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Convolutions with Hypergeometric Functions

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Abstract. In this paper we study the behaviour of $[I_{a,b,c}(f)]\,(z) = zF\,(a,b,c,z) * f(z)$ where $F(a,b,c,z)$ is the Gaussian Hypergeometric function and the $*$ is usual Hadamard product. In the main result, we find conditions on a,b,c,A,B and β so that $[I_{a,b,c}(f)](z)$ belong to $S^*[A,B]$ whenever $f(z) \in R(\beta)$, $\beta < 1$.

1. Introduction

Let $\mathcal{A}$ denote the family of functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ that are analytic in the interior of unit disk $\Delta = \{z \in C : |z| < 1\}$. Let g be analytic and univalent in Δ and f be analytic in Δ then $f(z)$ is said to be subordinate to $g(z)$, written $f \prec g$ if $f(0) = g(0)$ and $f(\Delta) \subset g(\Delta)$.
For $-1 \le B < A \le 1$, let

$$S^*[A,B] = \left\{ f \in \mathcal{A} \;\middle|\; \frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz}, \quad z \in \Delta \right\}$$

For $A = 1, B = -1$ we get the well known family S^* of starlike functions. We further get $S[1-2\gamma, -1] = S^*(\gamma)$ and $S^*(\lambda, 0) = S^*_\lambda$. For $\beta < 1$, define

$$R(\beta) = \{ f \in \mathcal{A} \mid \exists\, \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) / \operatorname{Re}\left[e^{i\theta}\left(f'(z) - \beta\right)\right] > 0,\ z \in \Delta \}.$$

Note that when $\beta \ge 0$, we have $R(\beta) \subset S$, the class of univalent functions in $\mathcal{A}$. For each $\beta < 0$, $R(\beta)$ contains also nonunivalent functions.

For any complex number 'a' we define the ascending factorial notation $(a,n) = a(a+1)\cdots(a+n-1)$ for $n \geq 1$ and $(a,0)=1$ for $a \neq 0$. The triangle inequality for (a,n) is $|(a,n)| \leq (|a|, n))$. When 'a' is neither zero nor a negative integer, we can write $(a,n) = \Gamma(n+a)/\Gamma(a)$.

The Gaussian hypergeometric function is defined as

$$F(a,b,c,z) = \sum_{n=0}^{\infty} \frac{(a,n)(b,n)}{(c,n)(1,n)} z^n, \quad a,\ b,\ c \in C$$

where c is neither zero nor a negative integer. The following well known formula

$$F(a,b,c,1) = \frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)}, \quad \mathrm{Re}\,(c-a-b) > 0 \tag{1.1}$$

will be used frequently. Univalence, starlikeness and convexity properties of $zF(a,b,c,z)$ have been studied in [6] and [8].

For $f \in \mathcal{A}$, we consider the Hohlov convolution operator [2] $I_{a,b,c}(f)$ given by

$$[I_{a,b,c}(f)](z) = zF(a,b,c,z) * f(z)$$

where $*$ stands for the usual Hadamard product of power series.
For $\mathrm{Re}\, c > \mathrm{Re}\, b > 0$, it is known that

$$F(a,b,c,z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} \frac{dt}{(1-tz)^a}.$$

We can write

$$[I_{a,b,c}(f)](z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} \frac{f(tz)}{t}\, dt * \frac{z}{(1-z)^a}.$$

This operator reduces to Bernardi operator

$$B_f(z) = (1+\gamma) \int_0^1 t^{\gamma-1} f(tz)\, dt$$

for $a=1, b=1+\gamma$ and $c=2+\gamma$ with $\mathrm{Re}\ \gamma > -1$. For $\gamma = 1$ and 2, respectively we get Alexander transform and Libera transform. These three operators are all examples of the situation where $c = a+b$ in $I_{a,b,c}(f)$. Also we have

$\dfrac{z}{(1-z)^{n+1}} * f(z)=[I_{1,n+1,1}(f)](z)$, $n>-1$ which is known as Ruscheweyh differential, studied in [7]. It represents the case $c<a+b$ with $a=1$, $b=n+1$ and $c=1$. Some more special cases of the operator $I_{a,b,c}(f)$ can be found in [10].

P.T. Mocanu [3] obtained the range for γ so that the Bernardi operator $B_f \in S^*$ whenever $f\in R(0)$. As a natural extension, here we determine conditions on A, B, a, b, c and β, the transform by the hypergeometric function $F(a,b,c,z)$ on the class $R(\beta)$ so that $I_{a,b,c}(f) \in S^*[A,B]$.

2. Auxiliary lemmas

We shall state the following Lemmas [4] which may be used in proving the main theorems.

Lemma 2.1. *Let* $a,b,c>0$. *Then*

(i) *for* $c>a+b+1$,

$$\sum_{n=0}^{\infty} \frac{(n+1)\,(a,n)\,(b,n)}{(c,n)\,(1,n)} = \frac{\Gamma(c-a-b)\,\Gamma(c)}{\Gamma(c-a)\,\Gamma(c-b)}\left[\frac{ab}{c-1-a-b}+1\right]$$

(ii) *for* $c>a+b+2$,

$$\sum_{n=0}^{\infty} \frac{(n+1)^2\,(a,n)\,(b,n)}{(c,n)\,(1,n)} = \frac{\Gamma(c-a-b)\,\Gamma(c)}{\Gamma(c-a)\;\Gamma(c-b)}\left[1+\frac{(a,2)\,(b,2)}{(c-2-a-b,2)}+\frac{3ab}{c-1-a-b}\right].$$

Lemma 2.2. *Let* $a,b,c>0$ *and for* $a\neq 1, b\neq 1, c\neq 1$ *with* $c>\max\{0, a+b-1\}$,

$$\sum_{n=0}^{\infty} \frac{(a,n)(b,n)}{(c,n)(1,n+1)} = \frac{1}{(a-1)(b-1)}\left[\frac{\Gamma(c+1-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)}-(c-1)\right].$$

Lemma 2.3. *Let* $a,b,c>0$. *For* $b\neq 1$ *and* $c>1+b$,

$$\sum_{n=0}^{\infty} \frac{(b,n)}{(c,n)}\frac{1}{(n+1)} = \frac{c-1}{(b-1)}\left(\psi(c-1)-\psi(c-b)\right)$$

where $\psi(x)=\Gamma'(x)/\Gamma(x)$.

3. Main theorems

Now let us study the action of the hypergeometric function on the classes $R(\beta)$ and S.

Theorem 3.1. *Let $a, b \in C\setminus\{0\}$, $|a| \neq 1$, $|b| \neq 1$, $c \neq 1$ and $c > |a| + |b|$.*

For $-1 \le B < A \le 1$, assume that

$$\frac{\Gamma(c-|a|-|b|)\Gamma(c)}{\Gamma(c-|a|)\Gamma(c-|b|)}\left\{(1-B) - \frac{(1-A)\,(c-|a|-|b|)}{(|a|-1)\,(|b|-1)}\right\}$$

$$\le (A-B)\left\{1 + \frac{1}{2(1-\beta)}\right\} - \frac{(1-A)\,(c-1)}{(|a|-1)\,(|b|-1)} \qquad (3.1)$$

Then the operator $I_{a,b,c}(f)$ maps $R(\beta)$ into $S^*[A,B]$.

Proof. Let $a,\ b \in C\setminus\{0\}$ and $c > |a|+|b|$, $|a| \neq 1$, $|b| \neq 1$ and $c \neq 1$. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be a function in $R(\beta)$. Then, it is well-known that $|a_n| \le \frac{2(1-\beta)}{n}$.

Consider $zF(a,b,c,z) * f(z) = z + \sum_{n=2}^{\infty} B_n z^n$ where $B_1 = 1$ and for $n \ge 1$,

$$B_n = \frac{(a,n-1)\,(b,n-1)}{(c,n-1)\,(1,n-1)}\, a_n$$

A special case of Theorem 3 in [1] gives a sufficient condition for $f \in S^*[A,B]$ is that $\sum_{n=2}^{\infty} \{n(1-B)-(1-A)\}\,|a_n| \le A-B$. Then we have to show that

$$T = \sum_{n=2}^{\infty} \{n\,(1-B)-(1-A)\}|B_n| \le A-B.$$

We have

$$T \le \sum_{n=2}^{\infty} \{n(1-B)-(1-A)\}\, \frac{(|a|,\, n-1)\,(|b|,\, n-1)}{(c,\, n-1)\,(1,\, n-1)}\, \frac{2(1-\beta)}{n}$$

$$= 2(1-\beta)\left\{(1-B)\sum_{n=1}^{\infty} \frac{(|a|,\, n)\,(|b|,\, n)}{(c,\, n)(1,\, n)} - (1-A)\sum_{n=1}^{\infty} \frac{(|a|,n)\,(|b|,n)}{(c,n)\,(1,\, n+1)}\right\} := T_1 \qquad (3.2)$$

Using the formula (1.1) and Lemma 2.2. we observe that

$$T_1 = 2(1-\beta)\left\{ (1-B)\, \frac{\Gamma(c-|a|-|b|)\,\Gamma(c)}{\Gamma(c-|a|)\,\Gamma(c-|b|)} - (1-B) \right.$$

$$\left. -(1-A)\, \frac{\Gamma(c-|a|-|b|)\,\Gamma(c)\,(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)(|a|-1)(|b|-1)} + \frac{(1-A)\,(c-1)}{(|a|-1)(|b|-1)} + (1-A) \right\}$$

$$= 2(1-\beta)\left\{ \frac{\Gamma(c-|a|-|b|)\,\Gamma(c)}{\Gamma(c-|a|)\,\Gamma(c-|b|)} \left[(1-B) - \frac{(1-A)\,(c-|a|-|b|)}{(|a|-1)(|b|-1)} \right] \right.$$

$$\left. + \frac{(1-A)\,(c-1)}{(|a|-1)(|b|-1)} - (A-B) \right\}.$$

Then under the hypothesis (3.1) of the theorem we get

$$T \le T_1 \le 2\,(1-\beta)\, \frac{(A-B)}{2\,(1-\beta)} = A-B\,,$$

thereby showing that $f \in S^*[A,B]$.

Note. For $A=\lambda,\ B=0$ we get, as a special case, Theorem 2.1 of [4].

Theorem 3.2. *Let* $b \in C\setminus\{0\}$, $c>0$, $|b| \ne 1$ *and* $c > 1+|b|$. *For* $-1 \le B < A \le 1$, *assume that*

$$\frac{(1-B)\,(c-1)}{(c-|b|-1)} - (1-A)\left(\frac{c-1}{b-1}\right)(\psi\,(c-1)-\psi\,(c-|b|)) \le \frac{A-B}{2(1-\beta)} + (A-B) \qquad (3.3)$$

where $\psi(x)=\Gamma'(x)/\Gamma(x)$. *Then the operator* $I_{1,b,c}(f)$ *maps* $R(\beta)$ *into* $S^*[A,B]$.

Proof. Putting $a=1$ in (3.2) we get

$$T_1 = 2(1-\beta)\left\{ (1-B)\sum_{n=1}^{\infty} \frac{(|b|,n)}{(c,n)} - (1-A)\sum_{n=1}^{\infty} \frac{(|b|,n)}{(c,n)\,(n+1)} \right\}.$$

Using (1.1) and Lemma 2.3. we get

$$T_1 = 2(1-\beta)\left\{ \frac{(1-B)\,(c-1)}{(c-|b|-1)} - (1-A)\left(\frac{c-1}{b-1}\right)(\psi\,(c-1) - \psi\,(c-|b|)) - (A-B) \right\}.$$

Thus under the hypothesis (3.3) of the theorem we get $T \le T_1 \le (A-B)$, there by showing that the operator $I_{1,b,c}(f)$ maps $R(\beta)$ into $S^*[A,B]$.

Note. For $A=\lambda$, $B=0$, we get as a special case, Theorem 2.2. of [4].

From the proof of Theorems 3.1 and 3.2, we observe that for $A=1$, $B=0$. We need not treat the case $a=1$ separately neither we need the aestrictions $|b|\neq 1$ and $c\neq 1$. In this case, we have the following result.

Corollary 3.3. *Let* $a,b \in C\setminus\{0\}$ *and* $c > |a| + |b|$. *Assume that*

$$\frac{\Gamma(c-|a|-|b|)\,\Gamma(c)}{\Gamma(c-|a|)\,\Gamma(c-|b|)} \le 1 + \frac{1}{2(1-\beta)}.$$

Then the operator $I_{a,b,c}(f)$ maps $R(\beta)$ into $S^*[1,0]$.

Let $\pi : [0,1] \to R$ be a nonnegative function normalized so that $\int_0^1 \pi(t)dt = 1$ and define

$$[V_\pi(f)](z) = \int_0^1 \pi(t)\,\frac{f(tz)}{t}\,dt,\; f \in \mathcal{A}.$$

Let $\Pi(t) = \int_0^1 \pi(s)\frac{ds}{s}$ and assume that $t\Pi(t) \to 0$ when $t \to 0+$. It is shown in [9] that the class $S^*[A,B]$, $-1 \le B < A \le 1$ can be characterized interms of convolutions that

$$f \in S^*[A,B] \Leftrightarrow \frac{f(z)}{z} * \frac{h_{(A,B)}(z)}{z} \neq 0$$

where

$$h_{(A,B)}(z) = \frac{z\left[1-\frac{A-x}{A-B}z\right]}{(1-z)^2};\quad |x| = 1.\quad \text{Choose } G(t) = \frac{(A-B) - (1-A)t}{(A-B)(1+t)^2}.$$

From $tg'(t)+g(t)+1 = 2G(t)$, we get

$$g(t) = \frac{2(1-B)-(A-B)(1+t)}{(A-B)(1+t)} - \frac{2(1-A)}{(A-B)}\,\frac{\log(1+t)}{t}$$

An application of Theorem 2.1 in [5] gives the following result.

Theorem 3.4. *Let β be given by*

$$\frac{\beta}{1-\beta} = -\int_0^1 \pi(t)\left[\frac{2(1-B)-(A-B)(1+t)}{(A-B)(1+t)} - \frac{2(1-A)}{(A-B)}\,\frac{\log(1+t)}{t}\right]dt\,.$$

Then,

$$V_\pi\,(R(\beta)) \subset S^*[A,B] \Leftrightarrow L_\Pi\,(e^{-i\theta}h_{(A,B)}(e^{i\theta}z)) \geq 0, \quad z \in \Delta$$

Where

$$L_\Pi(h) = \inf_{z\in\Delta} \int_0^1 \Pi(t)\left[\operatorname{Re}\left(\frac{h(tz)}{tz}\right) - \frac{(A-B)+(A-1)t}{(A-B)(1+t)^2}\right]dt\,.$$

Note. The operator $I_{1,b,c}(f)$ corresponds to $V_\pi(f)$ with $\pi(t)=\pi_{b,c}(t)$ $= \dfrac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\,t^{b-1}(1-t)^{c-b-1}$ where $\int_0^1 \pi_{b,c}(t)\,dt = 1$. The cases $A=\lambda$, $B=0$ and $A=1-2\gamma$, $B=-1$ were treated in [4] and [5] respectively.

Next we determine the condition on a,b,c and A,B when $f(z)$ is in S instead of $f(z)\in R(\beta)$.

Theorem 3.5. *Let* $a,b \in C\setminus\{0\}$, $c > 2 + |a| + |b|$. *Suppose that a, b and* $-1 \leq B < A \leq 1$ *satisfy the condition that*

$$\frac{\Gamma(c-|a|-|b|)\,\Gamma(c)}{\Gamma(c-|a|)\,\Gamma(c-|b|)}\left[\frac{(1-B)\,|a(a+1)\,b(b+1)|}{(c-2-|a|-|b|)\,(c-1-|a|-|b|)} + (A+2-3B)\,\frac{|ab|}{c-1-|a|-|b|} + (A-B)\right] \leq 2\,(A-B) \qquad (3.4)$$

Then the operator $I_{a,b,c}(f)$ *maps S into* $S^*[A,B]$.

Proof. Let $a \in C\,|\{0\}$, $c > 2+|a|+|b|$ and $-1 \leq B < A \leq 1$. Let $f(z) = z+\sum_{n=2}^{\infty} a_n z^n \in S$. Then we have that $|a_n| \leq n$. Consider $zF(a,b,c,z) * f(z) = z + \sum_{n=2}^{\infty} B_n z^n$ where

$$B_n = \frac{(a,n-1)(b,n-1)}{(c,n-1)(1,n-1)}a_n\,.$$

It is enough to show that

$$T = \sum_{n=2}^{\infty} \{n(1-B) - (1-A)\} | B_n | \le A-B.$$

We have

$$T = \sum_{n=2}^{\infty} \{n(1-B) - (1-A)\} \frac{|(a,n-1)(b,n-1)|}{(c,n-1)(1,n-1)} |a_n|$$

$$\le \sum_{n=2}^{\infty} \{(n+1)^2 (1-B) - (n+1)(1-A)\} \frac{(|a|,n)(|b|,n)}{(c,n)(1,n)}$$

$$= (1-B) \sum_{n=1}^{\infty} \frac{(n+1)^2 (|a|,n)(|b|,n)}{(c,n)(1,n)} - (1-A) \sum_{n=1}^{\infty} \frac{(n+1)(|a|,n)(|b|,n)}{(c,n)(1,n)} := T_2$$

From Lemma 2.1. we get

$$T_2 = \frac{\Gamma(c-|a|-|b|)\Gamma(c)}{\Gamma(c-|a|)\Gamma(c-|b|)} \Bigg[(1-B) + \frac{(1-B)|(a,2)(b,2)|}{(c-2-|a|-|b|,2)} + \frac{3(1-B)|ab|}{c-1-|a|-|b|}$$

$$- \frac{(1-A)\,|ab|}{(c-1-|a|-|b|)} - (1-A) \Bigg] - (1-B) + (1-A)$$

$$= \frac{\Gamma(c-|a|-|b|)\Gamma(c)}{\Gamma(c-|a|)\Gamma(c-|b|)} \Bigg[\frac{(1-B)\,|a(a+1)b(b+1)|}{(c-2-|a|-|b|)(c-1-|a|-|b|)} + \frac{A|ab|}{c-1-|a|-|b|}$$

$$- \frac{|ab|}{(c-1-|a|-|b|)} + \frac{3(1-B)|ab|}{(c-1-|a|-|b|)} + (A-B) \Bigg] - (A-B)$$

$$= \frac{\Gamma(c-|a|-|b|)\Gamma(c)}{\Gamma(c-|a|)\Gamma(c-|b|)} \Bigg[\frac{(1-B)\,|a(a+1)b(b+1)|}{(c-2-|a|-|b|)(c-1-|a|-|b|)}$$

$$+ \frac{(A+2-3B)|ab|}{c-1-|a|-|b|} + (A-B) \Bigg] - (A-B).$$

Then, under the hypothesis (3.4) of the theorem we get $T \le T_2 \le A-B$. Therefore the operator $I_{a,b,c}(f)$ maps S into $S^*[A,B]$.

Note. When $A = \lambda$, $B = 0$, this reduces to Theorem 2.6. in [4].

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