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Entire Functions and their Derivatives Share Two Finite Sets

MING LIANG FANG

Department of Mathematics, Nanjing Normal University, Nanjing 210097, P. R. China

State Key Laboratory of Millimeter Waves, Southeast University, Nanjing 210096, P. R. China

e-mail: mlfang@pine.njnu.edu.cn

Abstract. In this paper, we study the uniqueness of entire functions. We mainly obtain the following result: Let $f(z)$ and $g(z)$ be two non-constant entire functions, $n \geq 5$, k two positive integers, and let $S_1 = \{z : z^n = 1\}, S_2 = \{a, b, c\}$ where a, b, c are nonzero finite distinct constants satisfying $a^2 \neq bc, b^2 \neq ac, c^2 \neq ab$. If $E(S_1, f) = E(S_1, g)$, $E(S_2, f^{(k)}) = E(S_2, g^{(k)})$, then $f(z) \equiv g(z)$.

1. Introduction and main results

Let $f(z)$ be a non-constant meromorphic function in the whole complex plane. In this paper we use the following standard notations of value distribution theory,

$$T(r, f),\ m(r, f),\ N(r, f),\ \overline{N}(r, f),\ N\left(r, \frac{1}{f}\right),\ \cdots$$

(see Hayman [7], Yang [9]). We denote by $S(r, f)$ any function satisfying

$$S(r, f) = o\{T(r, f)\},$$

as $r \to +\infty$, possibly outside of a set with finite measure.

Let S be a set of complex numbers. Set

$$E(S, f) = \bigcup_{a \in S} \{z : f(z) - a = 0\},$$

where a zero point with multiplicity m is counted m times in the set.

In 1977, Gross [5] posed the following question.

Question 1. Can one find two finite sets $S_j\,(j=1,2)$ such that any two non-constant entire functions *f* and *g* satisfying $E(S_j, f)=E(S_j, g)$ for $j=1,2$ must be identical?

Yi [11] gave a positive answer to the question. He proved

Theorem A. *Let $f(z)$ and $g(z)$ be two non-constant entire functions, $n\ge 5$ a positive integer, and let $S_1=\{z: z^n=1\}, S_2=\{a\}$, where $a\ne 0$ is a constant satisfying $a^{2n}\ne 1$. If $E(S_j, f)=E(S_j, g)$ for $j=1,2$, then $f(z)\equiv g(z)$.*

In this paper, we have proved

Theorem 1. *Let $f(z)$ and $g(z)$ be two non-constant entire functions, $n\ge 5$, k two positive integers, and let $S_1=\{z: z^n=1\}, S_2=\{a,b,c\}$, where a, b, c are nonzero finite distinct constants satisfying $a^2\ne bc,\ b^2\ne ac,\ c^2\ne ab$.* If $E(S_1, f)=E(S_1, g)$, $E(S_2, f^{(k)})=E(S_2, g^{(k)})$, then $f(z)\equiv g(z)$.

Remark 1. The following example shows that the condition that *a, b, c* are nonzero finite distinct constants satisfying $a^2\ne bc,\ b^2\ne ac,\ c^2\ne ab$ in Theorem 1 is necessary.

Example 1. Let $S_1=\{z: z^n=1\}, S_2=\{a,b,\sqrt{ab}\}$, where *a, b* are two distinct nonzero constants. Taking $f(z)=e^{\sqrt{-ab}z}$, $g(z)=e^{-\sqrt{-ab}z}$. Obviously, $E(S_1, f)=E(S_1, g)=\{z: e^{n\sqrt{-ab}z}=1\}$, $E(S_2, f')=E(S_2, g')=\left\{z: e^{\sqrt{-ab}z}=a/\sqrt{-ab}\ a \text{ or } e^{\sqrt{-ab}z}=\frac{b}{\sqrt{-ab}} \text{ or } e^{\sqrt{-ab}z}=\frac{\sqrt{ab}}{\sqrt{-ab}}\right\}$, but $f(z)\not\equiv g(z)$.

When S_2 has two elements or one element, we have the following results.

Theorem 2. *Let $f(z)$ and $g(z)$ be two non-constant entire functions, $n\ge 5$, k two positive integers, and let $S_1=\{z: z^n=1\}, S_2=\{a,b\}$, where a, b are two nonzero finite distinct constants. If $E(S_1, f)=E(S_1, g)$, $E(S_2, f^{(k)})=E(S_2, g^{(k)})$, then one of the following cases must occur:*

(1) $f(z)\equiv g(z)$;

(2) $b=-a$, $f(z)=e^{cz+d}, g(z)=te^{-cz-d}$, *where c, d, t are three constants satisfying $t^n=1$ and $(-1)^k tc^{2k}=a^2$;*

(3) $f(z)=e^{cz+d}$, $g(z)=te^{-cz-d}$, *where c, d, t are three constants satisfying* $t^n=1$ *and* $(-1)^k tc^{2k}=ab$ *;*

(4) $b=-a$, $f(z)\equiv -g(z)$.

Theorem 3. *Let* $f(z)$ *and* $g(z)$ *be two non-constant entire functions,* $n\geq 5$, k *two positive integers, and let* $S_1=\{z:z^n=1\}, S_2=\{a\}$ *, where* $a\neq 0,\infty$. *If* $E(S_1,f)=E(S_1,g)$, $E(S_2,f^{(k)})=E(S_2,g^{(k)})$, *then one of the following cases must occur:*

(1) $f(z)\equiv g(z)$;

(2) $f(z)=e^{cz+d}$, $g(z)=te^{-cz-d}$, *where c, d, t are three constants satisfying* $t^n=1$ *and* $(-1)^k tc^{2k}=a^2$.

2. Some lemmas

For the proof of our theorems we need the following lemmas.

Lemma 1 ([3]). *Let* $f(z)$ *be a non-constant entire function, and let* $k\geq 2$ *be a positive integer. If* $f(z)f^{(k)}(z)\neq 0$, *then* $f=e^{az+b}$ *, where* $a\neq 0$, b *are constants.*

Lemma 2 ([11]). *Let* $f(z)$ *and* $g(z)$ *be two transcendental entire functions,* $n\geq 5$ *a positive integer, and let* $S=\{z:z^n=1\}$. *If* $E(S,f)=E(S,g)$, *then either* $f(z)g(z)\equiv t$ *or* $f(z)\equiv tg(z)$, *where t is a constant satisfying* $t^n=1$.

3. Proof of Theorem 1

First we consider the case when $f(z)$ and $g(z)$ are two transcendental entire functions. By Lemma 2 we know that either $f(z)g(z)\equiv t$ *or* $f(z)\equiv tg(z)$, where *t* is a constant satisfying $t^n=1$. Next we divide two cases.

Case 1. $f(z)g(z)\equiv t$, where *t* is a constant satisfying $t^n=1$. Obviously, $f\neq 0$. Hence we have

$$f(z)=e^{h(z)},\qquad g(z)=te^{-h(z)} \tag{3.1}$$

where $h(z)$ is a non-constant entire function. Thus we have

$$f^{(k)} = P(h', \cdots, h^{(k)})\, e^{h}, \quad g^{(k)} = tQ(h', \cdots, h^{(k)})\, e^{-h} \tag{3.2}$$

where P, Q are polynomials of $h', h'', \cdots, h^{(k)}$. Set

$$P(h', h'', \cdots, h^{(k)})\,(z) = P(h'(z), h''(z), \cdots, h^{(k)}(z)),$$
$$Q(h', h'', \cdots, h^{(k)})\,(z) = Q(h'(z), \cdots, h^{(k)}(z)).$$

Obviously there exists z_0 such that $f^{(k)}(z_0) = a$. Then by $E(S_2, f^{(k)}) = E(S_2, g^{(k)})$ and (3.2) we deduce that one of the following cases must occur:

(i) $g^{(k)}(z_0) = a, P(h', h'', \cdots, h^{(k)})(z_0) Q(h', h'', \cdots, h^{(k)})(z_0) - \frac{a^2}{t} = 0$;

(ii) $g^{(k)}(z_0) = b, P(h', h'', \cdots, h^{(k)})(z_0) Q(h', h'', \cdots, h^{(k)})(z_0) - \frac{ab}{t} = 0$;

(iii) $g^{(k)}(z_0) = c, P(h', h'', \cdots, h^{(k)})(z_0) Q(h', h'', \cdots, h^{(k)})(z_0) - \frac{ac}{t} = 0$.

Next we consider four sub-cases.

Case 1.1. $P(h'(z), h''(z), \cdots, h^{(k)}(z))\, Q(h'(z), h''(z), \cdots, h^{(k)}(z)) - \frac{a^2}{t} \not\equiv 0$,

$P(h'(z), h''(z), \cdots, h^{(k)}(z)) Q(h'(z), \cdots, h^{(k)}(z)) - \frac{ab}{t} \not\equiv 0$,

$P(h'(z), h''(z), \cdots, h^{(k)}(z)) Q(h'(z), h''(z), \cdots, h^{(k)}(z)) - \frac{ac}{t} \not\equiv 0$.

Then by $E(S_2, f^{(k)}) = E(S_2, g^{(k)})$ we obtain

$$\begin{aligned}\overline{N}\left(r, \frac{1}{f^{(k)} - a}\right) \le\; & \overline{N}\left(r, \frac{1}{P(h', h'', \cdots, h^{(k)}) Q(h', h'', \cdots, h^{(k)}) - \frac{a^2}{t}}\right) \\ & + \overline{N}\left(r, \frac{1}{P(h', h'', \cdots, h^{(k)}) Q(h', h'', \cdots, h^{(k)}) - \frac{ab}{t}}\right) \\ & + \overline{N}\left(r, \frac{1}{P(h', h'', \cdots, h^{(k)}) Q(h', h'', \cdots, h^{(k)}) - \frac{ac}{t}}\right).\end{aligned} \tag{3.3}$$

By Logarithmic derivative lemma (see [7,9]), we have

$$T(r,h')=m(r,h')=m\left(r,\frac{f'}{f}\right)=S(r,f)\,.$$

Obviously,

$$T(r,h^{(j)})\le T(r,h')+S(r,h')=S(r,f),\ \ (j=2,\cdots,k)\,.$$

Hence we get

$$T(r,P(h',h'',\cdots,h^{(k)}))=S(r,f)\,,\ T(r,Q(h',h'',\cdots,h^{(k)}))\ =\ S(r,f)\,. \qquad (3.4)$$

Thus by (3.3), (3.4) and Nevanlinna first fundamental theorem we have

$$\begin{aligned}\overline{N}\left(r,\frac{1}{f^{(k)}-a}\right)\ &\le T\left(r,P(h',h'',\cdots,h^{(k)})Q(h',h'',\cdots,h^{(k)})-\frac{a^2}{t}\right)\\ &+T\left(r,P(h',h'',\cdots,h^{(k)})Q(h',h'',\cdots,h^{(k)})-\frac{ab}{t}\right)\\ &+T\left(r,P(h',h'',\cdots,h^{(k)})Q(h',h'',\cdots,h^{(k)})-\frac{ac}{t}\right)\ +\ O(1)\ \le\ S(r,f)\,.\end{aligned}$$

By Milloux's inequality (see [7,9]) we obtain

$$T(r,f)\ \le\ \overline{N}(r,f)+N\left(r,\frac{1}{f}\right)\ +\ \overline{N}\left(r,\frac{1}{f^{(k)}-a}\right)\ +\ S(r,f)\,.$$

Hence by the above two formulas and (3.3)-(3.4) we deduce a contradiction: $T(r,f)=S(r,f)$.

Case 1.2. $P(h',h'',\cdots,h^{(k)})\,Q(h',h'',\cdots,h^{(k)})-\frac{a^2}{t}\equiv 0$. Then by (3.2) we deduce that $f^{(k)}(z)g^{(k)}(z)\equiv a^2$ and $f^{(k)}(z)=a$ if and only if $g^{(k)}(z)=a$. Thus we obtain that if $f^{(k)}(z)=b$, then either $g^{(k)}(z)=b$ or $g^{(k)}(z)=c$. If $g^{(k)}(z)=b$ if and only if $f^{(k)}(z)=b$, then we get $g^{(k)}(z)=c$ if and only if $f^{(k)}(z)=c$. Hence we deduce that $a^2=b^2$, $a^2=c^2$. Thus we get either $a=b$ or $a=c$ or $b=c$, which is a contradiction. If there exists z_1 such that $f^{(k)}(z_1)=b$, $g^{(k)}(z_1)=c$, then we get $a^2=bc$, a contradiction.

Case 1.3. $P(h',h'',\cdots,h^{(k)})Q(h',h'',\cdots,h^{(k)}) - \frac{ab}{t} \equiv 0$. Then by (3.2) we deduce that $f^{(k)}(z)g^{(k)}(z) \equiv ab$ and $f^{(k)}(z)=a$ if and only if $g^{(k)}(z)=b$, $f^{(k)}(z)=b$ if and only if $g^{(k)}(z)=a$. Hence by $E(S_2, f^{(k)}) = E(S_2, g^{(k)})$ we deduce that $f^{(k)}(z)=c$ if and only if $g^{(k)}(z)=c$. Thus by (3.2) we get $c^2 = ab$, a contradiction.

Case 1.4. $P(h',h'',\cdots,h^{(k)})Q(h',h'',\cdots,h^{(k)}) - \frac{ac}{t} \equiv 0$. In this case, by using the same argument as do in Case 1.3 we get a contradiction.
Hence we deduce that $f(z)g(z) \equiv t$ is impossible.

Case 2. $f(z) \equiv tg(z)$, where t is a constant satisfying $t^n = 1$. Hence we have $f^{(k)} \equiv tg^{(k)}$.
We claim that $t=1$. Without loss of generality, we assume that there exist z_1 and z_2 such that $f^{(k)}(z_1)=a$ and $f^{(k)}(z_2)=b$. Suppose that $t \neq 1$, then by $E(S_2, f^{(k)}) = E(S_2, g^{(k)})$ and $f^{(k)} \equiv tg^{(k)}$ we deduce that either $g^{(k)}(z_1)=b$ or $g^{(k)}(z_1)=c$ and that either $g^{(k)}(z_2)=a$ or $g^{(k)}(z_2)=c$. Now we discuss the following four cases.

(2.1) $g^{(k)}(z_1)=b$, $g^{(k)}(z_2)=a$. Then by $f^{(k)}(z) \equiv tg^{(k)}(z)$ we get $a=tb$ and $b=ta$. Thus we get $b=-a, t=-1$. If there exists z_3 such that $f^{(k)}(z_3)=c$ then $g^{(k)}(z_3)=-c$. Hence by $E(S_2, f^{(k)}) = E(S_2, g^{(k)})$ we deduce that $-c=a$ or $-c=b$ or $-c=c$. Thus by $b=-a$ we get $c=b$ or $c=a$ or $c=0$, which is a contradiction. If there exists z_3 such that that $g^{(k)}(z_3)=c$, then we can similarly deduce a contradiction. If $f^{(k)}(z) \neq c$ and $g^{(k)}(z) \neq c$, then by $f^{(k)}(z) = -g^{(k)}(z)$ we get $f^{(k)}(z) \neq c, -c$, which contradicts Picard's theorem.

(2.2) $g^{(k)}(z_1)=b$, $g^{(k)}(z_2)=c$. Then by $f^{(k)}(z) \equiv tg^{(k)}(z)$ we get $a=tb$ and $b=tc$. Thus we get $b^2 = ac$, a contradiction.

(2.3) $g^{(k)}(z_1)=c$, $g^{(k)}(z_2)=a$. Then by $f^{(k)}(z) \equiv tg^{(k)}(z)$ we get $a^2 = bc$, a contradiction.

(2.4) $g^{(k)}(z_1)=c$, $g^{(k)}(z_2)=c$. Then by $f^{(k)}(z) \equiv tg^{(k)}(z)$ we get $a=b$, a contradiction.

Hence we deduce that $t=1$, that is $f(z) \equiv g(z)$.

Now we consider the case when $f(z)$ and $g(z)$ are two polynomials. Thus by $E(S_1, f) = E(S_1, g)$ we have

$$f^n(z) - 1 \equiv k[g^n(z) - 1], \tag{3.5}$$

where k is a constant. Hence we have

$$f^{n-1}(z) f'(z) \equiv k g^{n-1}(z) g'(z). \tag{3.6}$$

Thus by (3.6) and $n \geq 5$ we deduce that there exists z_0 such that $f(z_0) = g(z_0) = 0$. Substituting this into (3.5) we get $k = 1$, that is $f^n(z) \equiv g^n(z)$. Hence we get

$$f(z) \equiv t g(z), \tag{3.7}$$

where t is a constant satisfying $t^n = 1$. Thus we have

$$f^{(k)}(z) \equiv t g^{(k)}(z). \tag{3.8}$$

Next by using the similar argument to Case 2 we get $f(z) \equiv g(z)$. The proof of the theorem is complete.

4. Proofs of theorems 2-3

As the proof of Theorem 2 and Theorem 3 is similar, we only give the

Proof of Theorem 2. First we consider the case when $f(z)$ and $g(z)$ are two transcendental entire functions.

By Lemma 2 we know that either $f(z)g(z) \equiv t$, or $f(z) \equiv t g(z)$, where t is a constant satisfying $t^n = 1$. Next we divide two cases.

Case 1. $fg \equiv t$. Obviously, $f \neq 0$. Hence we have

$$f(z) = e^{h(z)}, g(z) = t e^{-h(z)}, \tag{4.1}$$

where $h(z)$ is a non-constant entire function. In the following we consider two sub-cases.

Case 1.1. $k = 1$. Thus we by (4.1) have

$$f'(z) = h'(z) e^{h(z)}, \ g(z) = -t h'(z) e^{-h(z)}. \tag{4.2}$$

Obviously there exists z_0 such that $f'(z_0)=a$. Then by $E(S_2, f')=E(S_2, g')$ and (4.2) we deduce that one of the following cases must occur:

(i) $g'(z_0)=a,\ [h'(z_0)]^2+\frac{a^2}{t}=0$;
(ii) $g'(z_0)=b,\ [h'(z_0)]^2+\frac{ab}{t}=0$.

Next we consider three sub-cases.

Case 1.1.1. $(h'(z))^2+\frac{a^2}{t}\not\equiv 0$, $(h'(z))^2+\frac{ab}{t}\not\equiv 0$. Then by using the same argument as do in Case 1.1 of the proof of Theorem 1 we deduce a contradiction.

Case 1.1.2. $(h'(z))^2+\frac{a^2}{t}\equiv 0$. Then we have $h(z)=cz+d$, where *c*, *d* are two constants satisfying $-tc^2=a^2$. Thus we get

$$f(z)=e^{cz+d},\ g(z)=te^{-cz-d}.$$

Hence we have

$$f'(z)=ce^{cz+d},\ g'(z)=-tce^{-cz-d}.$$

Obviously, $f'(z)=a$ if and only if $g'(z)=a$. Thus by $E(S_2, f')=E(S_2, g')$ we deduce that $f'(z)=b$ if and only if $g'(z)=b$. Hence we get $a^2=b^2$, that is $b=-a$. Thus the conclusion (2) occurs.

Case 1.1.3. $(h'(z))^2+\frac{ab}{t}\equiv 0$. Then we have $h(z)=cz+d$, where *c*, *d* are two constants satisfying $-tc^2=ab$. Thus we get

$$f(z)=e^{cz+d},\ g(z)=te^{-cz-d}.$$

Hence we have

$$f'(z)=ce^{cz+d},\ g'(z)=-tce^{-cz-d}.$$

Obviously, $f'(z)=a$ if and only if $g'(z)=b$, $f'(z)=b$ if and only if $g'(z)=a$. Thus the conclusion (3) occurs.

Case 1.2. $k\geq 2$. Then by (4.1) we have

$$f^{(k)}=P(h',\cdots,h^{(k)})e^h,\ g^{(k)}=tQ(h',\cdots,h^{(k)})e^{-h} \tag{4.3}$$

where *P*, *Q* are polynomials of $h', h'', \cdots, h^{(k)}$.

Obviously there exists z_0 such that $f^{(k)}(z_0)=a$. Then by $E(S_2, f^{(k)})=E(S_2, g^{(k)})$ and (4.3) we deduce that one of the following cases must occur:

(i) $g^{(k)}(z_0)=a, P(h', h'', \cdots, h^{(k)})(z_0)Q(h', h'', \cdots, h^{(k)})(z_0)-\frac{a^2}{t}=0$
(ii) $g^{(k)}(z_0)=b, P(h', h'', \cdots, h^{(k)})(z_0)Q(h', h'', \cdots, h^{(k)})(z_0)-\frac{ab}{t}=0$.

Next we consider three sub-cases.

Case 1.2.1. $P(h'(z), h''(z), \cdots, h^{(k)}(z))Q(h'(z), h''(z), \cdots, h^{(k)}(z))-\frac{a^2}{t}\not\equiv 0$, and $P(h'(z), h''(z), \cdots, h^{(k)}(z))Q(h'(z), h''(z), \cdots, h^{(k)}(z))-\frac{ab}{t}\not\equiv 0$. Then by using the same argument as do in Case 1.1 of the proof of Theorem 2 we deduce a contradiction.

Case 1.2.2. $P(h', h'', \cdots, h^{(k)})Q(h', h'', \cdots, h^{(k)})-\frac{a^2}{t}\equiv 0$. Thus by (4.3) we deduce that $f^{(k)}(z)g^{(k)}(z)\equiv a^2$ and $f^{(k)}(z)=a$ if and only if $g^{(k)}(z)=a$. Hence we obtain $f^{(k)}(z)\neq 0$, thus by Lemma 1 we deduce that $f(z)=e^{cz+d}$. Considering $f(z)g(z)\equiv t$, we get $g(z)=te^{-cz-d}$. Thus we have

$$f^{(k)}(z)=c^k e^{cz+d}, g^{(k)}(z)=(-1)^k tc^k e^{-cz-d}, \tag{4.4}$$

Obviously, c, d satisfies $(-1)^k tc^{2k}=a^2$. Thus the conclusion (2) occurs.

Case 1.2.3. $P(h', h'', \cdots, h^{(k)})Q(h', h'', \cdots, h^{(k)})-\frac{ab}{t}\equiv 0$. Thus by (4.3) we deduce that $f^{(k)}(z)g^{(k)}(z)\equiv ab$ and $f^{(k)}(z)=a$ if and only if $g^{(k)}(z)=b$. Hence we have $f^{(k)}(z)\neq 0$, thus by Lemma 1 we deduce that $f(z)=e^{cz+d}$. Considering $f(z)g(z)\equiv t$, we get $g(z)=te^{-cz-d}$. Hence we have

$$f^{(k)}(z)=c^k e^{cz+d}, g^{(k)}(z)=(-1)^k tc^k e^{-cz-d}, \tag{4.5}$$

Obviously, c,d satisfies $(-1)^k tc^{2k}=ab$. Thus the conclusion (3) occurs.

Case 2. $f\equiv tg$. Then $f^{(k)}\equiv tg^{(k)}$. Without loss of generality, we assume that there exists z_1 such that $f^{(k)}(z_1)=a$. Suppose that $t\neq 1$, then by $E(S_2, f^{(k)})=E(S_2, g^{(k)})$ and $f^{(k)}\equiv tg^{(k)}$ we deduce that $g^{(k)}(z_1)=b$. Hence we deduce that $f^{(k)}(z)=a$ if and only if $g^{(k)}(z)=b$ and that $f^{(k)}(z)=b$ if and only if $g^{(k)}(z)=a$.

If $f^{(k)}(z)=b$ has solution, then by $f^{(k)}(z)\equiv tg^{(k)}(z)$ we get $a=tb$ and $b=ta$. Hence we get $b=-a$ and $t=-1$. That is $f(z)\equiv -g(z)$, the conclusion (4) occurs.

If $f^{(k)}(z)\neq b$, then $g^{(k)}(z)\neq a$. Hence by $f^{(k)}(z)\equiv tg^{(k)}(z)$ we get $f^{(k)}(z)\neq b$, ta. If $b\neq ta$, then by Picard's theorem we get a contradiction. If $b=ta$, then by $f^{(k)}(z_1)=a$ and $g^{(k)}(z_1)=b$ we get $a=tb$. Hence we get $b=-a$ and $t=-1$. That is $f(z)\equiv -g(z)$, the conclusion (4) occurs.

Now we consider the case when $f(z)$ and $g(z)$ are two polynomials. Then by using same argument as do in Theorem 1 we get $f(z)\equiv tg(z)$. Thus we obtain $f^{(k)}(z)\equiv tg^{(k)}(z)$. Next by using the similar argument to Case 2 we obtain $f(z)\equiv g(z)$. The proof of the theorem is complete.

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