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Some Results on Congruences on Semihypergroups

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Abstract. The purpose of this paper is to introduce the basic concepts and theorems of congruences on semihypergroups and H_v-semigroups. They are important tools for the discription of H_v-semigroups.

1. Introduction

The concept of hyperstructure was introduced in 1934 by Marty [9] and has been studied in the following decades and nowadays by many mathematicians among whom, Krasner, Prenowitz, Mittas, Corsini, Sureou, Comer, Jantosciak, Vougiouklis.

The last of these, during the fourth A.H.A. congress (1990) [14], introduced the notion of H_v-structure. The concept of H_v-structure constitute a generalization of the well-known algebraic hyperstructures (semihypergroup, hypergroup, hyperring and so on). To be more precise some axioms concerning the above hyperstructures are replaced by their corresponding weak axioms. Since then many papers concerning various H_v-structures have appeared in the literature, for example [3,4,5,6,7,8,10,11,13,15].

In this paper we give some properties of homomorphisms and congruences on semihypergroups. They are important tools in the discription of H_v-semigroups.

2. Basic facts about *Hv*-semigroups

First of all we will recall some algebraic definitions which will be used in the paper.

Definition 2.1. *A hyperstructure is a set H together with a map* $\cdot : H \times H \longrightarrow P^*(H)$, *where* $P^*(H)$ *denotes the set of all the non-empty subsets of H.*

Definition 2.2. *A hyperstructure* $(H,\cdot)$ *is called a semihypergroup if* $(x\cdot y)\cdot z = x\cdot(y\cdot z)$ *for all x, y, z in H.*

In the above definition if $x\in H$ *and* $A,B\subseteq H$ *then*

$$A\cdot B = \bigcup_{a\in A, b\in B} a.b\,,\quad A\cdot x = A\cdot\{x\},\ \text{and}\quad x\cdot B = \{x\}\cdot B\,.$$

If a semihypergroup $(H,\cdot)$ *satisfies the reproduction axiom,* $a\cdot H = H\cdot a = H$ *for all* $a\in H$*, then it is called a hypergroup.*

A non-empty subset B of the semihypergroup H is called a subsemihypergroup of H if $B\cdot B\subseteq B$ *and H is called in this case supersemihypergroup of B.*

Definition 2.3. *Let H be a non-empty set and* $\cdot : H\times H \longrightarrow P^*(H)$ *be a hyperoperation. The* $\cdot$ *in H is called weak associative if*

$$(x,y)\cdot z \cap x\cdot(y\cdot z) \neq \phi \text{ for all } x,y,z\in H\,.$$

The hyperstructure $(H,\cdot)$ *is called an* H_v*- semigroup [12] if* $\cdot$ *is weak associative.*

Let $(H,\cdot)$ *be a semihypergroup,* σ *an equivalence relation on H and* $\sigma(x)$ *the* σ*-equivalence class of the element x in H. In* H/σ *consider the hyperoperation* $\otimes$ *defined in the usual manner*

$$\sigma(x)\otimes\sigma(y) = \left\{\sigma(x)\ \middle|\ z\in\sigma(x)\cdot\sigma(y)\right\} \text{ for all } x,y\in H.$$

Then $(H/\sigma,\otimes)$ *is an* H_v*-semigroup which is not always a semihypergroup. Indeed, for all* $x,y,z\in H$ *we have*

$$(x,y)\cdot z \subseteq (\sigma(x)\cdot\sigma(y))\cdot\sigma(z)\,,$$
$$x\cdot(y\cdot z)\subseteq\sigma(x)\cdot(\sigma(y)\cdot\sigma(z))\,.$$

Thus $\cdot$ *is weak associative.*

Clearly, every semihypergroup is an H_v*-semigroup.*

Definition 2.4. *Let* H_1 *and* H_2 *be* H_v*-semigroups (semihypergroups). A function* $f: H_1 \longrightarrow H_2$ *is called an inclusion homomorphism if it satisfies the condition*

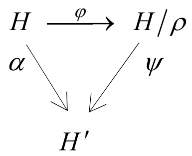
$$f(x\cdot y)\subseteq f(x)\cdot f(y) \text{ for all } x,y\in H.$$

f *is a strong homomorphism if*

$$f(x\cdot y) = f(x)\cdot f(y) \text{ for all } x,y\in H.$$

3. Congruences and homomorphisms

Let α be a strong homomorphism from a semihypergroup H into semihypergroup H'. The relation $\alpha o\, \alpha^{-1}$ is an equivalence ρ on H ($a\rho b$ if and only if $\alpha(a)=\alpha(b)$) known as the kernel of α. The natural mapping associated with ρ is $\varphi: H \longrightarrow H/\ker\alpha$, where $\varphi(a)=\rho(a)$. The mapping $\psi: H/\rho \longrightarrow H'$, where $\psi(\rho(a))=\alpha(a)$ is then the unique bijection that makes the following diagram commute:



Theorem 3.1. *Let* ϕ_1 *and* ϕ_2 *be strong homomorphisms of a semihypergroup (H_v-semigroup) H upon semihypergroups (H_v -semigroups)* H_1 *and* H_2 *respectively, such that* $\phi_1 o \phi_1^{-1} \subseteq \phi_2 o \phi_2^{-1}$. *Then, a unique strong homomorphism* θ *of* H_1 *upon* H_2 *such that* $\theta\phi_1 = \phi_2$, *exists.*

Proof. Let $a_1 \in H_1$ and let a be an element of H such that $\phi_1(a)=a_1$. Define $\theta(a_1)=\phi_2(a)$. This is single valued, because if $\phi_1(b)=a_1$ (b in H), we have $(a,b)\in \phi_1 o \phi_1^{-1} \subseteq \phi_2 o \phi_2^{-1}$, so that $\phi_2(a)=\phi_2(b)$. It is clear that $\theta\phi_1=\phi_2$ and the assertion that θ is a strong homomorphism follows from this

$$\theta(\phi_1(a)\cdot\phi_1(b) = \theta(\phi_1(a\cdot b)) = \phi_2(a\cdot b) = \phi_2(a)\cdot\phi_2(b) = \theta(\phi_1(a))\cdot\theta(\phi_1(b))$$

The uniqueness of θ is evident, indeed, if θ is to satisfy $\theta\phi_1=\phi_2$ we are compelled to define θ as above.

Definition 3.2. *Let H be a semihypergroup. By a congruence on H we mean an equivalence relation* ρ *such that* $x\rho y$ *iff for every* $a\in H$ *and for every* $u\in x\cdot a$, *there exists* $v\in y\cdot a$ *such that* $u\rho v$.

Theorem 3.3. *Let* $\alpha: H \longrightarrow H'$ *be a strong homomorphism of semihypergroups. Then* $\rho=\ker\alpha$ *is a congruence and there exists a strong homomorphism* $f: H/\rho \longrightarrow H'$ *such that* $f\varphi=\alpha$. *(Note that H/ρ is an H_v-semigroup).*

Proof. Suppose that $a\rho b$, then for every $c \in H$ we have

$$\alpha(a \cdot c) = \alpha(a) \cdot \alpha(c) = \alpha(b) \cdot \alpha(c) = \alpha(b \cdot c) .$$

Therfore for every $x \in a \cdot c$, there exists $y \in b.c$ such that $\alpha(x) = \alpha(y)$ and so $x\rho y$. Therefore ρ is a congruence on H.

Now let $\rho(a) \in H / \rho$ and define $f(\rho(a)) = \alpha(a)$. We note that if $b \in \rho(a)$ then $\alpha(a) = a(b)$, therefore f is well-defined. f is a strong homomorphism because if $\rho(a)$, $\rho(b) \in H / \rho$ then

$$\begin{aligned} f(\rho(a) \otimes \rho(b)) &= f\left(\left\{ \rho(z) \,\middle|\, z \in \rho(a) \cdot \rho(b) \right\} \right) \\ &= \left\{ \alpha(z) \,\middle|\, z \in \rho(a) \cdot \rho(b) \right\} \\ &= \alpha\left(\bigcup_{x \in \rho(a), y \in \rho(b)} x \cdot y \right) \\ &= \bigcup_{x \in \rho(a), y \in \rho(b)} \alpha(x \cdot y) \\ &= \bigcup_{x \in \rho(a), y \in \rho(b)} \alpha(x) \cdot \alpha(y) \\ &= \alpha(a) . \alpha(b) \\ &= f(\rho(a)) \cdot f(\rho(b)) . \end{aligned}$$

Therefore f is a strong homomorphism. Obviously, f is one to one.

Theorem 3.4. *If ρ_1 and ρ_2 are congruences on a semihypergroup H such that $\rho_1 \subseteq \rho_2$, then a strong homomorphism from H / ρ_1 upon H / ρ_2, exists.*

Proof. Let $\pi_1 : H \longrightarrow H / \rho_1$ and $\pi_2 : H \longrightarrow H / \rho_2$ be canonical homomorphisms. Since $\rho_1 = \pi_1 o \pi_1^{-1}$ and $\rho_2 = \pi_2 o \pi_2^{-1}$, the hypotheses of Theorem 3.1 are satisfied and we conclude that there is a strong homomorphism θ of H / ρ_1 upon H / ρ_2.

Lemma 3.5. *Let I be an ideal of a semihypergroup H. We consider the Rees relation on H as follows:*

$$a\rho b \Leftrightarrow a = b \text{ or } (a \in I \text{ and } b \in I).$$

Then ρ is a congruence on H.

Proof. Obviously, ρ is an equivalence relation. Now let $x\rho y$ and $a \in H$. If both *x,y* belong to *I*, then $x \cdot a \subseteq I$ and $y \cdot a \subseteq I$. Therefore for every $u \in x \cdot a$ and $v \in y \cdot a$ we have $u\rho v$. If $x = y$ then $x \cdot a = y \cdot a$ and so for every $u \in x \cdot a$ we have $(u,u) \in \rho$. Therefore ρ is a congruence on *H*.

Theorem 3.6. *Let A be an ideal and B a subsemihypergroup of a semihypergroup H. Then* $A \cap B$ *is an ideal of B,* $A \cap B$ *is a subsemihypergroup of H, and there is an inclusion homomorphism from* $\frac{B}{A \cap B}$ *upon* $\frac{A \cap B}{A}$.

Proof. Since

$$(A \cup B)^2 = A^2 \cup AB \cup BA \cup B^2 \subseteq P^*(A \cup B),$$

$A \cup B$ is a subsemihypergroup of *H*. It is evident that $A \cap B$ is an ideal of *B*, and *A* is an ideal of $A \cup B$. Hence the quotients $(A \cup B)/A$ and $B/(A \cap B)$ are defined. Now, if we place

$$\text{on } B: x\rho_1 y \Leftrightarrow x = y \text{ or } (x \in A \cap B \text{ and } y \in A \cap B),$$
$$\text{on } A \cup B: x\rho_2 y \Leftrightarrow x = y \text{ or } (x \in A \text{ and } y \in A).$$

For every $x \in B$ we have $\rho_1(x) \subseteq \rho_2(x)$. Now, we consider the map $f : \frac{B}{A \cap B} \longrightarrow \frac{A \cup B}{A}$ with $f(\rho_1(x)) = \rho_2(x)$. It is easy to see that *f* is well defined. We show that *f* is an inclusion homomorphism. For every $x, y \in B$ we have

$$\begin{aligned} f(\rho_1(x)\rho_2(x)) &= \{ f(\rho_1(z)) \mid z \in \rho_1(x) \cdot \rho_1(y) \} \\ &= \{ \rho_2(z) \mid z \in \rho_1(x) \cdot \rho_1(y) \} \\ &\subseteq \{ \rho_2(z) \mid z \in \rho_2(x) \cdot \rho_2(y) \} \\ &= \rho_2(x) \otimes \rho_2(y). \end{aligned}$$

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