Convexity of the Stability Region of Hurwitz Systems of Differential Equations

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Abstract. In a recent paper, the problem of stability of systems of differential equations was approached from a geometric perspective. In the present work, we continue in this line and we show the convexity of the family of systems of differential equations which are Hurwitz stable and whose characteristic polynomials are mapped into the space of Markov parameters.

1. Introduction

A system of differential equations is said to be Hurwitz (Schur) stable if all of its eigenvalues lie in the left-half (unit disc) of the complex plane. The Hurwitz and Schur stability properties are some of the essential concepts in a variety of mathematical and engineering fields: control theory, spectral analysis, numerical computations and digital signal processing to name a few. The existing methods for testing the Hurwitz (Schur) stability of a system of differential equations are mainly restricted to the real case and they are due to contributions by Hurwitz, Cohn, Naimark, Marden and others. For a variety of references in this context, see [2], [3], [5], [6], [7]. For some recent work on the stability of systems with complex coefficients see [9], [10], [11] and [12].

In a recent paper [11], a completely new approach to the stability problem has been advocated. We discussed the geometry of the stability region of systems of differential equations that are Schur stable and we proved that while the stability region of all linear Schur stable systems of dimension 2 is convex, this nice property is not true for all $n \geq 3$. In the present work, we intend to push further the discussion on the geometry of stability regions.

Most stability results in the theory of differential equations are expressed in polynomial coefficient space, that is, the established stability criteria are formulated in terms of the coefficients of the characteristic polynomial of the system under discussion. In [1, p. 341], a theorem on determinants due to Markov seems to have been overlooked. This interesting theorem leads to simple stability results for a family of systems of differential equations. It is an interesting fact that Markov's theorem does not work in

the standard polynomial coefficient space. Rather, polynomials are expressed in terms of their associated Markov parameters. As an illustration, consider the polynomial

$$f(z) = z^4 + 5z^3 + 8z^2 + 8z + 3 = (z^4 + 8z^2 + 3) + z(5z^2 + 8)$$
$$= h(z^2) + z\,g(z^2)$$

where $h(z) = z^2 + 8z + 3$ and $g(z) = 5z + 8$.

From the expression:

$$\frac{g(z)}{h(z)} = 5z^{-1} - 32z^{-2} + 241z^{-3} - 1832z^{-4} + \cdots.$$

The four leading coefficients in this series (5, 32, 241, 1832) are the Markov parameters for this fourth-degree polynomial. For a general reference on Markov's work, refer to [4] and [8].

   In this paper, we shall be working in the space of Markov parameters and this must not be seen as an inconvenience, because variations in physical parameters which lead to polynomial families do not respect whether an analyst models a polynomial family in its coefficient space or in its space of Markov parameters.

   This paper is structured as follows. In section 2, we give some notations. In section 3, we establish the relationships between the space of polynomial coefficients and that of Markov parameters. The convexity of the stability region in the space of Markov parameters is established in section 4.

2.   Definitions and notations

Consider a real system of differential equations with characteristic polynomial:

$$f(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n.$$

The polynomial $f(z)$ may be viewed as a vector $(a_0, a_1, \cdots, a_n)$ in its (n+1)st dimensional space of coefficients. Alternatively, we can express $f(z)$ in terms of its Markov parameters. First we write

$$f(z) = h(z^2) + z\,g(z^2)$$

where we assume that $h(z)$ and g(z) are coprime, then we expand the irreducible rational fraction $\frac{g(z)}{h(z)}$ in a series of decreasing powers of z

$$\frac{g(z)}{h(z)} = b_{-1} + \frac{b_0}{z} - \frac{b_1}{z^2} + \frac{b_2}{z^3} - \frac{b_3}{z^4} + \cdots \tag{1}$$

It is to be noted that if $f(z)$ is a Hurwitz polynomial, then its zeros have negative real parts. It results from [1, Theorem 13, P. 228] that (h, g) forms a positive pair, and therefore must have only interlacing zeros. It follows that $h(z)$ and $g(z)$ are coprime.

Definition 1. *Let* $m = \left[\frac{n}{2}\right]$, *then the Markov parameters of* $f(z)$ *are defined to be:*

$$(b_0, \ b_1, \ \cdots, \ b_{2m-1}), \ \text{if } n = 2m$$

or

$$(b_{-1}, \ b_0, \ b_1, \ \cdots, \ b_{2m-1}), \ \text{if } n = 2m+1.$$

3. Polynomial coefficients and Markov parameters

It is of interest to study the relationship that may exist between the space of polynomial coefficients $(a_0, \ a_1, \ \cdots, \ a_n)$ and the space of Markov parameters $(b_{-1}, \ b_0, \ b_1, \cdots, \ b_{2m-1})$. Such a relationship will make it possible to study stability analysis in the space of Markov parameters. Without loss of generality, we may assume that n is even, leading to $b_{-1} = 0$.

Let $P \subset \mathfrak{R}^{n+1}$ denote the set of all polynomial coefficients for which $h(z)$ and $g(z)$ are coprime. Therefore, if $(a_0, \ a_1, \cdots, \ a_n) \in P$, then $h(z)$ and $g(z)$ are coprime and

$$a_0 z^n + a_1 z^{n-1} + \ \cdots + a_n = h(z^2) + z \, g(z^2) \tag{2}$$

Define the mapping $\quad \Phi: P \subset \mathfrak{R}^{n+1} \to \ R(\Phi) \subset \mathfrak{R}^n$ by

$$\Phi\left[(a_0, a_1, \cdots, a_n)\right] = (b_0, \ b_1, \ \cdots, \ b_{2m-1})$$

where $R(\Phi)$ denotes the range of Φ and b_k, $k = 0, 1, \cdots, 2m - 1$ are the leading $2m$ coefficients in the expansion (1) and h and g are given in (2). Before we carry further, we need the following 2 lemmas.

Lemma 1. *The coefficients* b_{2m}, b_{2m+1}, $\cdots$ *in series* (1) *are uniquely determined by the* $2m$ *numbers* b_0, b_1, $\cdots$, b_{2m-1}.

Proof. Each vector $(b_0, b_1, \cdots, b_{2m-1})$ in $R(\Phi)$ generates a finite Hankel matrix

$$H_m = \begin{bmatrix} b_0 & b_1 & \dots & b_{m-1} \\ b_1 & b_2 & & \\ \vdots & & & \\ b_{m-1} & b_m & \dots & b_{2m-2} \end{bmatrix} \tag{3}$$

of rank m [1, Theorem 8, p. 207]. This fact combined with [1, Theorem 7, P. 205] leads to the desired result.

It follows from the above lemma that an nth-degree polynomial $f(z)$, with h and g coprime, is uniquely determined (up to a constant factor) by its Markov parameters.

Lemma 2. *If $M \subset \Re^n$ is the set of all vectors $(b_0, b_1, \ldots, b_{2m-1})$ for which the corresponding matrix H_m in (3) has rank m, then $R(\Phi) = M$, i.e. Φ is onto M.*

Proof. Consider an arbitrary vector $(b_0, b_1, \cdots, b_{2m-1})$ in M. Since H_m in (3) has rank m, then these numbers uniquely determine the coefficients $r_1, r_2, \cdots, r_m$ through the relations:

$$b_k = \sum_{j=1}^{m} r_j \, b_{k-j}, \quad k = m, m+1, \cdots 2m-1$$

We generate the numbers $b_{2m}, b_{2m+1}, \cdots$ from this recursion, i.e.

$$b_k = \sum_{j=1}^{m} r_j \, b_{k-j}, \quad k = 2m, 2m+1, \cdots$$

By [1, Theorem 8, P. 207] the sum of the series

$$S(z) = \frac{b_0}{z} - \frac{b_1}{z^2} + \frac{b_2}{z^3} - \frac{b_3}{z^4} + \cdots$$

is a rational function in z. Therefore, we can write $S(z) = \dfrac{g(z)}{h(z)}$ and form

$$f(z) = h(z^2) + z \, g(z^2) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n \tag{4}$$

It follows that $\Phi\big[(a_0, a_1, \cdots, a_n)\big] = (b_0, b_1, \cdots, b_{2m-1})$ and Φ is onto M.

Now, suppose that the polynomials g and h of (4) are given by

$$g(z) = c_1 z^{m-1} + c_2 z^{m-2} + \cdots + c_m$$

and

$$h(z) = d_0 z^m + d_1 z^{m-1} + \cdots + d_m,$$

then the c_k and d_k are related to the Markov parameters by the following relations:

$$\begin{bmatrix} d_m/d_0 \\ d_{m-1}/d_0 \\ \vdots \\ d_1/d_0 \end{bmatrix} = \begin{bmatrix} b_0 & -b_1 & \cdots & (-1)^{m-1} b_{m-1} \\ -b_1 & b_2 & & \\ \vdots & & & \\ (-1)^{m-1} b_{m-1} & (-1)^m b_m & & b_{2m-2} \end{bmatrix}^{-1} \begin{bmatrix} (-1)^m b_m \\ (-1)^{m+1} b_{m+1} \\ \vdots \\ (-1)^{2m} b_{2m-1} \end{bmatrix}$$

$c_1 = d_0 b_0;$

$c_2 = -d_0 b_1 + d_1 b_0;$

  $\vdots$

$c_m = (-1)^{m-1} d_0 b_{m-1} + (-1)^{m-2} d_1 b_{m-2} + \cdots + d_{m-1} b_0$

4. Convexity of the stability region

We prove in this section that the stability region in the space of Markov parameters is convex.

Definition 2. *The Markov parameters* $(b_0, b_1, \cdots, b_{2m-1})$ *for* $n = 2m$ *or* $(b_{-1}, b_0, b_1, \cdots, b_{2m-1})$ *for* $n = 2m+1$ *are said to be Hurwitz if the polynomial* $f(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n$ *is Hurwitz where* $\Phi\big[(a_0, a_1, \cdots, a_n)\big] = (b_0, b_1, \cdots, b_{2m-1}).$

We need the next lemma which is essentially Theorem 17 of [1, P. 235].

Lemma 3. *The Markov parameters* $(b_0, b_1, \cdots, b_{2m-1})$ *for* $n = 2m$ *or* $(b_{-1}, b_0, b_1, \cdots, b_{2m-1})$ *for* $n = 2m+1$ *are Hurwitz if and only if the two finite Hankel matrices*

$$H_m = \begin{bmatrix} b_0 & b_1 & \ldots & b_{m-1} \\ b_1 & b_2 & & \\ \vdots & & & \\ b_{m-1} & b_m & \ldots & b_{2m-2} \end{bmatrix} \text{ and } H_m^{(1)} = \begin{bmatrix} b_1 & b_2 & \ldots & b_m \\ b_2 & b_3 & & \\ \vdots & & & \\ b_m & b_{m+1} & \ldots & b_{2m-1} \end{bmatrix} \tag{5}$$

are positive definite and $b_{-1} > 0$ when $n = 2m+1$.

Theorem. *The space of all Hurwitz Markov parameters is convex.*

Proof. Since the space of positive-definite matrices is convex and since the entries of H_m and $H_m^{(1)}$ in (5) depend linearly on the Markov parameters, then the result follows immediately.

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