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Some Results on the Non-Commutative Neutrix Product of Distributions and $\Gamma^{(r)}(x)$

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Abstract. The Gamma function $\Gamma(x)$ and the associated Gamma functions $\Gamma(x_{\pm})$ are defined as distributions and neutrix product $\Gamma^{(s)}(x_-) \circ x_+^r \, \ell n \, x_+$ is evaluated .

J.G. van der Corput developed the neutrix calculus having noticed that, in study of the asymptotic behaviour of integrals, functions of certain type could be neglected. This idea was also used by Fisher (see [3]) in order to define the neutrix product of the distributions. The neutrix product of distributions generalizes the definition of the product of distributions by Gelfand and Shilov and applicable to broader class of distributions.

In the following, we let *N* be the neutrix, see van der Corput [1], having domain $N' = \{1,2.., n, \cdots\}$ and range the real numbers, with negligible functions finite linear sums of the functions

$$n^{\lambda} \ell n^{r-1} n, \quad \ell n^r n : \quad \lambda > 0 \, , r = 1, 2, \cdots$$

and all functions which converge to zero in the normal sense as *n* tends to infinity.

We now let $\rho(x)$ be any infinitely differentiable function having the following properties:

(i) $\rho(x) = 0$ for $|x| \geq 1$,

(ii) $\rho(x) \geq 0$,

(iii) $\rho(x) = \rho(-x)$,

(iv) $\int_{-1}^{1} \rho(x) \, dx = 1$

Putting $\delta_n(x) = n \, \rho(nx)$ for $n = 1, 2, \cdots$, it follows that $\{\delta_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the Dirac delta - function $\delta(x)$.

Now let D be the space of infinitely differentiable functions with compact support and let D' be the space of distributions defined on D. Then if f is an arbitrary distribution in D', we define

$$f_n(x) = (f * \delta_n)(x) = \langle f(t), \delta_n(x-t) \rangle$$

for $n = 1, 2, \cdots$. It follows that $\{f_n\}$ is a regular sequence of infinitely differentiable functions converging to the distribution f.

A first extension of the product of a distribution and an infinitely differentiable function is the following, see for example [2].

Definition 1. *Let f and g be distributions in* D' *for which on the interval* (a,b), f *is the k-th derivative of a locally summable function F in* $L^p(a,b)$ *and* $g^{(k)}$ *is a locally summable function in* $L^q(a,b)$ *with* $\frac{1}{p} + \frac{1}{q} = 1$. *Then the product* $fg = gf$ *of f and g is defined on the interval* (a,b) *by*

$$fg = \sum_{i=0}^{k} \binom{k}{i} (-1)^i \left[F g^{(i)} \right]^{(k-i)}$$

The following definition for the non-commutative neutrix product of two distributions was given in [4] and generalizes Definition 1.

Definition 2. *Let f and g be distributions in* D' *and let* $g_n = g * \delta_n$. *We say that the neutrix product* $f \circ g$ *of f and g exists and is equal to the distribution h on the interval* (a,b) *if*

$$N_{n\to\infty} - \lim \langle f g_n, \phi \rangle = \langle h, \phi \rangle,$$

for all functions ϕ *in D with support contained in the interval* (a,b). *Note that if*

$$\lim_{n\to\infty} \langle f g_n, \phi \rangle = \langle h, \phi \rangle,$$

we simply say that the product $f \cdot g$ *exists and equals h.*

This definition of the neutrix product is in general non-commutative. A commutative neutrix product, denoted by $f \rightleftharpoons g$, was considered in [3].

It is obvious that if the product $f \cdot g$ exists then the neutrix product $f \circ g$ exists and $f \cdot g = f \circ g$. Further, it was proved in [4] that if the product $f g$ exists by Definition 1 then the product $f \circ g$ exists by Definition 2 and $fg = f \circ g$.

The following two theorems hold in [6] and [8] respectively.

Theorem 1. *Let f and g be distributions in* D' *and suppose that the neutrix products* $f \circ g^{(i)}$ *(or* $f^{(i)} \circ g$ *) exist on the interval* (a,b) *for* $i = 0, 1, 2, \cdots, r$. *Then the neutrix products* $f^{(k)} \circ g$ *(or* $f \circ g^{(k)}$ *) exist on the interval* (a,b) *for* $k = 1, 2, \cdots, r$ *and*

$$f^{(k)} \circ g = \sum_{i=o}^{k} \binom{k}{i} (-1)^i (f \circ g^{(i)})^{(k-i)} \tag{1}$$

or

$$f \circ g^{(k)} = \sum_{i=0}^{k} \binom{k}{i} (-1)^i (f^{(i)} \circ g)^{(k-i)} \tag{2}$$

on the interval (a,b) *for* $k = 1, 2, \cdots, r$.

Theorem 2. *The neutrix products* $\ell n\, x_+ \circ x_-^{-s}$ *and* $x_-^{-s} \circ \ell n\, x_+$ *exist and*

$$\ell n\, x_+ \circ x_-^{-s} = \frac{1}{(s-1)!}\left(c_2 - \frac{\pi^2}{12}\right)\delta^{(s-1)}(x) + \\ -\sum_{i=1}^{s-1} \frac{(-1)^i c_1}{(s-i-1)!\,i!\,i}\,\delta^{(s-1)}(x) = x_-^{-s} \circ \ell n\, x_+ \tag{3}$$

for $r = 0, 1, 2, \cdots, s-1$ *and* $s = 1, 2, \cdots$, *where*

$$c_1(\rho) = \int_0^1 \ell n\ t\ \rho(t)\ dt, \qquad c_2(\rho) = \int_0^1 \ell n^2 t\ \rho(t)\, dt$$

Now let us consider the Gamma function $\Gamma(x)$. This function is defined for $x > 0$ by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$$

and it follows that $\Gamma(x+1) = x\Gamma(x)$ for $x > 0$. $\Gamma(x)$ is then defined by

$$\Gamma(x) = x^{-1} + \Gamma(x+1)$$

for $-1 < x < 0$. Further we can express this function as follows

$$\begin{aligned}\Gamma(x) &= x^{-1} + f(x)\\ &= x^{-1} + \sum_{i=1}^{\infty} \frac{\Gamma^{(i)}(1)}{i!}\, x^{i-1},\end{aligned}$$

where x^{-1} is interpreted in the distributional sense. The distribution $\Gamma(x)$ is of course an ordinary summable function for $x > 0$.

The related distribution $\Gamma(x_+)$ by equation

$$\begin{aligned}\Gamma(x_+) &= x_+^{-1} + f(x_+)\\ &= x_+^{-1} + \sum_{i=1}^{\infty} \frac{\Gamma^{(i)}(1)}{i!}\, x_+^{x-1}, \qquad (4)\end{aligned}$$

and the distribution $\Gamma(x_-)$ by equation

$$\begin{aligned}\Gamma(x_-) &= x_-^{-1} + f(x_-)\\ &= x_-^{-1} + \sum_{i=1}^{\infty} \frac{\Gamma^{(i)}(1)}{i!}\, x_-^{i-1}, \qquad (5)\end{aligned}$$

where x_+^{-1}, x_-^{-1} are interpreted in the distributional sense, see [9]. It follows that

$$\Gamma(x) = \Gamma(x_+) - \Gamma(x_-) \qquad (6)$$

Differentiating equation (4) s times we have

$$\begin{aligned}\Gamma^{(s)}(x_+) &= (-1)^s\ s!\, x_+^{-s-1} + f^{(s)}(x_+)\\ &= (-1)^s\, s!\, x_+^{-s-1} + \sum_{i=0}^{\infty} \frac{\Gamma^{(s+i+1)}(1)}{(s+i+1)i!}\, x_+^{i} \qquad (7)\end{aligned}$$

and differentiating equation (5) s times we have

$$\begin{aligned}\Gamma^{(s)}(x_-) &= s!\,x_-^{-s-1} + f^{(s)}(x_-)\\ &= (-1)^s s!\,x_-^{-s-1} + \sum_{i=0}^{\infty} \frac{\Gamma^{(s+i+1)}(1)}{(s+i+1)i!}\, x_-^i\,. \end{aligned} \tag{8}$$

As an immediate consequence we have the following theorem.

Theorem 3. *The neutrix products* $\ell n\, x_+ \circ \Gamma^{(s)}(x_-)$ *and* $\Gamma^{(s)}(x_-) \circ \ell n\, x_+$ *exist and*

$$\ell n\, x_+ \circ \Gamma^{(s)}(x_-) = \left(c_2 + c_1\,\psi(s) - \frac{\pi^2}{12}\right)\delta^{(s)}(x)\,, \tag{9}$$

$$= \Gamma^{(s)}(x_-) \circ \ell n\, x_+ \tag{10}$$

$$= (-1)^s\, \ell n\, x_- \circ \Gamma^{(s)}(x_+) = (-1)^s\, \Gamma^{(s)}(x_+) \circ \ell n\, x_- \tag{11}$$

for $s = 0, 1, 2, \cdots$ *where*

$$\psi(s) = \begin{cases} 0 & , \quad s = 0\,, \\ \displaystyle\sum_{i=1}^{s} \frac{1}{i} & , \quad s \ge 1. \end{cases}$$

Proof. The product of the functions $\ell n\, x_+$ and x_-^i is just a straightforward product of functions in $L^2(a,b)$ for every bounded interval (a,b) and so

$$\ell n\, x_+ \circ x_-^i = \ell n\, x_+\, x^i{}_- = 0 \tag{12}$$

for $i = 0, 1, 2, \cdots$

Equations (9) and (10) follow from equations (8) and (12) on noting that

$$-\psi(s) = \sum_{i=1}^{s} \binom{s}{i} \frac{(-1)^i}{i},$$

for $s = 1, 2, \cdots$ and equation (11) follows from equations (9) and (10) on replacing x by $-x$.

The existence of neutrix product $x_+^{-1} \circ \Gamma^{(s)}(x_-)$ follows from equation (10) and by differentiating this equation we have

$$x_+^{-1} \circ \Gamma^{(s)}(x_-) = -\frac{c_1}{s+1} \delta^{(s+1)}(x). \quad (13)$$

More generally, the neutrix product $x_+^{-r} \circ \Gamma^{(s)}(x_-)$ exists and in the form of

$$x_+^{-r} \circ \Gamma^{(s)}(x_-) = L_{rs} \delta^{(s+r)}(x) . \quad (14)$$

The following two theorems were proved in [6] and [5] respectively.

Theorem 4 . *The neutrix products* $x_+^r \circ x_-^{-s}$ *and* $x_-^{-s} \circ x_+^r$ *exist and*

$$x_+^r \circ x_-^{-s} = x_+^r x_-^{-s} = 0,$$
$$x_-^{-s} \circ x_+^r = x_-^{-s} x_+^r = 0$$

for $r = s, s+1, \cdots$ *and* $s = 1, 2, \cdots$ *and*

$$x_+^r \circ x_-^{-s} = \sum_{i=r+1}^{s} \binom{s}{i} \frac{(-1)^{i-1} r!}{(s-1)!} c_1(\rho) \delta^{(s-r-1)}(x) , \quad (15)$$

$$x_-^{-s} \circ x_+^r = \sum_{i=r+1}^{s} \binom{s}{i} \frac{(-1)^{i-1} r!}{(s-1)} ! \left[c_1(\rho) + \frac{1}{2} \psi(i-r-1) \right] \delta^{(s-r-1)}(x) \quad (16)$$

for $r = 0, 1, \cdots, s-1$ *and* $s = 1, 2, \cdots$

Theorem 5. *The neutrix products* $x_+^{-r} \circ x_-^{-s}$ *and* $x_-^{-s} \circ x_+^{-r}$ *exist and*

$$x_+^{-r} \circ x_-^{-s} = \frac{(-1)^r c_1(\rho)}{(r+s-1)!} \delta^{(r+s-1)}(x), \quad (17)$$

$$x_-^{-s} \circ x_+^{-r} = \frac{(-1)^{r-1} c_1(\rho)}{(r+s-1)!} \delta^{(r+s-1)}(x) \quad (18)$$

for $r, s = 1, 2, \cdots$

We now prove the following theorem.

Theorem 6. *The neutrix products* $x_+^r \,\ell n\, x_+ \circ \Gamma^{(s)}(x_-)$ *and* $\Gamma^{(s)}(x_-)\circ x_+^r \ell n\, x_+$ *exist and*

$$(x_+^r \,\ell n x_+)\circ \Gamma^{(s)}(x_-) = (x_+^r \ell n x_+)\Gamma^{(s)}(x_-) = 0, \tag{19}$$

$$\Gamma^{(s)}(x_-)\circ (x_+^r \ell n\, x_+) = \Gamma^{(s)}(x_-)(x_+^r \ell n\, x_+) = 0 \tag{20}$$

for $r = s, s+1, s+2\cdots$ *and* $s = 1, 2, \cdots$ *and*

$$\begin{aligned}(x_+^r \,\ell n x_+) \circ \Gamma^{(s)}(x_-) &= \frac{(-1)^r s!}{(s-r)!}\left(c_2 - \frac{\pi^2}{12}\right)\delta^{(s-r)}(x) + \\ &- \sum_{i=r+1}^{s}\binom{s}{i}\frac{(-1)^i r! c_1}{(i-r)}\delta^{(s-r)}(x) + \\ &- \psi(r)\sum_{i=r+1}^{s+1}\binom{s+1}{i}(-1)^i\ r! c_1 \delta^{(s-r)}(x)\end{aligned}$$

$$\begin{aligned}\Gamma^{(s)}(x_-) \circ (x_+^r \,\ell n\, x_+) &= \frac{(-1)^r s!}{(s-r)!}\left(c_2 - \frac{\pi^2}{12}\right)\delta^{(s-r)}(x) + \\ &- \sum_{i=r+1}^{s}\binom{s}{i}\frac{(-1)^i r! c_1}{(i-r)}\delta^{(s-r)}(x) + \\ &- \psi(r)\sum_{i=r+1}^{s+1}\binom{s+1}{i}(-1)^i\ r!\left[c_1 + \frac{1}{2}\psi(i-r-1)\right]\delta^{(s-r)}(x)\end{aligned} \tag{22}$$

for $r = 0, 1, 2, \cdots s-1$ *and* $s = 0, 1, 2, \cdots$

Proof. We define the function $f(x_+, r)$ by

$$f(x_+, r) = \frac{x_+^r \ell n\, x_+ - \psi(r)\, x_+^r}{r!}$$

and it follows easily by induction that

$$f^{(i)}(x_+, r) = f^{(i)}(x_+, r-i)\ ,$$

for $i = 0, 1, \cdots, r$. In particular,

$$f^{(r)}(x_+, r) = \ell n\, x_+\ ,$$

so that

$$f^{(i)}(x_+, r) = (-1)^{i-r-1}(i-r-1)!\, x_+^{-i-r},$$

for $i = r+1, r+2, \cdots$. Now the product of the function x_+^i and $x_+^i \ell n\, x_+$ and the distribution $\Gamma^{(s)}(x_-)$ exists by Definition 1 and it is easily seen that

$$x_+^i\, \Gamma(x_-) = (x_+^i \ell n\, x_+)\Gamma(x_-) = 0, \tag{23}$$

for $i = 1, 2, \cdots, r$. Using equation (9) we have

$$f^{(r)}(x_+, r) \circ \Gamma(x_-) = \left(c_2 - \frac{\pi^2}{12}\right)\delta(x), \tag{24}$$

and using equation (10) we have

$$f^{(i)}(x_+, r) \circ \Gamma(x_-) = -\frac{c_1}{(i-r)}\delta^{(i-r)}(x),$$

for $i = r+1, r+2, \cdots$

Using equations (2) and (12) we now have

$$\begin{aligned} f(x_+, r)\, \Gamma^{(s)}(x_-) &= \sum_{i=0}^{s} \binom{s}{i} (-1)^i \left[f^{(i)}(x_+, r)\, \Gamma(x_-)\right]^{(s-i)} \\ &= \frac{1}{r!}\left[x_+^r\, \ell n\, x_+ - \psi(r) x_+^r\right]\Gamma^{(s)}(x_-) \\ &= 0 \end{aligned}$$

for $r = s, s+1, s+2, \cdots$ and $s = 1, 2, \cdots$. Equations (19) follow on using equations (23). When $r < s$ we have

$$\begin{aligned} f(x_+, r) \circ \Gamma^{(s)}(x_-) &= \sum_{i=0}^{s} \binom{s}{i} (-1)^i \left[f^{(i)}(x_+, r) \circ \Gamma(x_-)\right]^{(s-i)} \\ &= \binom{s}{r}(-1)^r \left(c_2 - \frac{\pi^2}{12}\right) \delta^{(s-r)}(x) + \\ &\quad - \sum_{i=r+1}^{s} \binom{s}{i} (-1)^{r-1} \frac{c_1 (i-r-1)!}{(i-r)} \delta^{(s-r)}(x) \end{aligned}$$

on using equations (2), (23), (24) and (25). It now follows that

$$(x_+^r \ell n\, x_+) \circ \Gamma^{(s)}(x_-) = r!\, f(x_+, r) \circ \Gamma^{(s)}(x_-) + \psi(r)\, x_+^r \circ \Gamma^{(s)}(x_-)$$

and equation (21) follows on using equation (15).
We now consider the product $\Gamma^{(s)}(x_-) \circ (x_+^r\, \ell n\, x_+)$. As above, we have

$$\Gamma^{(s)}(x_-)\, x_+^i = \Gamma^{(s)}(x_-)(x_+^i\, \ell n\, x_+) = 0, \tag{26}$$

for $i = 0, 1, \cdots, r-1$. Using equation (20) we have

$$\Gamma^{(s)}(x_-) \circ f^{(r)}(x_+, r) = \left(c_2 - \frac{\pi^2}{12}\right) \delta(x) \tag{27}$$

and using equation (18) we have

$$\Gamma^{(s)}(x_-) \circ f^{(i)}(x_+, r) = \frac{c_1}{i-r}\, \delta^{(i-r)}(x), \tag{28}$$

for $i = r+1, r+2, \cdots$. Equations (20) follow as above on using equations (1) and (26) and equations (22) follow on using equations (1), (9), (18), (27) and (28).

Corollary 1. *The neutrix products* $(x_-^r\, \ell n\, x_-) \circ \Gamma^{(s)}(x_+)$ *and* $\Gamma^{(s)}(x_+) \circ (x_-^r\, \ell n\, x_-)$ *exist and*

$$(x_-^r\, \ell n\, x_-) \circ \Gamma^{(s)}(x_+) = (x_-^r\, \ell n\, x_-)\, \Gamma^{(s)}(x_+) = 0, \tag{29}$$

$$\Gamma^{(s)}(x_+) \circ (x_-^r \ell n\, x_-) = \Gamma^{(s)}(x_+)(x_-^r\, \ell n\, x_-) = 0, \tag{30}$$

for $r = s, s+1, s+2, \cdots$ *and* $s = 1, 2, \cdots$ *and*

$$\begin{aligned}(x_-^r\, \ell n\, x_-) \circ \Gamma^{(s)}(x_+) = {} & \frac{(-1)^s s!}{(s-r)!}\left(c_2 - \frac{\pi 2}{12}\right)\delta^{(s-r)}(x) + \\ & - \sum_{i=r+1}^{s} \binom{s}{i} \frac{(-1)^{s-r+i} r!\, c_1}{(i-r)}\, \delta^{(s-r)}(x) + \\ & - \psi(r) \sum_{i=r+1}^{s+1} \binom{s+1}{i} (-1)^{s-r+i}\, r!\, c_1\, \delta^{(s-r)}(x),\end{aligned} \tag{31}$$

$$\Gamma^{(s)}(x_+) \circ (x_-^r \ \ell n \ x_-) = \frac{(-1)^r s!}{(s-r)!}\left(c_2 - \frac{\pi^2}{12}\right)\delta^{(s-r)}(x) + \\ - \sum_{i=r+1}^{s} \binom{s}{i} \frac{(-1)^{s-r+i} r! c_1}{(i-r)} \delta^{(s-r)}(x) + \\ -\psi(r) \sum_{i=r+1}^{s+1} \binom{s+1}{i} (-1)^{s-r+i} \ r! \left[c_1 + \frac{1}{2}\psi(i-r-1)\right]\delta^{(s-r)}(x) \quad (32)$$

for $r = 0, 1, 2, \cdots \ s-1$ *and* $s = 0, 1, 2, \cdots$

Proof . The results follow immediately on replacing *x* by –*x* in equations (19), (20), (21) and (22).

References

1. J.G. van der Corput, Introduction to the neutrix calculus, *J. Analyse Math.*, **7** (1959–60), 291–398.
2. B. Fisher, The product of distribution, *Quart. J. Math. Oxford* (2), **22** (1971), 291– 298.
3. B. Fisher, The neutrix distribution product $x_+^{-r}\delta^{(r-1)}(x)$, *Studia Sci. Math. Hungar*, **9** (1974), 439– 441.
4. B. Fisher, A non-commutative neutrix product of distribution, *Math. Nachr.*, **108** (1982), 117 – 127.
5. B. Fisher and A. Kilicman, The non-commutative neutrix product of x_+^{-r} and x_-^{-s}", *Math. Balkanica*, **8** (1994), 251– 258.
6. B. Fisher, E. Savas, S. Pehlivan, and E. Ozcag, Results on the non-commutative neutrix product of distributions, *Math. Balkanica*, **7** (1993), 347–356.
7. I.M. Gel'fand and G.E. Shilov, *Generalized Function*, Vol. I, Academic Press, 1964.
8. A. Kilicman and B. Fisher, On the non-commutative neutrix product $(x_+^r \ell n \ x_+) \circ x_-^{-s}$, *Georgian Math. J.*, **3** (1995), 131–140.
9. A. Kilicman and B. Fisher, The commutative neutrix product of $\Gamma^{(r)}(x)$ and $\delta^{(s)}(x)$, *Punjab J. Math.*, **31** (1998), 1–12.

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