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P(*r*, *m*) Near-Rings

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Abstract. In this paper we introduce the concept of *P*(*r, m*) near-rings where *r, m* are positive integers. Under certain conditions, we (i) obtain a complete characterization of such near-rings (ii) discuss their properties vis-a-vis subdirect irreducibility, semiprimeness etc. and (iii) obtain a structure theorem for such near-rings.

1. Introduction

A right near-ring $(N, +, \cdot)$ is an algebraic system with two binary operations such that (i) $(N,+)$ is a group-not necessarily abelian-with 0 as its identity element, (ii) $(N,\cdot)$ is a semigroup (we write xy for $x \cdot y$ for all x, y in *N*) and (iii) $(x+y)z = xz + yz$ for all x, y, z in *N*. Because of (iii) $0n = 0$ for all *n* in *N*. As we do not stipulate the left distributive law, "$n0 = 0$" need not hold good for all *n* in *N*. We say that *N* is zero-symmetric if $n0 = 0$ for all *n* in *N*. *N* is called an *S*-near-ring or an *S'*-near-ring according as $x \in Nx$ or $x \in xN$ for all $x \in N$. A subgroup *M* of *N* is called an *N*-subgroup if $NM \subset M$ and an invariant *N*-subgroup if, in addition, $MN \subset M$.

An ideal *I* of *N* is called a semiprime ideal if for all ideals *J* of *N*, $J^2 \subset I \Rightarrow J \subset I$. If $\{0\}$ is a semiprime ideal, then *N* is called a semiprime near-ring. An ideal *I* of *N* is called completely semiprime if $x \in I$ whenever $x^2 \in I$. *N* is called a strictly prime near-ring if $\{0\}$ is a strictly prime ideal i.e. if *A* and *B* are *N*-subgroups of *N* such that $AB = \{0\}$, then either $A = \{0\}$ or $B = \{0\}$.

The concept of a mate function in *N* has been introduced in [4] with a view to handle the regularity structure in a near-ring with considerable ease. A map *m* from *N* into *N* is called a mate function for *N*, if $x = xm(x)x$ for all *x* in *N*. $m(x)$ is called a mate of *x*.

Basic concepts and terms used but not defined in this paper can be found in Pilz [3]. Throughout this paper *N* stands for a near-ring – more precisely a right near-ring – with at least two elements.

As in p.249 Pilz [3], "if *N* is a near-field then either *N* is isomorphic to $M_c(Z_2)$ or *N* is zero-symmetric" (For the concept of $M_c(Z_2)$ one may refer to Example 1.4(a), p.8 and 1.15, p.12 of Pilz [3]. Obviously $M_c(Z_2)$ is a near-field of order 2 and is not zero-symmetric). All the near-fields in this paper are zero-symmetric.

1.2. Notations

(a) *E* denotes the set of all idempotents of *N*.
(b) *L* is the set of all nilpotent elements of *N*.
(c) $N_d = \{ n \in N / n(x+y) = nx + ny, \text{ for all } x, y \in N \}$ – the set of all distributive elements of *N*.
(d) $N_0 = \{ n \in N / n0 = 0 \}$ – the zero-symmetric part of *N* (It is worth noting that *N* is zero-symmetric if $N = N_0$).
(e) $C(N) = \{ n \in N / nx = xn \text{ for all } x \in N \}$.

1.3. Preliminary results

We freely make use of the following results from [4], [3] and [2] and designate them as *K*(1), *K*(2) etc. (*K* for 'known results').

K(1): If *N* has a mate function m, then for every $x \in N, xm(x), m(x)x \in E$ and $Nx = Nm(x)x$ and $xN = xm(x)N$. (Lemma 3.2 of [4]).

K(2): If $L = \{0\}$ and $N = N_0$, then (i) $xy = 0 \Rightarrow yx = 0$ (for x, y in *N*) and (ii) *N* has "Insertion of Factors Property" – IFP for short – i.e. for x, y in *N*, $xy = 0 \Rightarrow xny = 0$ for all *n* in *N*. (In this paper we write that *N* has (*, IFP) if *N* has both (i) and (ii)) (Lemma 2.3 of [4]).

K(3): A zero-symmetric near-ring *N* has IFP if and only if $(0 : S)$ is an ideal, where *S* is any non-empty subset of *N*. (9.3, p.289 of [3]).

K(4): A near-ring *N* has no non-zero nilpotent elements if and only if $x^2 = 0 \Rightarrow x = 0$ for all *x* in *N* (Prob. 14 , p.9 of [2]).

2. *P*(*r*, *m*) near-rings

We shall start with the following.

Definition 2.1. *A near-ring N is said to have the property* $P(r, m)$ *if* $x^r N = Nx^m$ *for all x in N where r and m are positive integers.*

Examples 2.2.

(a) The direct product of any two near fields is a $P(r,m)$ near-ring for all positive integers *r* and *m*.

(b) A Boolean $P(1,1)$ near-ring is a $P(r,m)$ near-ring for all positive integers *r* and *m*.

(c) Let N be an arbitrary near-ring and let I be the ideal generated by $\{ a^r n - ba^m \,/\, a, n, b$ are in *N* and *r, m* are fixed positive integers$\}$. Obviously then the factor near-ring $\overline{N} = N / I$ is a $P(r,m)$ near-ring. (This incidentally serves as a device for manufacturing $P(r,m)$ near-rings for all positive integers r, m from an arbitrary near-ring.)

(d) Let $(N,+)$ be the Klein's four group with $N = \{0, a, b, c\}$. The near-ring $(N, +, \cdot)$ where '.' is defined as (per scheme (12) p. 408 of Pilz [3] which forms part of Clay [1])

•	0	a	b	c
0	0	0	0	0
a	0	a	0	a
b	0	0	0	0
c	0	a	0	a

is a $P(r,m)$ near-ring for all positive integers *r,m*.

(e) Let $(N,+)$ be the group of integers modulo 8. We define '.' as per scheme (48) p. 413 of Pilz [3] as follows:

•	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	2	4	2	0	2	4	6
2	0	4	0	4	0	4	0	4
3	0	6	4	6	0	6	4	2
4	0	0	0	0	0	0	0	0
5	0	2	4	2	0	2	4	6
6	0	4	0	4	0	4	0	4
7	0	6	4	6	0	6	4	2

$(N, +, \cdot)$ is a $P(1,1)$ near-ring but is neither $P(1,2)$ nor $P(2,1)$. It is worth noting that this near-ring does not admit mate functions.

(f) Consider $(N, +, \cdot)$ where $(N, +)$ is the Klein's four group and '.' is defined as per scheme (4) p.408 of Pilz [3] as follows:

.	0	a	b	c
0	0	0	0	0
a	0	0	a	a
b	0	a	c	b
c	0	a	b	c

This near-ring (actually a Commutative Ring)is obviously $P(1,1)$. But it is neither $P(1,2)$ nor $P(2,1)$. Thus "$P(1,1)$" need not imply "$P(1,2)$" or "$P(2,1)$" even in the Ring Theory. (Incidentally this near-ring has the property $P(r,r)$ i.e. $x^r N = Nx^r$ for any positive integer r).

Remark 2.3. It is easy to see that a $P(r,m)$ near-ring is zero-symmetric.

Properties of $P(1, 2)$ and $P(2, 1)$ near-rings

We shall obtain a complete characterization for $P(1,2)$ and $P(2,1)$ near-rings and obtain structure theorems for such near-rings – under certain conditions.

Proposition 2.4. *Let N be a $P(1,2)$ near-ring.*
(i) If N has no non-zero nilpotent elements then N is an S-near-ring.
(ii) If N is an S'-near-ring then N has no non-zero nilpotent elements.

Proof.
(i) Since $xN = Nx^2$ and $x^2 \in xN$ for all x in N, we have $x^2 = yx^2$ for some y in N. Therefore $(x - yx)x = 0$. By $K(2)$, this implies that $x(x - yx) = 0$ and $yx(x - yx) = 0$. Consequently $(x - yx)^2 = 0$. Since $L = \{0\}, x - yx = 0$, forcing $x = yx$. Thus $x \in Nx$ i.e. N is an S-near-ring.
(ii) If N is an S'-near-ring then $x \in xN$ and since $xN = Nx^2$ we get $x = nx^2$ for some $n \in N$. Therefore $x^2 = 0 \Rightarrow x = 0$. $\therefore$ N has no non-zero nilpotent elements, from $K(4)$.

Corollary 2.5. *If N is a $P(1,2)$ near-ring without non-zero nilpotent elements, then from $K(2)$, we see that N has $(*, IFP)$.*

It is obvious that the property $P(r,m)$ is preserved by near-ring homomorphisms. Consequently, we have

Proposition 2.6. *Any homomorphic image of a $P(1,2)$ $(P(2,1))$ near-ring is $P(1,2)$ $(P(2,1))$.*

As an immediate consequence of Proposition 2.6 we have the following theorem:

Theorem 2.7. *Every* $P(1,2)$ $(P(2,1))$ *near-ring N is isomorphic to a subdirect product of subdirectly irreducible* $P(1,2)$ $(P(2,1))$ *near-rings.*

Proof. By Theorem 1.62, p.26 of Pilz [3], *N* is isomorphic to a subdirect product of subdirectly irreducible near-rings N_i's, say, and each N_i is a homomorphic image of *N* under the projection map π_i. The desired result now follows from Proposition 2.6.

We shall now discuss the behaviour of *N*-subgroups and ideals of $P(1,2)$ near-rings. To start with we have the following:

Proposition 2.8. *Let N be a* $P(1,2)$ *near-ring. Then every N-subgroup of N is invariant.*

Proof. If *A* is an *N*-subgroup of *N*, then $NA \subset A$. Whenever $an \in AN$, we have $an \in aN = Na^2 \Rightarrow an = n'a^2$ for some $n' \in N$. This forces $an \in NA \subset A \Rightarrow AN \subset A$. Hence *A* is an invariant *N*-subgroup.

Proposition 2.9. *Let N be a* $P(1,2)$ *near-ring. Then every left ideal of N is an ideal.*

Proof. Let *A* be a left ideal of *N*. Since *N* is zero-symmetric, $NA \subset A$ i.e. *A* is an *N*-subgroup of *N*. Proceeding as in Proposition 2.8 we get $AN \subset A$. Hence *A* becomes an ideal.

It is easy to observe the following:

Corollary 2.10. *Every left ideal (and therefore every ideal) of a* $P(1,2)$ *near-ring N is an invariant N-subgroup of N.*

Proposition 2.11. *If N is a* $P(1,2)$ *or a* $P(2,1)$ *near-ring then N has strong IFP.*

Proof. Let *N* be a $P(1,2)$ near-ring. In view of Proposition 9.2 Pilz [3], we need only to establish that for all ideals *I* of *N* and for all $a, b, n \in N$, $ab \in I \Rightarrow anb \in I$. Since *I* is an ideal, $IN \subset I$ and since *N* is zero-symmetric, *I* is an *N*-subgroup of *N* i.e. $NI \subset I$. Now $an \in aN = Na^2 \Rightarrow an = n'a^2$ for some $n' \in N \Rightarrow anb = (n'a^2)b$ $= (n'a)(ab) \in NI \Rightarrow anb \in I$.

When *N* is a $P(2,1)$ near-ring we observe that $nb \in Nb = b^2N \Rightarrow nb = b^2n'$ for some $n' \in N \Rightarrow anb = a(b^2n') = (ab)(bn') \in IN \Rightarrow anb \in I$.

Notation 2.12. If a $P(r,m)$ near-ring *N* is an *S* (or *S'*)-near-ring then we write that *N* is an $S - P(r,m)$ near-ring (or $S' - P(r,m)$ near-ring).

Remark 2.13. For an $S-P(2,1)$ near-ring, we see that for all x in N, $x \in Nx = x^2N$ $\Rightarrow x = x^2n$ for some $n \in N$. Hence $x^2 = 0 \Rightarrow x = 0$ and $K(4)$ demands that $L = \{0\}$.

Proposition 2.14. *In a* $P(1,2)$ $(P(2,1))$ *near-ring,* $E \subset C(N)$.

Proof. For $e \in E, eN = Ne^2\,(e^2N = Ne) \Rightarrow eN = Ne \Rightarrow Ne = eNe = eN$. It follows that $en = (ene =)ne$ for all $n \in N$ i.e. $E \subset C(N)$.

Remark 2.15. It is worth noting that we do not stipulate that N admits mate functions for the validity of the above results.

Proposition 2.16.

(i) *Let N be a* $P(1,2)$ *near-ring. Then N has a mate function if and only if N is an S'-near-ring.*

(ii) *Let N be a* $P(2,1)$ *near-ring with* $N = N_d$. *Then N has a mate function if and only if N is an S-near-ring.*

Proof.

(i) When N has a mate function `m' for all $x \in N, x = xm(x)x \in xN$ and obviously N is an S'-near-ring.
Conversely let N be an S' near-ring. $x \in xN = Nx^2$

$$x = nx^2 \text{ for some } n \in N \tag{1}$$

$\Rightarrow x^2 = xnx^2 \Rightarrow (x - xnx)x = 0.$ Using Proposition 2.4(ii) and Corollary 2.5 we get $x(x - xnx) = 0$ and $xnx(x - xnx) = 0$ and consequently $(x - xnx)^2 = 0.$ Since $L = \{0\}$ we get $x - xnx = 0$ i.e. $x = xm(x)x$ where we set $m(x) = n.$ This guarantees that $m : N \to N$ is a mate function for N.

(ii) When N has a mate function, it is obvious that N is an S-near-ring.
Conversely if N is an $S-P\,(2,1)$ near-ring with $N = N_d, x \in Nx = x^2N$

$$x = x^2n \text{ for some } n \text{ in } N \tag{2}$$

$\Rightarrow x^2 = x^2nx = x(x - xnx) = 0 \Rightarrow xnx(x - xnx) = 0.$ Rest of the proof is as in (i).

Remarks 2.17.

(a) If N has property $P(2,1)$ and a mate function 'm', then $L = \{0\}$. This is obvious from Remark 2.13 and the fact that when N has a mate function 'm' it is an S-near-ring.

(b) As in Corollary 2.5, $K(2)$ yields that a $P(2,1)$ near-ring with a mate function possesses $(*, IFP)$.

(c) A $P(2,1)$ Ring has a mate function if and only if it is an S-Ring.

Proposition 2.18. *Let N be an $S' - P(1,2)$ near-ring (or an $S - P(2,1)$ near-ring with $N = N_d$). Then N has a P_3 mate function [5].*

Proof. When N is an $S'{-}P(1,2)$ near-ring it admits a mate function 'm'. From stage (1) of Proposition 2.16 we have $x = m(x)x^2 \Rightarrow (xm(x) - m(x)x)x = 0$ $\Rightarrow (xm(x) - m(x)x)^2 = 0$ (since N has $(*, IFP)$ $\Rightarrow$ $xm(x) - m(x)x = 0 \Rightarrow xm(x) = m(x)x$ i.e. $m(x) \in C(x)$ i.e. 'm' is a P_3 mate function.

(When N is an $S - P(2,1)$ near-ring starting with stage (2) of Proposition 2.16 (ii) we can show that $m(x) \in C(x)$).

Propositions 2.16(i), 2.4 and Corollary 2.5 readily yield the following:

Proposition 2.19. *If N has property $P(1,2)$ and a mate function 'm' then $L = \{0\}$ and N has $(*, IFP)$.*

We now give a complete characterization of $P(1,2)$ and $P(2,1)$ near-rings when they admit mate functions.

Theorem 2.20. *Let N be a near-ring with a mate function 'm'. Then the following statements are equivalent:*

(i) *N is P(1,2)*

(ii) $E \subset C(N)$

(iii) *N is* $P(2,1)$.

Proof. *(ii)* $\Rightarrow$ *(i)*

For $a \in N$, $an \in aN$ for all $n \in N$ and since $E \subset C(N)$,

$$\begin{aligned} an &= a(m(a)an) \\ &= an(m(a)a) \\ &= an[m(a)am(a)a] \quad (\text{since } m(a)a \in E) \\ &= anm(a)(m(a)a)a \\ &= anm(a)^2 a^2 \\ &\in Na^2 \end{aligned}$$

$$\therefore\ aN \subset Na^2 \qquad \text{(A)}$$

Also $na^2 \in Na^2$

$$\begin{aligned} \Rightarrow na^2 &= na(am(a)a) \\ &= (am(a))na^2 \\ &= a(m(a)na^2) \\ &\in aN \end{aligned}$$

$$\therefore Na^2 \subset aN \qquad \text{(B)}$$

From (A) and (B) we get $aN = Na^2$ for all a in N and (i) follows.
Proof of '(i) $\Rightarrow$ (ii)' and that of '(iii) $\Rightarrow$ (ii)' are taken care of by Proposition 2.14.
Proof of '(ii) $\Rightarrow$ (iii)' runs parallel to that of '(ii) $\Rightarrow$ (i)' and the theorem follows.

Remark 2.21. Let N admit a mate function 'm' and let $E \subset C(N)$. It is easy to observe that for every x in N, $x = xm(x)x \Rightarrow x = m(x)x^2$. Consequently Proposition 2.18 guarantees that m is a P_3 mate function.

As a consequence of Theorem 2.20 and Remark 2.21 we have the following:

Theorem 2.22: Let N admit a mate function 'f'. Then N is a $P(r,m)$ near-ring for all positive integers r,m if and only if N is a $P(1,2)$ near-ring.

Proof. N is a $P(1,2)$ -near-ring $\Rightarrow E \subset C(N)$ (from Theorem 2.20)
Let $a \in x^r N$. $\therefore a = x^r n$ for some n in N.
Now

$$\begin{aligned} x^r n &= (xf(x)x)^r n \\ &= x^r (f(x)x)^r n \qquad (\text{since } f(x)x \in E \subset C(N)) \\ &= x^r (f(x)xn) \\ &= x^r n(f(x)x) \qquad (\text{since } E \subset C(N)) \\ & x^r n(f(x)x)^m \qquad (\text{since } f(x)x \in E) \end{aligned}$$

$= x^r n(f(x))^m x^m$ (since f is a P_3 mate function from Propositions 2.16 and 2.18)

$= (x^r n(f(x))^m)x^m$

$\in Nx^m$

$\therefore x^r N \subset Nx^m$.

In a similar fashion, we get $Nx^m \subset x^r N$. Hence $x^r N = Nx^m$ and N is a $P(r,m)$ near-ring for all positive integers *r* and *m*. The converse is obvious – it follows by taking $r = 1$ and $m = 2$.

As an immediate consequence, we have

Corollary 2.23. *If N is a* $P(1,2)$ *near-ring with a right identity, then it is a* $P(r,m)$ *near-ring for all positive integers r, m.*

Proof. N is a $P(1,2)$ near-ring with a right identity $\Rightarrow$ N is an $S'{-}P(1,2)$ near-ring $\Rightarrow$ N is a $P(1,2)$ near-ring with a mate function (from Proposition 2.16 (i)) $\Rightarrow$ N is a $P(r,m)$ near-ring for all positive integers r, m (from Theorem 2.22).

Remark 2.24. In view of Theorem 2.22, to discuss the properties of a *P*(*r, m*)-near-ring we need only to concentrate on $P(1,2)$ near-rings with mate functions.

To start with we have the following:

Theorem 2.25. *Every N-subgroup of N is an ideal in an* $S' - P(1,2)$ *near-ring.*

Proof: Since N is an $S'{-}P(1,2)$ near-ring, it admits a mate function '*m*' (from Proposition 2.16(i)) and $L = \{0\}$ (from Proposition 2.4(ii)). It is clear from *K*(2) that *N* has (*, *IFP*). Again for any non empty $S \subset N, (0:S)$ is an ideal of *N* (by *K*(3)).
If *M* is any *N*-subgroup of *N*, then $M = \sum_{x \in M} Nx$. We first show that each *Nx* is an ideal.
Let $S = (0:Nx)$. We claim that $Nx = (0:S)$.
Clearly

$$Nx \subset (0:S) \tag{3}$$

Now if $y \in (0:S)$ then $yS = \{0\}$.
Also

$$(y - ym(x)x)m(x)x = 0 \tag{4}$$

$\Rightarrow (y - ym(x)x)Nm(x)x = \{0\}$
$\Rightarrow (y - ym(x)x)Nx = \{0\}$ (using $K(1)$)
$\Rightarrow (y - ym(x)x) \in (0 : Nx) = S$
Since

$$yS = \{0\}, y(y - ym(x)x) = 0 \tag{5}$$

Using the fact that N has (*, *IFP*), it is easy to get from (4) and (5), $(y - ym(x)x)^2 = 0.$ Since $L = \{0\}$ we get $y - ym(x)x = 0 \Rightarrow y = ym(x)x$ $\Rightarrow y = ym(x)x \Rightarrow y = ym(x)x \Rightarrow y \in Nx$. Therefore

$$\therefore (0 : S) \subset Nx \tag{6}$$

From (3) and (6) we get $Nx = (0 : S)$ and hence Nx is an ideal. The desired result now follows.

Remarks 2.26.

(a) It is worth noting that in a $P(1,2)$ near-ring with mate functions the concepts of N-subgroups, left ideals, right ideals and ideals are equivalent.

(b) Recall that the nilradical of N is the greatest nil ideal of N. Since $L = \{0\}$, for an $S' - P(1,2)$ near-ring N, it follows that the nilradical of $N = \{0\}$.

Proposition 2.27. *Let N be an $S' - P(1,2)$ near-ring. Then any N-subgroup of N is a completely semiprime ideal.*

Proof. Suppose I is an N-subgroup of N. From Theorem 2.25 it follows that I is an ideal. Let $x^2 \in I$. Since N has strong *IFP*, $xm(x)x \in I$ i.e. $x \in I$. Hence I is a completely semiprime ideal.

From p.289 of Pilz [3] we have the following:

Definition 2.28. *A near-ring N has property P_4 if for all ideals I of N, $xy \in I \Rightarrow yx \in I$.*

Proposition 2.29. *An $S' - P(1,2)$ near-ring has property P_4.*

Proof. Let I be an ideal of N and let $xy \in I$. Now $(yx)^2 = (yx)(yx)$ $= y(xy)x \in NIN \subset I$ (using Remark 2.26 (a)) $\Rightarrow (yx)^2 \in I$. Using Proposition 2.27 we get $yx \in I$ i.e. N has property P_4.

Definition 2.30. *For any subset A of N we define* $\sqrt{A} = \{x \in N / x^k \in A$ *for some positive integer k*$\}$.

Proposition 2.31. *Let N be an* $S' - P(1,2)$ *near-ring. Then* $A = \sqrt{A}$ *for any N-subgroup A of N.*

Proof. Let $x \in \sqrt{A}$. Then there exists some positive integer k such that $x^k \in A.$. Since N is an $S' - P(1,2)$ near-ring, $x \in xN = Nx^2 \Rightarrow x = nx^2$ for some $n \in N \Rightarrow x = nx^2 = n^2x^3 = \cdots = n^{k-1}x^k \in NA \subset$ i.e. $x \in A$. $\therefore \sqrt{A} \subset A$. But obviously $A \subset \sqrt{A}$ and hence the desired result.

Proposition 2.32. *If N is an* $S' - P(1,2)$ *near-ring, then*

(i) *N is a semiprime near-ring*
(ii) $A \cap B = AB$ *for all N-subgroups A,B of N*
(iii) $Na \cap Nb = Nab$ *for all* $a, b \in N$
(iv) *every ideal I fulfills* $I = I^2$.

Proof. Since *N* is an $S' - P(1,2)$ near-ring, it admits a mate function '*m*'.

(i) Let *A* be an *N*-subgroup of *N*. Then *A* is an ideal of *N*. Let *I* be any ideal of *N* such that $I^2 \subset A$. If $a \in I$, then $a = a(m(a)a) \in I(NI) \subset I^2 \subset A$ $\Rightarrow a \in A \Rightarrow I \subset A$. Thus any *N*-subgroup *A* of *N* is a semiprime ideal. In particular $\{0\}$ is a semiprime ideal and therefore *N* is a semiprime near-ring.

(ii) Let *A* and *B* be two *N*-subgroups of *N*. By Proposition 2.8 both are invariant *N*-subgroups and consequently

$$AB \subset A \cap B. \tag{7}$$

To prove the reverse inclusion, we note that for any $x \in A \cap B$, $x = xm(x)x = xm(x)x = (m(x)x)x \in (NA)B \subset AB \Rightarrow x \in AB$, hence

$$A \cap B \subset AB \tag{8}$$

(1) and (2) yield (ii).

(iii) Clearly *Na* and *N*b are *N*-subgroups of *N*. Now $Na \cap Nb = N(aN)b$. From Theorem 2.22, *N* is $P(1,1)$ also. Hence, $Na \cap Nb = NaNb = N(Na)b = Nab$. Therefore (iii) follows.

(iv) Take $A = B = I$ in (i) and appeal to Remark 2.26 (a).

Definition 2.33. *An N-subgroup* $A \neq \{0\}$ *of N is called essential if* $A \cap B = \{0\}$, *where B is any N-subgroup of N, implies* $B = \{0\}$.

Proposition 2.34. *Let N be an* $S' - P(1,2)$ *near-ring. If N has no non-zero zero-divisors, then every N-subgroup of N is essential.*

Proof. Let $A \neq \{0\}$ be an *N*-subgroup of *N*. Suppose there exists an *N*-subgroup *B* of *N* such that $A \cap B = \{0\}$. This implies $AB = \{0\}$ (using Proposition 2.31 (i)). Since *N* has no non-zero zero-divisors, we get $B = \{0\}$ and the result follows.

3.

In this section we obtain a structure theorem for $P(1,2)$ near-rings. Throughout this section *N* denotes an $S' - P(1,2)$ near-ring and *m* is a mate function for *N*.

Theorem 3.1. *N is subdirectly irreducible if and only if N is a near-field.*

Proof. Suppose *N* is subdirectly irreducible. First we claim that no non-zero idempotent of *N* is a zero-divisor. Let *J* be the set of all non-zero idempotents which are zero-divisors and let $J \neq \phi$. Let $I = \bigcap_{e \in J} (0:e)$. Since *N* is subdirectly irreducible, $I \neq \{0\}$. Let $a \in I - \{0\}$. Thus

$$ae = 0 \text{ for all } e \text{ in } J. \tag{9}$$

This $\Rightarrow m(a)ae = 0 \Rightarrow em(a)a = 0$ (using $K(2)$) $\Rightarrow m(a)\,a \in J$.

From (9) we get $am(a)a = 0 \Rightarrow a = 0$. This contradiction implies that no non-zero idempotent of *N* is a zero-divisor.

We shall now prove that *N* has no non-trivial *N*-Subgroups. Let *M* be any *N*-subgroup of *N* such that $M \neq \{0\}$ and let $x(\neq 0) \in M$. The fact that *N* is a $P(r,k)$ near ring (from Theorem 2.22) for all positive integers r,k forces $Nx = Nx^2$ for all *x* in *N*. For any $n \in N$, there exists n_1 in *N* such that $nx = n_1x^2 \Rightarrow (n - n_1x)x = 0 \Rightarrow (n - n_1x)xm(m) = 0.$
$\Rightarrow n - n_1x = 0$ (from (ii)) $\Rightarrow n = n_1x \in NM \subset M$. $\therefore\ N \subset M$. i.e. $M = N$.
Thus *N* has no non-trivial *N*-subgroups. Clearly for $n \in N - \{0\}, Nn$ is an *N*-subgroup of *N*. Consequently $Nn = N$ for all

$$n \in N - \{0\} \tag{10}$$

Also, it is clear that $N_d \neq \{0\}$ (as $\left(E \subset C(N) \subset N_d\right)$. This and (10) guarantee that N is a near-field. (Theorem 8.3, Pilz [3]).

Converse is obvious.

As an immediate consequence of Theorem 3.1, we have the following:

Corollary 3.2. *N has no non-zero zero-divisors if and only if N is a near-field.*

We are now in a position to give a structure theorem for *N*.

Theorem 3.3. *N is isomorphic to a subdirect product of near-fields.*

Proof. From Theorem 2.7, *N* is isomorphic to a subdirect product of subdirectly irreducible $P(1,2)$ near-rings, N_i's, say. Obviously the existence of a mate function is preserved under homomorphisms. Hence each N_i admits a mate function. Appealing to Theorem 3.1 we get *N* is isomorphic to a subdirect product of near-fields.

Remark 3.4. From 8.11 of [3], the additive group of a near-field is abelian. It follows that for any $P(1,2)$ near-ring *N* with mate functions, $(N,+)$ is abelian.

Proposition 3.5. *Let N be a Boolean near-ring. Then N is* $P(1,2)$ *if and only if it is a commutative ring.*

Proof. We observe that identity function is a mate function for *N*. Appealing to Theorem 2.20 and Remark 3.4 we see that when *N* is a $P(1,2)$ near-ring, $N = E \subset C(N)$ and $(N,+)$ is abelian and hence *N* is a commutative ring.

Conversely *N* is Boolean and a commutative ring $\Rightarrow$ *N* is $P(1,1)$ $\Rightarrow xN = Nx = Nx^2$. Hence the result.

Proposition 3.6. *If N is distributively generated and has no non-zero zero-divisors then N is a division ring.*

Proof. Corollary 3.2 guarantees that *N* is a near-field. Also $(N,+)$ is abelian (by Remark 3.4). Since *N* is distributively generated, we see that *N* is a ring (from Theorem 6.6(c) of Pilz [3]) and hence the result.

Proposition 3.7. *If N is strictly prime then N is subdirectly irreducible. In fact N is a near-field.*

Proof. In view of Corollary 3.2 and Theorem 3.1, it is enough to show that *N* has no non-zero zero-divisors. Let $x, y \in N$ be such that $xy = 0$. Clearly Nx and Ny are *N*-subgroups of *N* and $NxNy = Nxy = N0 = \{0\}$. Since *N* is strictly prime we have either

$Nx = \{0\}$ or $Ny = \{0\}$. This forces $x = 0$ or $y = 0$. Hence N has no non-zero zero-divisors and the proof is complete.

Since the concepts of *N*-subgroups and ideals coincide in $S' - P(1, 2)$ near-rings we have the following obvious result:

Proposition 3.8. *N is strictly prime if and only if N is a prime near-ring.*

Proposition 3.9. *Any strictly prime ideal of N is maximal.*

Proof. Let *M* be a strictly prime ideal of *N*. Then *N/M* is a strictly prime near-ring. Since *N/M* is the image of *N* under the canonical homomorphism, it is also an $S' - P(1, 2)$ near-ring i.e. a $P(1, 2)$ near-ring with a mate function. Hence from Proposition 3.7, *N/M* is a near field. i.e. *M* is maximal.

We conclude our discussion of $S' - P(1, 2)$ near-rings with the following:

Theorem 3.10. *If N is 2 primitive then N is a near field.*

Proof. Since *N* is 2 primitive, from Corollary 4.4(c), Pilz [3], there exists a left ideal *I* of *N* such that *I* is 2-modular and $(I : N) = \{0\}$. By Corollary 3.24 Pilz [3], $(I : N) \subset I$. Since *N* is $P(1, 2)$, I is an ideal. $\therefore$ $IN \subset I \Rightarrow I \subset (I : N)$.

From above, $I = (I : N) = \{0\}$. $\therefore N(\cong N/I)$ is an N-group of Type 2. $\therefore$ N has no non-trivial *N*-subgroups. From stages (10) of Theorem 3.1 we get the desired result.

Remark 3.11. In view of Remark 2.24 we observe that all the results we have established for an $S' - P(1, 2)$ near-ring are valid for a $P(r, m)$ near-ring with mate functions where r and m are any two positive integers.

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