cells ℓ and ℓ' are equal and denoted by $\vec{j}^{\,e}$, and $\vec{E}^e$ represents the sum of the scattered electric fields at field points $\vec{r}_m$ and $\vec{r}_m^{\,'}$, then

$$\vec{\vec{G}}^e \cdot \vec{j}^{\,e} = \vec{E}^e \tag{9}$$

and

$$\vec{\vec{G}}^e \cdot (\vec{\vec{T}} \cdot \vec{j}^{\,e}) = \vec{\vec{T}} \cdot \vec{E}^e \tag{10}$$

Equations (9) and (10) show that for an even mode excitation (i.e., equal currents at cells ℓ and ℓ'), the relationship between the polarization currents and the scattered fields is given by

$$(j_x^e,\ j_y^e,\ j_z^e) \xrightarrow{\vec{\vec{G}}^e} (E_x^e,\ E_y^e,\ E_z^e)$$

$$(j_x^e,\ -j_y^e, j_z^e) \xrightarrow{\vec{\vec{G}}^e} (E_x^e,\ -E_y^e, E_z^e)$$

If the polarization currents in the cells ℓ and ℓ' are $\vec{j}^{\,o}$ and $-\vec{j}^{\,o}$, respectively, and $\vec{E}^o$ represents the sum of the scattered electric fields at field points $\vec{r}_m$ and $\vec{r}_m^{\,'}$,

then

$$\vec{\vec{G}}^o \cdot \vec{j}^{\,o} = \vec{E}^o \tag{11}$$

and

$$\vec{\vec{G}}^o \cdot (\vec{\vec{T}} \cdot \vec{j}^{\,o}) = -\vec{\vec{T}} \cdot \vec{E}^o \tag{12}$$

Equations (11) and (12) show that for an odd mode excitation (i.e., equal and opposite currents at cells ℓ and ℓ'), the relationship between the polarization currents and the scattered fields is given by

$$(j_x^o,\ j_y^o,\ j_z^o) \xrightarrow{\vec{\vec{G}}^o} (E_x^o,\ E_y^o,\ E_z^o)$$

$$(j_x^o,\ -j_y^o, j_z^o) \xrightarrow{\vec{\vec{G}}^o} (-E_x^o,\ E_y^o, -E_z^o)$$

For other planes of symmetry in the three dimensional space, equations similar to equation (7) to (12) can be derived by using various combinations of even and odd modes of excitations.

4. Unitary transformation matrices

We will derive the unitary transformation matrices, which block-diagonalize the scattering matrix for one, two and three planes of symmetry in three-dimensional space using group representation theory. Groups of transformation on points in three-dimensional spaces will be considered. A set *G* of transformations is a group if

- The identity transformation I is in G;
- The composition ST of two transformations S and T in G is again a transformation in G;
- If S, T and U are transformations in G then $(ST)U = S(TU)$, and for every T in G, there is a U in G such that $TU = UT = 1$.

We define a large group G generated by operators I, R_1, R_2 and R_3 where

$$R_1(x, y, z) = (-x, y, z)$$
$$R_2(x, y, z) = (x, -y, z)$$
$$R_3(x, y, z) = (x, y, -z)$$

The multiplication tables for the group are given in Table 1. This table shows that each row or column contains each element once and only once as stated in the Rearrangement Theorem [6].

Table 1. Group multiplication table

	I	R_1	R_2	R_3	R_1R_2	R_1R_3	R_2R_3	$R_1R_2R_3$
I	I	R_1	R_2	R_3	R_1R_2	R_1R_3	R_2R_3	$R_1R_2R_3$
R_1	R_1	I	R_1R_2	R_1R_3	R_2	R_3	$R_1R_2R_3$	R_2R_3
R_2	R_2	R_1R_2	I	R_2R_3	R_1	$R_1R_2R_3$	R_3	R_1R_3
R_3	R_3	R_1R_3	R_2R_3	I	$R_1R_2R_3$	$R1$	R_2	R_1R_2
R_1R_2	R_1R_2	R_2	R_1	$R_1R_2R_3$	I	R_2R_3	R_1R_3	R_3
R_1R_3	$R_1\mathrm{R}_3$	R_3	$R_1R_2R_3$	R_1	R_2R_3	I	R_1R_2	R_2
R_2R_3	R_2R_3	$R_1R_2R_3$	R_3	R_2	R_1R_3	R_1R_2	I	R_1
$R_1R_2R_3$	$R_1R_2R_3$	R_2R_3	R_1R_3	R_1R_2	R_3	R_2	R_1	I

We define three groups G_1, G_2 and G_3 such that

$$G_1 = \{ I,\ R_1, R_3,\ R_1R_3,\ R_2,\ R_1R_2,\ R_2R_3,\ R_1R_2R_3\}$$
$$G_2 = \{I, R_3, R_2, R_2R_3\}$$
$$G_3 = \{I, R_2\}$$

Group G_1 corresponds to three planes of symmetry $(xy, yz$ and xz planes$)$, G_2 corresponds to two planes of symmetry $(xy$ and xz planes$)$ and G_3 corresponds to one plane of symmetry (xz plane). The groups G_1, G_2 and G_3 are similar to the standard point symmetry groups D_{2h}, C_{2v} and C_{1h}, respectively. The character tables for the groups G_1, G_2 and G_3 are given in Table 2, 3 and 4 respectively.

Table 2. Character table for group G_1

G_1	I	R_1	R_3	R_1R_3	R_2	R_1R_2	R_2R_3	$R_1R_2R_3$
A_{1G}	1	1	1	1	1	1	1	1
B_{1G}	1	1	–1	–1	1	1	–1	–1
B_{2G}	1	–1	1	–1	1	–1	1	–1
B_{3G}	1	–1	–1	1	1	–1	–1	1
A_{1U}	1	1	1	1	–1	–1	–1	–1
B_{1U}	1	1	–1	–1	1	–1	1	1
B_{2U}	1	–1	1	–1	–1	1	–1	1
B_{3U}	1	–1	–1	1	-1	1	1	-1

Table 3. Character table for group G_2

G_2	I	R_3	R_2	R_2R_3
A_1	1	1	1	1
A_2	1	–1	–1	1
B_1	1	1	–1	–1
B_2	1	–1	1	–1

Table 4. Character table for group G_3

G_3	I	R_2
A'	1	1
A''	1	–1

A character table gives different unique combinations of the various symmetry operations, which are specified by the elements of the group. These unique combinations of symmetry operations are called irreducible representations. A set of irreducible representations for a group is defined such that any combination of the various symmetry operations is equivalent to one or direct sum of two or more irreducible representations. The sets of irreducible representations of the groups G_1, G_2 and G_3 are $\{A_{1G}, B_{1G}, B_{2G}, B_{3G}, A_{1U}, B_{1U}, B_{2U}, B_{3U}\}$, $\{A_1, A_2, B_1, B_2\}$ and $\{A', A''\}$ respectively.

In order to derive unitary transformation matrices for various planes of symmetry, we have to find matrix representations of different elements in the groups G_1, G_2 and G_3. We represent a $N \times N$ diagonal matrix $\ddot{C}$ by $\ddot{C} = \text{diag}(a_1, a_2, \cdots, a_N)$. Similarly, a $N' \times N'$ block diagonalized matrix with N'/M blocks of $M \times M$ matrices is denoted by

$$\overleftrightarrow{C} = \begin{bmatrix} \overleftrightarrow{C}_1 & & & \\ & \overleftrightarrow{C}_2 & & \\ & & \cdot & \\ & & & \overleftrightarrow{C}_K \end{bmatrix} = \text{Block diag.}(\overleftrightarrow{C}_1,\ \overleftrightarrow{C}_2, \cdots,\ \overleftrightarrow{C}_k)\,, \quad \text{where} \quad K = N'/M$$

Let

$$\overleftrightarrow{I}_o = \text{diag.}\ (1,\ 1,\ 1),\ \ \overleftrightarrow{I}_1 = \text{diag.}\ (-1,\ 1,\ 1), \overleftrightarrow{I}_2 = \text{diag.}\ (1, -1,\ 1),\ \text{ and }\ \overleftrightarrow{I}_3 = \text{diag.}\ (1,\ 1, -1).$$

For a body with N cells and n planes of symmetry $(n = 1, 2, 3)$, we define four matrices with $N' = 3N/2^n$, $M = 3$ and $K = N/2^n$ as follows,

$$\overleftrightarrow{I}_S = \text{Block diag.}\ (\overleftrightarrow{I}_o,\ \overleftrightarrow{I}_o, \cdots, \overleftrightarrow{I}_o)\,, \qquad \overleftrightarrow{I}_{S1} = \text{Block diag.}\ (\overleftrightarrow{I}_1,\ \overleftrightarrow{I}_1, \cdots, \overleftrightarrow{I}_1)$$
$$\overleftrightarrow{I}_{S2} = \text{Block diag.}\ (\overleftrightarrow{I}_2,\ \overleftrightarrow{I}_2, \cdots, \overleftrightarrow{I}_2)\,, \qquad \overleftrightarrow{I}_{S3} = \text{Block diag.}\ (\overleftrightarrow{I}_3,\ \overleftrightarrow{I}_3, \cdots, \overleftrightarrow{I}_3)$$

Now, it is observed that matrix representations $\overleftrightarrow{I}_S$, $\overleftrightarrow{I}_{S1}$, $\overleftrightarrow{I}_{S2}$, $\overleftrightarrow{I}_{S3}$, $-\overleftrightarrow{I}_{S2}$, $-\overleftrightarrow{I}_{S1}$, $-\overleftrightarrow{I}_{S3}$, and $-\overleftrightarrow{I}_S$ correspond to group elements I, R_1, R_2, R_3, R_1R_3, R_2R_3, R_1R_2, $R_1R_2R_3$, respectively. Substituting these matrix representations for various group elements in the character tables given by tables 2, 3 and 4, we get unitary transformation matrices for three planes, two planes and one plane of symmetry; $\vec{V}_1$, $\vec{V}_2$, and $\vec{V}_3$ where

$$\vec{V}_1 = \frac{1}{2\sqrt{2}} \begin{bmatrix}
\overleftrightarrow{I}_S & \overleftrightarrow{I}_{S1} & \overleftrightarrow{I}_{S3} & -\overleftrightarrow{I}_{S2} & \overleftrightarrow{I}_{S2} & \overleftrightarrow{I}_{S3} & -\overleftrightarrow{I}_{S1} & -\overleftrightarrow{I}_S \\
\overleftrightarrow{I}_S & \overleftrightarrow{I}_{S1} & -\overleftrightarrow{I}_{S3} & \overleftrightarrow{I}_{S2} & \overleftrightarrow{I}_{S2} & \overleftrightarrow{I}_{S3} & \overleftrightarrow{I}_{S1} & \overleftrightarrow{I}_S \\
\overleftrightarrow{I}_S & -\overleftrightarrow{I}_{S1} & \overleftrightarrow{I}_{S3} & \overleftrightarrow{I}_{S2} & \overleftrightarrow{I}_{S2} & \overleftrightarrow{I}_{S3} & -\overleftrightarrow{I}_{S1} & \overleftrightarrow{I}_S \\
\overleftrightarrow{I}_S & -\overleftrightarrow{I}_{S1} & -\overleftrightarrow{I}_{S3} & -\overleftrightarrow{I}_{S2} & \overleftrightarrow{I}_{S2} & \overleftrightarrow{I}_{S3} & \overleftrightarrow{I}_{S1} & -\overleftrightarrow{I}_S \\
\overleftrightarrow{I}_S & \overleftrightarrow{I}_{S1} & \overleftrightarrow{I}_{S3} & -\overleftrightarrow{I}_{S2} & -\overleftrightarrow{I}_{S2} & \overleftrightarrow{I}_{S3} & \overleftrightarrow{I}_{S1} & \overleftrightarrow{I}_S \\
\overleftrightarrow{I}_S & \overleftrightarrow{I}_{S1} & -\overleftrightarrow{I}_{S3} & \overleftrightarrow{I}_{S2} & -\overleftrightarrow{I}_{S2} & \overleftrightarrow{I}_{S3} & -\overleftrightarrow{I}_{S1} & -\overleftrightarrow{I}_S \\
\overleftrightarrow{I}_S & -\overleftrightarrow{I}_{S1} & \overleftrightarrow{I}_{S3} & \overleftrightarrow{I}_{S2} & -\overleftrightarrow{I}_{S2} & \overleftrightarrow{I}_{S3} & \overleftrightarrow{I}_{S1} & -\overleftrightarrow{I}_S \\
\overleftrightarrow{I}_S & -\overleftrightarrow{I}_{S1} & -\overleftrightarrow{I}_{S3} & -\overleftrightarrow{I}_{S2} & -\overleftrightarrow{I}_{S2} & -\overleftrightarrow{I}_{S3} & -\overleftrightarrow{I}_{S1} & \overleftrightarrow{I}_S
\end{bmatrix} \tag{13}$$

$$\vec{V}_2 = \frac{1}{2} \begin{bmatrix}
\overleftrightarrow{I}_S & \overleftrightarrow{I}_{S3} & \overleftrightarrow{I}_{S2} & -\overleftrightarrow{I}_{S1} \\
\overleftrightarrow{I}_S & -\overleftrightarrow{I}_{S3} & -\overleftrightarrow{I}_{S2} & -\overleftrightarrow{I}_{S1} \\
\overleftrightarrow{I}_S & \overleftrightarrow{I}_{S3} & -\overleftrightarrow{I}_{S2} & \overleftrightarrow{I}_{S1} \\
\overleftrightarrow{I}_S & -\overleftrightarrow{I}_{S3} & \overleftrightarrow{I}_{S2} & \overleftrightarrow{I}_{S1}
\end{bmatrix} \tag{14}$$

$$\vec{\vec{V}}_3 = \frac{1}{\sqrt{2}}\begin{bmatrix} \vec{\vec{I}}_S & \vec{\vec{I}}_{S2} \\ \vec{\vec{I}}_S & -\vec{\vec{I}}_{S2} \end{bmatrix} \tag{15}$$

All unitary transformation matrices are normalized so as to satisfy $\vec{\vec{V}}\,\vec{\vec{V}}^T = \vec{\vec{I}}$.

5. Block diagonalization of scattering matrix

We will derive the block diagonalized matrix for one plane of symmetry (*xz* plane) using the unitary transformation matrix obtained in section 4. Consider a body with *N* cells, which is surrounded by *N* receiving dipole location. The body and the receiving dipole locations are assumed to be symmetrical with respect to the *xz* plane, which passes through the origin of the three-dimensional *xyz* space. The positive *Y* region is denoted by region 1, while the negative *Y* is denoted by region 2. If $\vec{\vec{B}}_{ij}$ represents a $3N/2 \times 3N/2$ matrix which relates the polarization currents in the cells of region *j* to scattered electric fields at receiving dipole locations in the region *i*,

$$\vec{\vec{B}}_1 = \begin{bmatrix} \vec{\vec{B}}_{11} & \vec{\vec{B}}_{12} \\ \vec{\vec{B}}_{21} & \vec{\vec{B}}_{22} \end{bmatrix} \tag{16}$$

In region 1 and 2, if $\vec{j}_1$ and $\vec{j}_2$ are the polarization currents in the cells and $\vec{E}_1^S$ and $\vec{E}_2^S$ are the scattered electric fields at receiving dipole locations, then

$$\vec{j}^{\,p} = \begin{bmatrix} \vec{j}_1 \\ \vec{j}_2 \end{bmatrix} \qquad \text{and} \qquad \vec{E}^S = \begin{bmatrix} \vec{E}_1^S \\ \vec{E}_2^S \end{bmatrix} \tag{17}$$

Substituting into equation (1), we have

$$\begin{bmatrix} \vec{E}_1^S \\ \vec{E}_2^S \end{bmatrix} = \begin{bmatrix} \vec{\vec{B}}_{11} & \vec{\vec{B}}_{12} \\ \vec{\vec{B}}_{21} & \vec{\vec{B}}_{22} \end{bmatrix} \begin{bmatrix} \vec{j}_1 \\ \vec{j}_2 \end{bmatrix} \tag{18}$$

Using the symmetry in the Green's functions, it can be shown (see Appendix) that

$$\begin{aligned} \vec{\vec{B}}_{22} &= \vec{\vec{I}}_{S2} \cdot \vec{\vec{B}}_{11} \cdot \vec{\vec{I}}_{S2} \\ \vec{\vec{B}}_{21} &= \vec{\vec{I}}_{S2} \cdot \vec{\vec{B}}_{12} \cdot \vec{\vec{I}}_{S2} \end{aligned} \tag{19}$$

Using equations (3), (15) and (16), we have

$$\begin{aligned}
\overleftrightarrow{B}_{1b} &= \overleftrightarrow{V}\cdot\overleftrightarrow{B}_1\cdot\overleftrightarrow{V}^T \\
&= \frac{1}{\sqrt{2}}\begin{bmatrix}\overleftrightarrow{I}_S & \overleftrightarrow{I}_{S2}\\ \overleftrightarrow{I}_S & -\overleftrightarrow{I}_{S2}\end{bmatrix}\begin{bmatrix}\overleftrightarrow{B}_{11} & \overleftrightarrow{B}_{12}\\ \overleftrightarrow{B}_{21} & \overleftrightarrow{B}_{22}\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}\overleftrightarrow{I}_S & \overleftrightarrow{I}_S\\ \overleftrightarrow{I}_{S2} & -\overleftrightarrow{I}_{S2}\end{bmatrix} \\
&= \frac{1}{2}\begin{bmatrix}2\left(\overleftrightarrow{B}_{11}+\overleftrightarrow{B}_{12}\cdot\overleftrightarrow{I}_{S2}\right) & \overleftrightarrow{0}\\ \overleftrightarrow{0} & 2\left(\overleftrightarrow{B}_{11}-\overleftrightarrow{B}_{12}\cdot\overleftrightarrow{I}_{S2}\right)\end{bmatrix}
\end{aligned}$$

Therefore,

$$\overleftrightarrow{B}_{1b} = \begin{bmatrix}\overleftrightarrow{B}_{11}+\overleftrightarrow{B}_{12}\cdot\overleftrightarrow{I}_{S2} & \overleftrightarrow{0}\\ \overleftrightarrow{0} & \overleftrightarrow{B}_{11}-\overleftrightarrow{B}_{12}\cdot\overleftrightarrow{I}_{S2}\end{bmatrix} \tag{20}$$

where $\overleftrightarrow{0}$ is a $3N/2\times 3N/2$ null matrix.

Using equations (4), (15) and (17), we have

$$\vec{j}_b^{\,p} = \overleftrightarrow{V}\cdot\vec{j}^{\,p}$$

$$\vec{j}_b^{\,p} = \frac{1}{\sqrt{2}}\begin{bmatrix}\overleftrightarrow{I}_S & \overleftrightarrow{I}_{S2}\\ \overleftrightarrow{I}_S & -\overleftrightarrow{I}_{S2}\end{bmatrix}\begin{bmatrix}\vec{j}_1\\ \vec{j}_2\end{bmatrix}\ \vec{j}_b^{\,p} = \frac{1}{\sqrt{2}}\begin{bmatrix}\vec{j}_1+\overleftrightarrow{I}_{S2}\,\vec{j}_2\\ \vec{j}_1-\overleftrightarrow{I}_{S2}\,\vec{j}_2\end{bmatrix} \tag{21}$$

Using equations (5), (15) and (18), we have

$$\vec{E}_b^S = \overleftrightarrow{V}\cdot\vec{E}^S$$

$$\vec{E}_b^S = \frac{1}{\sqrt{2}}\begin{bmatrix}\overleftrightarrow{I}_S & \overleftrightarrow{I}_{S2}\\ \overleftrightarrow{I}_S & -\overleftrightarrow{I}_{S2}\end{bmatrix}\begin{bmatrix}\vec{E}_1^S\\ \vec{E}_2^S\end{bmatrix}$$

$$\vec{E}_b^S = \frac{1}{\sqrt{2}}\begin{bmatrix}\vec{E}_1^S+\overleftrightarrow{I}_{S2}\vec{E}_2^S\\ \vec{E}_1^S-\overleftrightarrow{I}_{S2}\vec{E}_2^S\end{bmatrix} \tag{22}$$

After substituting equations (20) to (22) into equation (2), it is observed that we have to consider two $3N/2\times 3N/2$ matrices instead of a $3N\times 3N$ scattering matrix $\overleftrightarrow{B}_1$. Equations similar to equations (20) to (22) can be derived for other planes of symmetries by using appropriate unitary transformation matrices given in equations (13) to (15).

6. Conclusion

The group representation theory is a powerful tool for achieving reduction in matrix formation and matrix inversion time in electromagnetic imaging problems. The block diagonalized matrix $\vec{\vec{B}}_{1b}$ has 2^n blocks of $3N/2^n \times 3N/2^n$ matrices where n is the number of planes of symmetry. For the $3N \times 3N$ scattering matrix, time for inversion is proportional to $(3N)^3$ and formation time is proportional to $(3N)^2$. Thus the time for inversion of block diagonalized matrix $\vec{\vec{B}}_{1b}$ is reduced by a factor of 2^{2n} as compared with time for inversion of $\vec{\vec{B}}_1$. The matrix formation time is reduced accordingly by a factor of 2^n.

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Appendix

For evaluating the matrix elements, we need to solve the following volume integral involving Green's functions. Assume that

$$\vec{\vec{G}}_{m\ell} = k_\varepsilon^2 \int_{V_\ell} \left[\vec{\vec{I}} + \frac{\nabla\nabla}{k_\varepsilon^2} \right] \frac{e^{-jk_\varepsilon R}}{4\pi R}\, dv'$$

where $\vec{\vec{G}}_{m\ell}$ is a 3×3 matrix relating three orthogonal components of scattered electric fields at the mth receiver location to three orthogonal components of polarization current in the ℓth cell. By going through steps similar to equation 15 to 44 of [7], we have

$$\vec{\vec{G}}_{m\ell} = \vec{\vec{G}}_{m\ell}{}^{V} + \vec{\vec{G}}_{m\ell}{}^{S}$$

For $\vec{\vec{B}}_1$ matrix,

$$\vec{\vec{G}}_{m\ell}{}^{V} = \frac{e^{-jk_\varepsilon R_{m\ell}}}{k_\varepsilon R_{m\ell}} [\sin(k_\varepsilon a) - k_\varepsilon a \cos(k_\varepsilon a)] \cdot \vec{\vec{I}} \tag{A1}$$

where $R_{m\ell} = [(x_m - x_\ell)^2 + (y_m - y_\ell)^2 + (z_m - z_\ell)^2]^{1/2}$

$$a = \left(\frac{3}{4\pi}\right)^{1/3} \Delta\ell$$

$\Delta\ell$ is the cell size of the ℓth cell and $\vec{\vec{I}}$ is a 3×3 identity matrix.

$$\vec{\vec{G}}_{m\ell}{}^{S} = \sum_{i=1}^{2} (-1)^i \int_{S_n} \gamma_3 G_\phi(\vec{r}_m, \vec{r}\,')\, ds'$$

where G_ϕ in dyadic notation,

$$G_\phi(\vec{r}_m, \vec{r}\,') = \begin{bmatrix} (x_m - x')\hat{x}\,\hat{x} + (x_m - x')\hat{x}\,\hat{y} + (x_m - x')\hat{x}\,\hat{z} \\ (y_m - y')\hat{y}\,\hat{x} + (y_m - y')\hat{y}\,\hat{y} + (y_m - y')\hat{y}\,\hat{z} \\ (z_m - z')\hat{z}\,\hat{x} + (z_m - z')\hat{z}\,\hat{y} + (z_m - z')\hat{z}\,\hat{z} \end{bmatrix} \tag{A2}$$

$$\vec{r}\,' = \vec{r}_\ell + (-1)^i \frac{\Delta\ell}{2} \vec{U}_j$$

$U_j \big| j = 1, 2, 3$ unit vectors in $\hat{x}$, $\hat{y}$ and $\hat{z}$ directions, respectively.

$$\gamma_3 = \left[\frac{1 + jk_\varepsilon R'}{4\pi R'^3} \right] e^{-jk_\varepsilon R'}$$

$$R' = [(x_m - x')^2 + (y_m - y')^2 + (z_m - z')^2]^{1/2}$$

From equation (A1) and (A2), it is observed that the off-diagonal elements are determined by equation (A2) only. If there is one plane of symmetry (*xz* plane), then the off-diagonal elements will be negative for the Green's function of $\overleftrightarrow{G}(\vec{r}_m', \vec{r}_\ell')$. This is due to $\hat{y}$ becoming $-\hat{y}$ and the change in sign for the term $(y_m - y')$ in equation (A2). Thus,

$$\overleftrightarrow{G}(\vec{r}_m', \vec{r}_\ell') = \overleftrightarrow{T} \cdot \overleftrightarrow{G}(\vec{r}_m, \vec{r}_\ell) \cdot \overleftrightarrow{T}$$

$$\overleftrightarrow{G}(\vec{r}_m, \vec{r}_\ell') = \overleftrightarrow{T} \cdot \overleftrightarrow{G}(\vec{r}_m', \vec{r}_\ell) \cdot \overleftrightarrow{T}$$

Because $\overleftrightarrow{I}_{S2}$ consists of blocks of $\overleftrightarrow{T}$ matrix, it is observed that

$$\overleftrightarrow{B}_{22} = \overleftrightarrow{I}_{S2} \cdot \overleftrightarrow{B}_{11} \cdot \overleftrightarrow{I}_{S2}$$

$$\overleftrightarrow{B}_{21} = \overleftrightarrow{I}_{S2} \cdot \overleftrightarrow{B}_{12} \cdot \overleftrightarrow{I}_{S2}$$