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Uniquely *N*-colorable and Chromatically Equivalent Graphs

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Abstract. For each integer $n \geq 3$, we present a uniquely *n*-colorable graph with $2n+1$ vertices, and their generalizations. For each integer $n \geq 3$, we present two uniquely $(n+1)$-colorable graphs which are chromatically equivalent with $2n+2$ vertices, and with $2n+3$ vertices, and their generalizations.

1. Introduction

Let G be a simple graph, $V(G)$, be its vertex-set and $E(G)$ be its edge-set. An assignment of colors to the vertices of G in such a way that adjacent vertices are assigned with different colors is called a (proper) coloring of G. The minimum number of colors needed to color G, is called the chromatic number of G, and is denoted by $\chi(G)$. Let λ be a positive integer. Then a λ-coloring of G is a partition of $V(G)$ into λ color classes such that the vertices in the same color class are not adjacent. If every $\chi(G)$-coloring of G gives the same partition of G, then G is said to be a unique $\chi(G)$-colorable graph. A chromatic polynomial, $P(G,\lambda)$, in λ is the number of ways of λ-coloring of G. Two graphs G and H are said to be chromatically equivalent, if they are nonisomorphic and $P(G,\lambda) = P(H,\lambda)$.

In [1], some families of uniquely 3-colorable graphs without triangles were presented. In [4], the author, Osterweil, presented some families of uniquely 3-colorable graphs with triangles. His method was to use the complements of certain graphs. He also stated that the techniques used here seem applicable to the more general study of unique *n*-colorability in graphs. Recently, Chia in [2], by using the same method as Osterweil's, extended Osterweil's result to the case of uniquely *n*-colorable graphs. Here, we shall also consider the complements of certain graphs to prove the following

Theorem 1.

(a) For each integer $n \geq 3$, there exists a uniquely n-colorable graph with $2n$ vertices.

(b) For each integer $n \geq 3$, there exists a uniquely n-colorable graph with $2n+1$ vertices.

Using our Theorem 1, we shall prove

Theorem 2.

(a) *For each integer* $n \geq 3$, *there exist two uniquely* $(n+1)$*-colorable graphs with* $2n+2$ *vertices which are chromatically equivalent.*

(b) *For each integer* $n \geq 3$, *there exist two uniquely* $(n+1)$*-colorable graphs with* $2n+3$ *vertices which are chromatically equivalent.*

We need the following well known Theorem (see p. 55 in [3]) for our proofs and examples: Let G be a graph. Then

$$P(G,\lambda) = P(G-e,\lambda) - P(G/e,\lambda) \tag{1}$$

where $G-e$ is the graph obtained from G by deleting an edge e in G, and G/e is the graph obtained from G by contracting the edge e. Or

$$P(G,\lambda) = P(G+e',\lambda) + P((G+e')/e',\lambda) \tag{2}$$

where $e' \notin E(G)$ and $G+e'$ is the graph obtained from G by adding the edge e' into G.

2. Examples, proofs and generalizations

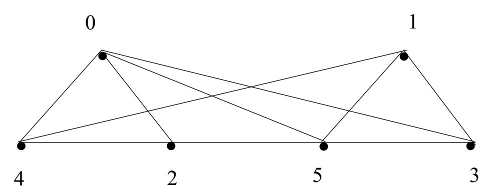
The following examples lead to the proof for the general case, i.e., they lead to the proofs for Theorem 1(a) and (b).

Example 1. Let P_5 be the following path of length 5,

P_5

and G_6 be the complement of P_5 in the complete graph, K_6, with 6 vertices. Thus, G_6 is the following graph:

G_6



Since G_6 contains the triangles $K_3' = <0,2,4>$ and $K_3'' = <1,3,5>$, $\chi(G_6) \geq 3$. We claim that G can be colored by 3 colors, α, β and γ with color indifference. We shall use the following notations: $i(\alpha)$ means the vertex i is colored by the color α, and $\rightarrow i(\alpha)$ means the vertex i is forced to be colored with the color α. We color $1(\alpha)$, $3(\beta)$ and $5(\gamma)$. Since the neighborhood of the vertex $0, N(0) \supset \{3(\beta), 5(\gamma)\}, \rightarrow 0(\alpha)$. Similarly, since $N(2) \supset \{0(\alpha), 5(\gamma)\}, \rightarrow 2(\beta)$, and since $N(4) = \{1(\alpha), 2(\beta), 0(\alpha)\}, \rightarrow 4(\gamma)$. Thus, $\chi(G_6) = 3$. Since there is no choice of colors for each vertex in G_6, G_6 is uniquely 3-colorable, and the 3 color classes are

$$\{0,1\},\ \{2,3\},\ \{4,5\}.$$

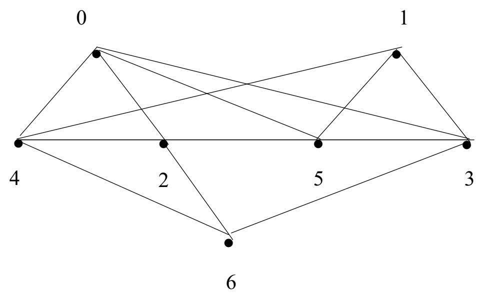
We also can show that G_6 is a uniquely 3-colorable graph by using its chromatic polynomial. The chromatic polynomial of G_6 is, by adding $e_1 = (0,1)$ to G_6 and using (2) and deleting $e_2(1,4)$ in $G_6 + e_1$ and using (1),

$$\begin{aligned} P(G_6,\lambda) &= (P(G_6 + e_1, \lambda) + P(G_6 / e_1, \lambda) \\ &= P((G_6 + e_1) - e_2, \lambda) - e_2, \lambda) - P(G_6 + e_1)/e_2, \lambda) + P(G_6 / e_1, \lambda) \\ &= \lambda(\lambda-1)(\lambda-2)^3(\lambda-3) - \lambda(\lambda-1)(\lambda-2)(\lambda-3)^2 + \lambda(\lambda-1)(\lambda-2)^3. \end{aligned}$$

The Lemma 1 in [1] states: Let $\chi(G) = k$. Then $P(G,k) = k! \cdot t$ for some positive integer t, and t is the number of ways of coloring G in exactly k colors with color indifference. Furthermore, $t = 1$ if and only if G is a uniquely k-colorable graph. Hence, with $\lambda = 3$, $P(G_6, 3) = 0 - 0 + 3 \cdot 2 \cdot 1^3 = 3!$, and G_6 is a uniquely 3-colorable graph.

Example 2. Let H_7' be the graph with 7 vertices consisting of a 7-cycle and a triangle, i.e., $V(H_7') = \{0, 1, \cdots, 6\}$ and $E(H_7') = \{(0,1), (1,2), (2,3), (3,4), (4,5), (5,6), (6,0), (6,1)\}$, and H_7 be the complement of H_7' in K_7. Thus, H_7 is the following graph:

H_7



Since H_7 contains triangles, $\chi(H_7) \geq 3$. We claim that H_7 can be colored by 3 colors α, β and γ with color indifference. We color $1(\alpha), 3(\beta)$ and $5(\gamma)$. Then since $N(0) \supset \{3(\beta), 5(\gamma)\}, \rightarrow 0(\alpha)$.

Similarly, since $N(2) \supset \{0(\alpha),\ 5(\gamma)\},\ \rightarrow\ 2(\beta)$. Since $N(4) \supset \{0(\alpha),\ 2(\beta),\ 1(\alpha)\},\ \rightarrow\ 4(\gamma)$. Since $N(6) = \{2(\beta),\ 3(\beta),\ 4(\gamma)\},\ \rightarrow\ 6(\alpha)$. Thus, $\chi(H_7) = 3$. Since there is no choice of colors for each vertex in H_7, H_7 is uniquely 3-colorable, and the 3 color classes are

$$\{0, 1, 6\},\ \{2, 3\},\ \{4, 5\}$$

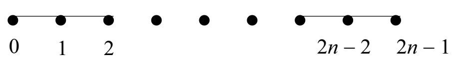
We also show that H_7 is a uniquely 3-colorable graph by using its chromatic polynomial. Repeatedly using (1) and (2), we have

$$P(H_7, \lambda) = P(H_7 - e_1, \lambda) - P(H_7 / e_1, \lambda) \qquad \text{(where } e_1 = (1, 4))$$
$$= (P(H_7 - e_1) + e_2, \lambda) + P(H_7 - e_1) / e_2, \lambda)) - (P(H_7 / e_1) + e_3, \lambda) + P(H_7 / e_1) / e_3, \lambda))$$
$$\text{where } e_2 = (0, 1) \text{ and } e_3 (2, 3))$$
$$= (P((H_7 - e_1 + e_2) - e_4, \lambda) - P((H_7 - e_1 + e_2) / e_4, \lambda)) -$$
$$(P((H_7 / e_1) + e_3 - e_4, \lambda) - P((H_7 / e_1 + e_3 / e_4, \lambda))) -$$
$$(P(((H_7 / e_1) / e_3) - e_4, \lambda) - P(((H_7 / e_1 / e_3) / e_4, \lambda)) \qquad \text{(where } e_4 = (6, 3))$$
$$= \lambda(\lambda-1)(\lambda-2)^4(\lambda-3) - \lambda(\lambda-1(\lambda-3)^3 + \lambda(\lambda-1)(\lambda-2)^4$$
$$- \lambda(\lambda-1)(\lambda-2)(\lambda-3)^2 - \lambda(\lambda-1)(\lambda-2)^2(\lambda-3)(\lambda-4) - \lambda(\lambda-1)^2(\lambda-2)(\lambda-3).$$

Hence, $P(H_7, 3) = 0 - 0\ +\ 3 \cdot 2 \cdot 1^4 - 0 - 0 - 0\ =\ 3!$, and H is uniquely 3-colorable.

The proof of Theorem 1(a) goes as follows. For any integer $n \geq 3$, let P_{2n-1} be the following (simple) path of length $2n-1$:

P_{2n-1}



and G_{2n} be the complement of P_{2n-1} in the complete graph $K_{2n} = \ < 0, 1, \cdots, (2n-1) >$. Thus, G_{2n} contains two complete subgraphs with n vertices, namely,

$$K_n' = \ < 0, 2, \cdots, 2t, \cdots, (2n-2) > \text{ and}$$
$$K_n'' = \ < 1, 3, \cdots, (2t+1), \cdots, (2n-1) >.$$

Clearly, $\chi(G_{2n}) \geq n$. We claim that G_{2n} can be colored by n colors, $\alpha_1, \alpha_2, \cdots, \alpha_n$ with color indifference. We color $1(\alpha_1)$, $3(\alpha_2)$, $\cdots, (2k-1)(\alpha_k)$, $\cdots$, $(2n-1)(\alpha_n)$. Since $N(0) \supset \{(2k-1)(\alpha_k)$ for $k = 2, 3, \cdots, n\}$, $\rightarrow 0(\alpha_1)$. Similarly, since $N(2t) \supset \{2p-2)(\alpha_p)$ for $p = 1, 2, \cdots, t$, and $(2k-1)(\alpha_k)$ for $k = t+2, t+3, \cdots, n\}$, $\rightarrow 2t(\alpha_{t+1})$ for $t = 1, 2, \cdots, n-2$. Also, since $N(2n-2) = \{(2p-2)(\alpha_p)\}$ for $p = 1, 2, \cdots, n-1$ and $(2k-1)(\alpha_k)$ for $1, 2, \cdots, n-1$, $\rightarrow (2n-2)(\alpha_n)$. Thus, $\chi(G_{2n}) = n$. Since there is no choice of colors for each vertex in G_{2n}, G_{2n} is uniquely n-colorable, and the *n* color classes are

$$\{0,1\},\ \{2,3\},\ \cdots, \{2k, 2k+1\}, \cdots, \{2n-2, 2n-1\}.$$

The proof of Theorem 1(b) goes as follows. For any integer $n \geq 3$, let H'_{2n+1} be the graph with $2n+1$ vertices consisting of $a(2n+1)$-cycle and a triangle, i.e., $V(H'_{2n+1}) = \{0, 1, \cdots, 2n\}$ and $E(H'_{2n+1}) = \{(0,1), (1,2), \cdots, (i, i+1), \cdots, (2n, 0), (2n, 1)\}$, and H_{2n+1} be the complement of H'_{2n+1} in the complete graph $K_{2n+1} = < 0, 1, \cdots, 2n >$. Thus, H_{2n+1} contains two complete subgraphs with *n* vertices, namely,

$$K'_n = < 0, 2, \cdots, 2t, \cdots, (2n-2) > \text{ and}$$

$$K''_n = < 1, 3, \cdots, (2t+1), \cdots, (2n-1) >.$$

Clearly, $\chi(H_{2n+1}) \geq n$. We claim that H_{2n+1} can be colored by *n* colors $\alpha_1, \alpha_2, \cdots, \alpha_n$ with color indifference. We color $1(\alpha_1)$, $3(\alpha_2), \cdots, (2k-1)(\alpha_k)$, $\cdots, (2n-1)(\alpha_n)$. Since $N(0) \supset \{(2k-1)(\alpha_k)$ for $k = 2, 3, \cdots, n\}, \rightarrow 0(\alpha_1)$. Similarly, since $N(2t) \supset \{2p - 2)(\alpha_p)$ for $p = 1, 2, \cdots, t$, and $(2k-1)(\alpha_k)$ for $k = t+2$, $t + 3, \cdots, n\}$, $\rightarrow 2t(\alpha_{t+1})$ for $t = 1, 2, \cdots, n-1$. Also, since $N(2n) = \{2p-2)(\alpha_p)$ for $p = 2, 3, \cdots, n$, and $(2k-1)(\alpha_k)$ for $k = 2, 3, \cdots, n-1\}$, $\rightarrow 2n(\alpha_1)$. Thus, $\chi(H_{2n+1}) = n$. Since there is no choice of colors for each vertex in H_{2n+1}, H_{2n+1} is uniquely *n*-colorable, and the *n* color classes are

$$\{0, 1, 2n\},\ \{2, 3\},\ \cdots,\ \{2k, 2k+1),\ \cdots, \{2n-2,\ 2n-1\}.$$

Following Osterweil's idea about 6-clique rings in [3], we let C_i be the complete graph with m_i vertices where $m_i \geq 1$. For $i \neq j$, we say that C_i and C_j are adjacent, denoted by $C_i \leftrightarrow C_j$, if every vertex in C_i is adjacent to all vertices in C_j.

For each integer $n \geq 3$, let $\overline{P}_{2n-1}$ be the graph with $V(\overline{P}_{2n-1}) = \bigcup_{i=1}^{2n-1} V(C_i)$ and $E(\overline{P}_{2n-1}) = \bigcup_{i=0}^{2n-2} E(C_i \leftrightarrow C_{i+1})$. Thus, $\overline{P}_{2n-1}$ has $m = \sum_{i=0}^{2n-1} m_i$ vertices and $E(\overline{P}_{2n-1})$ has $\sum_{i=0}^{2n-1} \binom{m_i}{2} + \sum_{i=0}^{2m-2} (m_i)(m_{i+1})$ edges. Let $\overline{G}_{2n}$ be the complement of $\overline{P}_{2n-1}$ in the complete graph K_m. Then we have:

Corollary 1(a). *$\overline{G}_{2n}$ is a uniquely n-colorable graph with m vertices.*

Proof. Let N_i be the null graph with m_i vertices for $i = 0, 1, \cdots, 2n-1$. The proof is similar to the proof of Theorem 1(a) by replacing the vertices i in G_{2n} by N_i for $i = 0, 1, \cdots, 2n-1$.

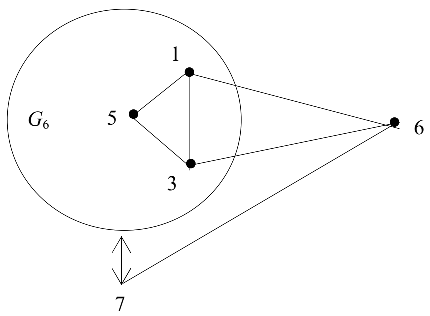
For each integer $n \geq 3$, let $\overline{H}'_{2n+1}$ be the graph with $V(\overline{H}'_{2n+1}) = \bigcup_{i=0}^{2n} V(C_i)$ and $E(\overline{H}'_{2n+1}) = (\bigcup_{i=0}^{2n-1} E(C_i \leftrightarrow C_{i+1})) \cup (E(C_{2n} \leftrightarrow C_0)) \cup (E(C_{2n} \leftrightarrow C_1))$, and $\overline{H}_{2n+1}$ be the complement of $\overline{H}'_{2n+1}$ in the complete graph K_q where $q = \sum_{i=0}^{2n} m_i$. Then we have:

Corollary 1(b). *$\overline{H}_{2n+1}$ is a uniquely n-colorable graph with q vertices.*

Proof. Let N_i be the null graph with m_i vertices for $i = 0, 1, \cdots, 2n$. The proof is similar to the proof of Theorem 1(b) by replacing the vertices i in H_{2n+1} by N_i for $i = 0, 1, \cdots, 2n$.

Example 3. Let G_6 be the graph in our Example 1, and Q_8 be the graph with $V(Q_8) = V(G_6) \cup \{6, 7\}$ and $E(Q_8) = E(G_6) \cup \{(6,1), (6,3)\} \cup \{(7, i)$ for $i = 0, 1, 2, 3, 4, 5, 6\}$. Thus, Q_8 is the following graph:

Q_8

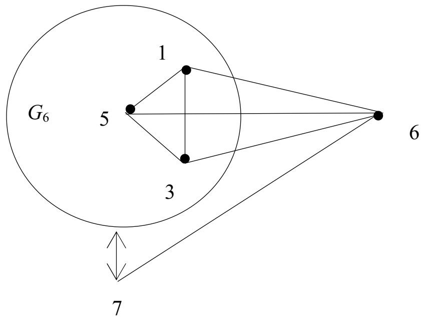


Since Q_8 contains a complete graph with 4 vertices, $\chi(Q_8) \geq 4$. We claim that Q_8 is uniquely 4-colorable. Since G_6 is a uniquely 3-colorable graph with colors α, β and γ, the vertex 7 in Q_8 has to be colored by a new color δ. Since $N(6) = \{1(\alpha), 3(\beta), 7(\delta)\}, \rightarrow 6(\gamma)$. Since there is no choice of colors for each vertex in Q_8, Q_8 is uniquely 4-colorable, and the 4 color classes are:

$$\{0, 1\},\ \{2, 3\},\ \{4, 5, 6\},\ \{7\}.$$

Let R_8 be the graph with $V(R_8) = V(G_6) \cup \{6, 7\}$ and $E(R_8) = E(G_6) \cup \{(6,1), (6,3), (6,5)\} \cup \{(7, i)$ for $i = 0, 1, 2, 3, 4, 5\}$. Thus, R_8 is the following graph:

R_8

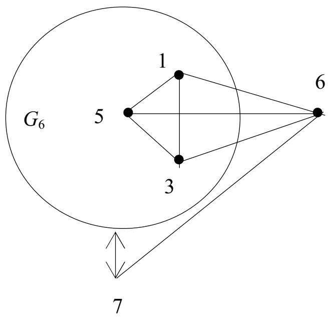


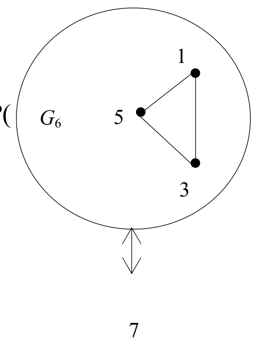
Since R_8 contains a complete graph with 4 vertices, $\chi(R_8) \geq 4$. We claim that R_8 is uniquely 4-colorable. Since G_6 is a uniquely 3-colorable graph with colors α, β and γ, $\rightarrow 7(\delta)$. Since there is no choice of colors for each vertex in R_8, R_8 is uniquely 4-colorable, and the 4 color classes are

$$\{0, 1\},\ \{2, 3\},\ \{4, 5\},\ \{6,7\}.$$

We claim that Q_8 and R_8 are chromatically equivalent. By using (2) and (1), we have

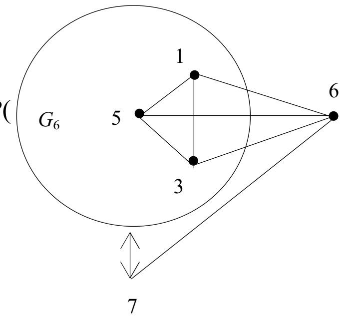
$P(Q_8, \lambda) = P(Q_8 + (5, 6), \lambda) + P(Q_8/(5, 6), \lambda)$, *i.e.,*

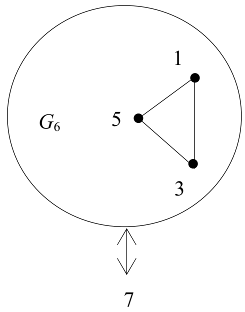
$P(Q_8, \lambda) = P($ 

$, \lambda) + P($ 

$, \lambda)$ (3)

$P(R_8,\lambda) = P(R_8 + (7,6),\lambda) + P(R_8/(7,6),\lambda)$, *i.e.,*

$P(Q_8,\lambda) = P($ 

$,\lambda) + P($ 

$,\lambda)$ (4)

Since the polynomials on the right sides of (3) and (4) are the same, $P(Q_8,\lambda) = P(R_8,\lambda)$. Since the degree of vertex 7 in Q_8 is 7 and no vertex in R_8 is of degree 7, $Q_8 \ncong R_8$ and Q_8 and R_8 are chromatically equivalent.

By using some of the properties of chromatic polynomials (in [3]), we have

$P(Q_8,\lambda) = \lambda(P(G_6,\lambda-1) \times ((\lambda-1)-2))$

$= \lambda(\lambda-1)(\lambda-2)(\lambda-3)^4(\lambda-4) - \lambda(\lambda-1)(\lambda-2)(\lambda-3)^2(\lambda-4)^2 + \lambda(\lambda-1)(\lambda-2)(\lambda-3)^4$

and

$P(R_8,\lambda) = (\lambda P(G_6,\lambda-1) \times (\lambda-3)$

$= [\lambda(\lambda-1)(\lambda-2)(\lambda-3)^3(\lambda-4) - \lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)^2 + \lambda(\lambda-1)(\lambda-2)(\lambda-3)]\,(\lambda-3)$

$= (\lambda(\lambda-1)(\lambda-2)(\lambda-3)^4(\lambda-4) - \lambda(\lambda-1)(\lambda-2)(\lambda-3)^2(\lambda-4)^2 + \lambda(\lambda-1)(\lambda-2)(\lambda-3))^4$

Thus, $P(Q_8,\lambda) = P(R_8,\lambda)$. Also, $P(Q_8,4) = P(R_8,4) = 4!$, i.e., Q_8 and R_8 are uniquely 4-colorable graphs, and they are chromatically equivalent.

Similarly, we may use the graph H_7 in our Example 2 to construct graphs Q_9 and R_9 as follows. Let $V(Q_9) = V(H_7) \cup \{7,8\}$, and $E(Q_9) = E(H_7) \cup \{(7,1),$ $(7,3),(7,5),\} \cup \{(8,i)$ for $i = 0,\cdots,7\}$. $\cup \{(7,1),(7,3)\} \cup \{(8,i)$ Also, let $V(R_9) = V(H_7) \cup \{7,8\}$, and $E(R_9) = E(H_7) \cup \{(7,1),(7,3),(7,5)\} \cup \{(8,i)$ for $i = 0,1,\cdots,6\}$. Then $P(Q_9,\lambda) = \lambda(P(H_7,\lambda-1)\cdot((\lambda-1)-2)$, and $P(R_9,\lambda)$ $= (\lambda P(H_7,\lambda-1))\cdot(\lambda-3)$. Clearly, Q_9 and R_9 are not isomorphic. We can show that $P(Q_9,\lambda) = P(R_9,\lambda)$ and $P(Q_9,4) = P(R_9,4) = 4!$, i.e., Q_9 and R_9 are uniquely 4-colorable, and they are chromatically equivalent.

The proof of Theorem 2 (a) goes as follows. The graph G_{2n} in Theorem 1 (a) is a uniquely n-colorable graph with $2n$ vertices containing two complete subgraphs with n vertices, namely,

$$K'_n = <0,2,\cdots,2t,\cdots,(2n-2)> \text{ and}$$

$$K''_n = <1,3,\cdots,(2t+1),\cdots,(2n-1)>.$$

We construct a graph Q_{2n+2} with $V(Q_{2n+2}) = V(G_{2n}) \cup \{2n, 2n+1\}$, and $E(Q_{2n+2}) = E(G_{2n}) \cup \{(2n, i)$ for $i = 1, 3, \cdots, (2t+1), \cdots, (2n-3)\} \cup \{(2n+1, j)$ for $j = 0, 1, \cdots, 2n\}$. Since G_{2n} is uniquely n-colorable and the vertex $2n$ is incident with every vertex in K_n'' except the vertex $(2n-1)(\alpha_n)$, $\rightarrow (2n)(\alpha_n)$. Since the vertex $2n+1$ is incident with every vertex i for $i = 0, 1, \cdots, 2n, \rightarrow (2n+1)(\alpha_{n+1})$. Thus, Q_{2n+2} is uniquely $(n+1)$-colorable with the color classes.

$$\{0, 1\}, \{2, 3\}, \cdots, \{2k, 2k+1\}, \cdots, \{2n-2, 2n-1, 2n\}, \{2n+1\}.$$

Let R_{2n+2} be the graph with $V(R_{2n+2}) = V(G_{2n}) \cup \{2n, 2n+1\}$, and $E(R_{2n+2}) = E(G_{2n}) \cup \{(2n, i)$ for $i = 1, 3, \cdots, (2t+1), \cdots, (2n-1)\} \cup \{(2n+1, j)$ for $j = 0, 1, \cdots, (2n-1)\}$.

Since G_{2n} is uniquely n-colorable and the vertex $2n$ is incident with every vertex in K_n'', $\rightarrow (2n)(\alpha_{n+1})$. Since the vertex $2n+1$ is incident with every vertex in G_{2n}, $\rightarrow (2n+1)(\alpha_{n+1})$. Thus, R_{2n+2} is uniquely $(n+1)$-colorable with the color classes.

$$\{0, 1\}, \{2, 3\}, \cdots, \{2k, 2k+1\}, \cdots, \{2n-2, 2n-1\}, \{2n, 2n+1\}.$$

We claim that Q_{2n+2} and R_{2n+2} are not isomorphic. In Q_{2n+2}, the degree of vertex $2n+1$ is $2n+1$, and in R_{2n+2}, none of the vertices is of degree $2n+1$. Hence, Q_{2n+2} and R_{2n+2} are not isomorphic.

We claim that Q_{2n+2} and R_{2n+2} are chromatically equivalent. By using (2) , we have

$P(Q_{2n+2}, \lambda) = P(Q_{2n+2} + (2n, 2n-1), \lambda) + P(Q_{2n+2}/(2n, 2n-1), \lambda)$, and
$P(R_{2n+2}, \lambda) = P(R_{2n+2} + (2n, 2n+1), \lambda) + P(R_{2n+2}/(2n, 2n+1), \lambda)$. Since
$P(Q_{2n+2} + (2n, 2n-1), \lambda) = P(R_{2n+2} + (2n, 2n+1), \lambda)$ and
$P(Q_{2n+2}/(2n, 2n-1), \lambda) = P(R_{2n+2}/(2n, 2n+1), \lambda), P(Q_{2n+2}, \lambda) = P(R_{2n+2}, \lambda)$.

Hence, Q_{2n+2} and R_{2n+2} are chromatically equivalent, and are uniquely $(n+1)$- colorable.

The proof of Theorem 2(b) goes as follows. The graph H_{2n+1} in Theorem 1(b) is a uniquely n-colorable graph with $2n+1$ vertices containing two complete subgraphs with n vertices, namely,

$$K_n' = \langle 0, 2, \cdots, 2t, \cdots, (2n-2) \rangle \text{ and}$$

$$K_n'' = \langle 1, 3, \cdots, (2t+1), \cdots, (2n-1) \rangle.$$

Let Q_{2n+3} be the graph with $V(Q_{2n+3)} = V(H_{2n+1}) \cup \{2n+1, 2n+2\}$ and $E(Q_{2n+3}) = E(H_{2n+1}) \cup \{(2n+1, i)$ for $i = 1, 3, \cdots, (2t+1), \cdots, (2n-3)\} \cup \{(2n+2, j)$ for $j = 0, 1, \cdots, 2n+1\}$, and R_{2n+3} be the graph with $V(R_{2n+3}) = V(H_{2n+1}) \cup \{2n+1, 2n+2\}$ and $E(R_{2n+3}) = E(H_{2n+1}) \cup \{(2n+1, i)$ for $i = 1, 3, \cdots, (2t+1), \cdots, (2n-1)\} \cup \{(2n+2, j)$ for $j = 0, 1, \cdots, 2n\}$. Similar to the proof of Theorem 2(a), Q_{2n+3} and R_{2n+3} are uniquely $(n+1)$-colorable with the color classes.

$$\{0, 1, 2n\},\ \{2, 3\}, \cdots, \{2k, 2k+1\}, \cdots, \{2n-2, 2n-1, 2n+1\}, \{2n+2\} \text{ in } Q_{2n+3},$$

and

$$\{0, 1, 2n\},\ \{2, 3\}, \cdots, \{2k, 2k+1\}, \cdots, \{2n-2, 2n-1\}, \{2n+1, 2n+2\} \text{ in } R_{2n+3}.$$

Also, similar to the proof in Theorem 2(a), Q_{2n+3} and R_{2n+3} are not isomorphic, and $P(Q_{2n+3}, \lambda) = P(R_{2n+3}, \lambda)$. Hence, Q_{2n+3} are R_{2n+3} chromatically equivalent, and are uniquely $(n+1)$-colorable.

In the corollary 1(a) and (b), both of $\overline{G}_{2n}$ and $\overline{H}_{2n}$ contain two "generalized complete" subgraphs $\overline{K}_n$ and $\overline{K}'_n$ with

$V(\overline{K}_n) = \{N_i,\ N_i$ is a null graph with m_i vertices for $i = 0, 2, \cdots, 2k, \cdots, (2n-2)\}$,
$E(\overline{K}_n) = \{N_i \leftrightarrow N_j;\ i \neq j,\ i, j = 0, 2, \cdots, 2k, \cdots, (2n-2)\}$,
$V(\overline{K}'_n) = \{N_i;\ N_i$ is a null graph with m_i vertices for $i = 0, 2, \cdots, 2k, \cdots, (2n-2)\}$,
and $E(\overline{K}'_n) = \{N_i \leftrightarrow N_j;\ i \neq j$ and $i, j = 1, 3, \cdots, (2k-1), \cdots, (2n-1)\}$.

Corollary 2(a). *Let $\overline{G}_{2n} = \langle N_0, N_1, \cdots, N_{2n-1} \rangle$ be the graph in corollary 1(a), Q_{2n+2} be the graph with $V(\overline{Q}_{2n+2}) = V(\overline{G}_{2n}) \cup \{N_{2n}, N_{2n+1}\}$ where N_{2n} and N_{2n+1} are null graphs with m_{2n} and m_{2n+1} vertices respectively, and $E(\overline{Q}_{2n+2}) = E(\overline{G}_{2n}) \cup \{N_{2n} \leftrightarrow N_i$ for $i = 1, 3, \cdots, (2t+1), \cdots, (2n-3)\} \cup \{N_{2n+1} \leftrightarrow N_j$ for $j = 0, 1, \cdots, 2n\}$ and $\overline{R}_{2n+2}$ be the graph with $V(\overline{R}_{2n+2}) = V(\overline{G}_{2n}) \cup \{N_{2n}, N_{2n+1}\}$ and $E(\overline{R}_{2n+2}) = E(\overline{G}_{2n}) \cup \{N_{2n} \leftrightarrow N_i$ for $i = 1, 3, \cdots, (2t+1), \cdots, (2n-1)\} \cup \{N_{2n+1} \leftrightarrow N_j$ for $j = 0, 1, \cdots, 2n-1\}$. Then $\overline{Q}_{2n+2}$ and $\overline{R}_{2n+2}$ are uniquely $(n+1)$-colorable, and are chromatically equivalent.*

Corollary 2(b). *Let* $\overline{H}_{2n+1} = < N_0, N_1, \cdots, N_{2n} >$ *be the graph in corollary* 1(*b*), Q_{2n+3} *be the graph with* $V(\overline{Q}_{2n+3}) = V(\overline{H}_{2n+1}) \cup \{N_{2n+1}, N_{2n+2}\}$ *where* N_{2n+1} *and* N_{2n+2} *are null graphs with* m_{2n+1} *and* m_{2n+2} *vertices respectively, and* $E(\overline{Q}_{2n+3}) = E(\overline{H}_{2n+1}) \cup \{N_{2n+1} \leftrightarrow N_i$ *for* $i = 1, 3, \cdots, (2t+1), \cdots, (2n-3)\} \cup \{N_{2n+2} \leftrightarrow N_j$ *for* $j = 0, 1, \cdots, 2n+1\}$, *and* $\overline{R}_{2n+3}$ *be the graph with* $V(\overline{R}_{2n+3} = V(\overline{H}_{2n+1}) \cup \{N_{2n+1}, N_{2n+2}\}$ *and* $E(\overline{R}_{2n+3}) = E(\overline{H}_{2n+1}) \cup \{N_{2n+1} \leftrightarrow N_i$ *for* $i = 1, 3, \cdots, (2t+1), \cdots, (2n-1)\} \cup \{N_{2n+2} \leftrightarrow N_j$ *for* $j = 0, 1, \cdots, 2n\}$. *Then* Q_{2n+3} *and* R_{2n+3} *are uniquely* $(n+1)$-*colorable and are chromatically equivalent.*

Proof.

(a) It is similar to the proof of Theorem 2(a) by replacing the vertices i in Q_{2n+2} and R_{2n+2} in the Theorem 2(a) by N_i for $i = 0, 1, \cdots, 2n+1$.

(b) It is similar to the proof of Theorem 2(b) by replacing the vertices i in Q_{2n+3} and R_{2n+3} by N_i for $i = 0, 1, \cdots, 2n+3$.

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