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Waves and Wave Groups in Deep Water

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Abstract. In this paper specific wave geometries are discussed which occur in deep water and are calculated by a numerical method based on Fourier transforms. Examples are presented of permanent waves and wave groups of permanent envelope in two and three dimensions without restriction on wave height.

1. Introduction

Analytic methods of modelling water waves of small but finite height are based on the linear theory and improved with weakly nonlinear theories [1]. An alternative is to develop, with computer assistance, water wave modes which are nonlinear in their lowest approximation and are valid for a range of heights up to the onset of wave breaking [2]. The present approach falls into the latter category, and is concerned with investigating wave geometries which occur in locally deep water.

Water waves propagating from a surface disturbance are subject to dispersion modified by nonlinear interactions. This property suggests that the numerical resolution into Fourier components of the nonlinear equations governing the evolution of a water wave system models the dispersion and its modification, and is therefore a natural method for investigating water waves properties. Forberg and Whitham [3] used this approach in studying certain nonliear model equations for wave phenomena. It is applied here to Laplace's equation with the nonlinear boundary conditions appropriate to irrotational gravity wave propagation in deep water.

Analytical solutions in the form of perturbation expansions exist for two dimensional water waves of permanent shape in deep water Stoke waves for which the dispersion and nonlinear modifications are in balance. A number of computer based methods have been used (Schwartz and Fenton [2]) to extend the calculations up to the highest waves of permanent form. The present method is demonstrated first in Section 2 for the calculation of two dimensional permanent waves. Three dimensional permanent waves have been found as perturbations to two dimensional permanent waves. The present method allows calculations of three dimensional waves independently of two dimensional waves, and one such example appears in Section 3.

Waves on ocean surface often occur locally as a wave group with an envelope that changes slowly as the waves propagate. Analytical solutions exist for weakly nonlinear wave groups of permanent envelope in two and three dimensions. The present method is applied to the calculation of wave groups of permanent envelope in two dimensions (Section 4).

Specific wave geometries which occur in deep water are calculated by a numerical method based on Fourier analysis. Examples are presented for wave parameters outside the range of validity of analytical methods. Wave properties, such as the form of the permanent waves of finite crest length, and the approach to wave breaking, are demonstrated. Although the method is applied here only to gravity waves in deep water, it may be generalised to further forms of nonlinear wave motion.

2. Method of calculation

The set of equations governing gravity waves in inviscid irrotational motion on the surface of deep water is

$$\phi_{xx} + \phi_{yy} + \phi_{zz} = 0, \quad z < \varepsilon\eta(x,y,t) \tag{2.1a}$$

$$\phi_x,\ \phi_y, \phi_z \to 0, \quad z \to -\infty \tag{2.1b}$$

$$\eta_t - \phi_z + \varepsilon\eta_x\phi_x + \varepsilon\eta_y\phi_y = 0, \quad z = \varepsilon\eta(x,y,t) \tag{2.1c}$$

$$\eta + \phi_t + \frac{\varepsilon}{2}(\phi_x^2 + \phi_y^2 + \phi_z^2) = 0, \quad z = \varepsilon\eta(x,y,t) \tag{2.1d}$$

The dimensional variables are the surface displacement $a\eta$, the velocity potential $\sqrt{gl}a\phi$ and $l_x, l_y, l_z, \sqrt{\frac{1}{g}}t$, where a is a measure of water wave amplitude, $2\pi l$ is a typical wave length, and $\varepsilon = \frac{a}{l}$ is a measure of wave steepness. The origin of coordinates lies in the mean water surface with the z-axis vertically upwards.

Symmetric two dimensional permanent wave solutions of the set of equations (2.1) exist for which

$$\eta = \sum_{k=1}^{N} a_k \cos\big(k(x-ct)\big) \tag{2.2a}$$

$$\phi = \sum_{k=1}^{N} b_k \sin\big(k(x-ct)\big)e^{kz} \tag{2.2b}$$

The permanent wave propagates with velocity $c\sqrt{gl}$, where c is an unknown function of ε and $2\pi l$ is the wavelength. The number of harmonics N is determined numerically by trial and error so that the set amplitudes a_k, b_k includes all those amplitudes greater in magnitude that some small prescribed value.

When the serves (2.2*a*,*b*) are substituted into the boundary conditions (2.1*c*,*d*) with c_k denoting the cosine in (2.2*a*) and s_k the sine in (2.2*b*) the resulting expression may be written as

$$F = \sum_k \left(kca_k s_k - kb_k e^{\varepsilon k\eta s} s_k\right) - \varepsilon\left[\left(\sum_k ka_k s_k\right)\left(\sum_k kb_k e^{\varepsilon k\eta} c_k\right)\right] = 0 \qquad (2.3a)$$

$$G = \sum_k \left(a_k s_k - kcb_k e^{\varepsilon k\eta} c_k\right) + \frac{1}{2}\varepsilon\left[\left(\sum_k kb_k e^{\varepsilon k\eta} c_k\right)^2 + \frac{1}{2}\varepsilon\left(\sum_k kb_k e^{\varepsilon k\eta} s_k\right)^2\right] = 0 \quad (2.3b)$$

where $e^{\varepsilon k\eta} = \exp\ (\varepsilon k \sum_k a_p c_p)$. If the measure of amplitude, *a*, is taken to be half the height of the wave crest above the wave trough, $\frac{1}{2}\left(\eta(0) - \eta(\tau)\right)$, then

$$H = \sum_{k\,odd} \left(a_k - 1\right) = 0 \qquad (2.3c)$$

Equations (2.3*a*,*b*) are transformed numerically to

$$F = \sum_m \left(F_m s_m\right) = 0 \qquad (2.4a)$$

$$G = \sum_m \left(G_m c_m\right) = 0 \qquad (2.4b)$$

from which $F_m = G_m = 0$ for all *m*. (2.5)

The Fourier components F_m, G_m are nonlinear functions of a_k, b_k and *c* for given ε. Equation (2.5) may be solved numerically by Newton's method, which for *F* is described by

$$\sum_k \left(\frac{\partial F}{\partial a_k}\right)_m (a_k - a_k') + \sum_k \left(\frac{\partial F}{\partial b_k}\right)_m (b_k - b_k') + \left(\frac{\partial F}{\partial c}\right)_m (c - c') = F_m\,, \text{ for all } m \qquad (2.6)$$

Each coefficient on the left of (2.6) is an *m* Fourier coefficient of a partial derivative of (2.3*a*) and the prime denotes new value of each variable. The coefficients and the height of (2.6) are evaluated at the odd values of the variables. There is a similar set of equations derived from *G* and a single equation derived from *H*. The complete set of linear equations is solved numerically for the differences $a_k - a'_k$, $b_k - b'_k$, $c - c'$, the new values of the variables are calculated and the procedure is repeated until the differences are less than some arbitrary number (usually 10^{-8}).

The range of permanent wave solutions may be explored as amplitude ratio ε is changed step by step. Since the number of harmonics increases rapidly as the limiting wave is approached, and the wave properties near the limiting wave are fully described already, the present calculations have been contained only up to 95% of the limiting wave height. Rienecker and Fenton [4] had developed a method similar to that described here except that Fourier transforms are not used and Equations (2.3) are solved directly by evaluating them at a number of points spaced along the wave profile. The Fourier transform method is now demonstrated for the more general wave geometries.

3. Three dimensional permanent waves

Three dimensional permanent wave solutions of the set of Equations (2.1) exist for which

$$\eta = \sum_{j=0}^{J} \sum_{k=k_1(j)}^{k_2(j)} a_{jk} \cos\left[k(x-ct) + \frac{jy}{r}\right] \tag{3.1a}$$

$$\phi = \sum_{j=0}^{J} \sum_{k=k_1(j)}^{k_2(j)} b_{jk} \exp\left[\sqrt{k^2 + \frac{j^2}{r^2}}\, z\right] \sin\left[k(x-ct) + \frac{jy}{r}\right] \tag{3.1b}$$

having wavelength $2\pi l$ in the x-direction and $2\pi rl$ in the y-direction, whose profile is steady relative to a frame of reference moving with velocity $c\sqrt{gl}$ in the x-direction. The bounds of summation are determined numerically by trial and error so that the set of amplitudes a_{jk} includes all these magnitudes greater in magnitude than some small prescribed value. Since η is chosen to have a zero mean and the argument is symmetric in k when $j = 0$, the lower bound $k_1(0)$ may be sent equal to 1 without loss of generality. Other lower bounds $k_1(j)$, $j > 0$, may be negative.

Yen and Lake [5] have considered three dimensional permanent waves which are perturbations to uniform two dimensional permanent waves, and which are steady to the two dimensional waves. Less restrictive assumptions are implicit in the analysis of Robers [6] and Roberts and Peregrine [7] where the perturbation expansion in wave steepness are developed.

When the series (3.1*a*,*b*) are substituted into the boundary conditions (2.1*c*,*d*), with c_{jk} denoting the cosine in (2.2*a*) and s_{jk} the sine in (2.2*b*), the resulting expression may be written as:

$$F = \sum_j \sum_k \left[kca_{jk} s_{jk} - K_{jk} b_{jk} e^{\varepsilon K_{jk}\eta} s_{jk} \right]$$

$$-\varepsilon \left(\sum_j \sum_k \left[ka_{jk} s_{jk} \right] \right) \left(\sum_j \sum_k \left[kb_{jk} e^{\varepsilon K_{jk}\eta} c_{jk} \right] \right) \qquad (3.2a)$$

$$-\varepsilon \left(\sum_j \sum_k \left[\frac{j}{r} a_{jk} s_{jk} \right] \right) \left(\sum_j \sum_k \left[\frac{j}{r} b_{jk} e^{\varepsilon K_{jk}\eta} c_{jk} \right] \right) = 0$$

$$G = \sum_j \sum_k \left[a_{jk} c_{jk} - ckb_{jk} e^{\varepsilon K_{jk}\eta} c_{jk} \right]$$

$$+ \frac{1}{2} \varepsilon \left(\sum_j \sum_k \left[kb_{jk} e^{\varepsilon K_{jk}\eta} c_{jk} \right] \right)^{2\,2} + \frac{1}{2} \varepsilon \left(\sum_j \sum_{kr} \left[\frac{k}{j} b_{jk} e^{\varepsilon K_{jk}\eta} c_{jk} \right] \right) \qquad (3.2b)$$

$$+ \frac{1}{2} \varepsilon \left(\sum_j \sum_k \left[K_{jk} b_{jk} e^{\varepsilon K_{jk}\eta} s_{jk} \right] \right)^2 = 0$$

where $K_{jk} = \sqrt{\frac{k^2 + j^2}{r^2}}$ and $e^{\varepsilon K_{jk}\eta} = \exp\left[\varepsilon K_{jk} \sum_p \sum_q a_{pq} c_{pq}\right]$. If the measure of the amplitude a is taken to be the central surface displacement $\eta(0, 0, 0)$, then

$$H = \sum_j \sum_k a_{jk} = 1 \qquad (3.2c)$$

Equations (3.2*a*, *b*) are transformed numerically to

$$F = \sum_m \sum_n F_{mn} s_{mn} = 0 \qquad (3.3a)$$

$$G = \sum_m \sum_n G_{mn} c_{mn} = 0 \qquad (3.3b)$$

for which $F_{mn} = G_{mn} = 0$, for all m, n.

The Fourier coefficients F_{mn}, G_{mn} are nonlinear functions of a_{jk}, b_{jk}, and c from given ε and *r*, and these harmonics and the wave velocity may be calculated by Newton's method. A three dimensional permanent wave example is presented now which lies outside the range of analytical solutions described previously. This example, drawn in Figure 1, has parameter values $r = 0.25, r = 10$, with $c = 1.0262$. The dominant harmonics have amplitudes $a_{11} = 0.65$ and $a_{01} = 0.44$ with all other harmonics having smaller amplitudes because they are the result of resonant nonlinear interactions between the dominant pair. The harmonic $j = 0, k = 1$ propagates in the *x*-direction with a velocity $c\sqrt{gl}$, while the harmonic $j = 1, k = 1$ propagates at an angle $\tan^{-1}(0.1)$ to the *x*-direction with a velocity component $c\sqrt{gl}$ in the *x*-direction. The net result is a long crested three dimensional permanent wave propagating at an angle $\theta = 0.06$ to the *x*-direction, whose wave height to wavelength ratio is 0.71. The wave speed of the permanent wave is $c\sqrt{gl}\cos\theta$, and its wavelength is $2\pi l\cos\theta$. The wave structure at the ends of the crest, drawn in detail in Figure 2, propagates in the *x*-direction with a velocity $c\sqrt{gl}$, and reduces the wave height to the wavelength ratio to 0.021 at its lower point. Relative to the frame of reference moving with the long crested permanent wave, the end structure propagates along the wave crests with a velocity $c\sqrt{gl}\sin\theta$ producing a three dimensional wave which has a steady profile relative to a reference moving with velocity $c\sqrt{gl}$ in the *x*-direction. The net result illustrated in Figure 3 is of the same form as that calculated analytically by Roberts and Peregrine [7] (Figure 4) for semi infinite long crested permanent waves.

The data for the present solution is as follows. Equation (3.1*a*) contains 173 harmonics in 17 wavebands $0 \le j \le 16$; the wave number range being $-6 \le k \le 9$. The maximum Fourier coefficients F_{mn}, G_{mn} not included in the calculations has magnitude 3.2×10^{-5}. The maximum magnitude of *F* and *G* over the (64×32) points used in the final calculation is 7.6×10^{-4} with a root mean square deviation of *F* and *G* from zero to 1.2×10^{-4}.

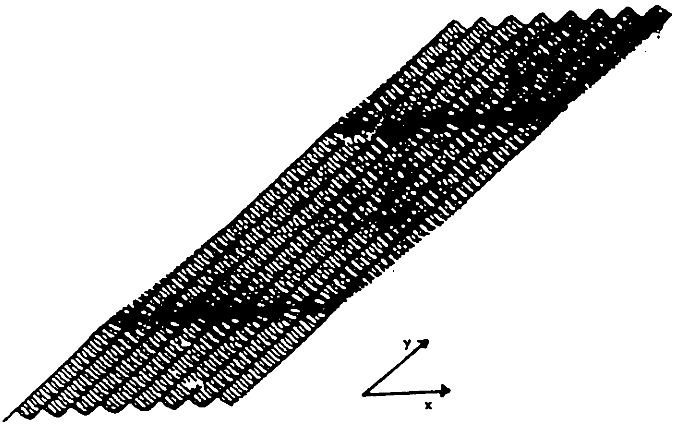

Figure 1. Perspective view of long-crested permanent waves showing 8 wave lengths in the x-direction and 2 wavelengths in the y-direction, with vertical magnification 5 when $\varepsilon = 0.25$ and $r = 10$.

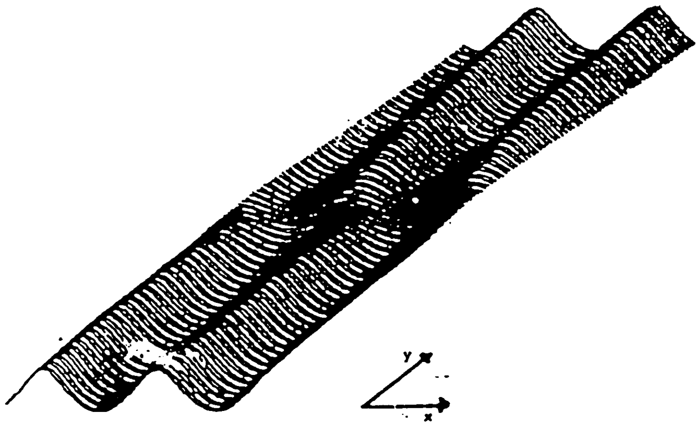

Figure 2. Detailed view of the end structure of the long-crested permanent waves in Figure 1, with vertical magnification 5.

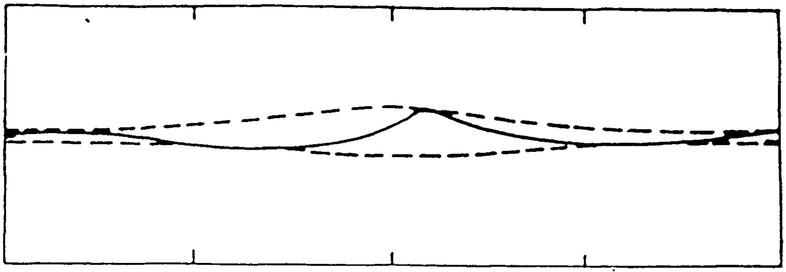

Figure 3. The wave group of permanent envelope with $\varepsilon = 0.470$ and $\nu = 0.6$, showing the water surface displacement at an instant (the solid curve) and the envelope (the dashed curves), drawn with the horizontal and vertical scales equal.

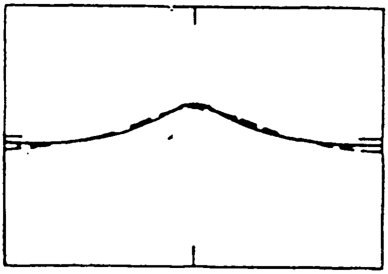

Figure 4. A wave at the center of the group of Figure 3 (the solid curves) compared with the permanent wave of the same height and length (the dashed curve), both drawn with the horizontal and vertical scales equal.

4. Two dimensional wave groups

Two dimensional wave groups with envelops of permanent shape, which are periodic in the x-direction with a group length $2\pi L$, are composed of harmonics with wave numbers $\frac{k}{L}$, where k has integer values only. If $2\pi l$ is the wavelength of a typical wave in the group with $k_o = \frac{L}{l}$ not necessarily an integer, then to a first approximation of the spectrum of the wave group in non-dimensional wave number space is centred on k_o. The velocity of the group, to a first approximation is $\frac{1}{2}\sqrt{gl}$ in deep water. The non-dimensional wave frequency, $\omega(k)$, expanded about the central wave number, k_o, is

$$\omega(k) = \omega(k_\circ) + (k-k_\circ)\left[\frac{d\omega}{dk}\right]_{k_\circ} + \frac{1}{2}(k-k_\circ)^2\left[\frac{d^2\omega}{dk^2}\right]_{k_\circ} + \cdots$$

$$= 1+\left(\frac{k-k_\circ}{2k_\circ}\right) + O\left(\frac{k-k_\circ}{2k_\circ}\right)^2 + \cdots \tag{4.1}$$

for waves in deep water with $\omega = \sqrt{\frac{k}{k_\circ}}$. The wave group is described then by

$$\eta = \sum_k a_k \cos\left\{\frac{kx}{k_\circ} - \omega t\right\} \tag{4.2a}$$

$$= \sum_k a_k \cos\left\{\frac{k-k_\circ}{k_\circ}\left(x-\frac{t}{2}\right) + x - (1+\beta)t\right\} \tag{4.2b}$$

where β is an unknown non-dimensional frequency correction. This form of a periodic wave group satisfies the nonlinear Schroedinger equation, which in the present non-dimensional notation is:

$$\frac{1}{2}\left(A_t + \frac{A_x}{2}\right) - \frac{1}{8}A_{xx} - \frac{1}{2}\varepsilon^2 |A|^2 A = 0 \tag{4.3a}$$

where

$$\eta = R\{A(x, t)\exp i(x-t)\} \tag{4.3b}$$

The nonlinear Schroedinger equation is valid for waves of small but finite height, $\varepsilon << 1$, whose spectrum lies in a narrow waveband $\frac{|k-k_\circ|}{k_\circ} \sim \varepsilon$. A generalisation of equation (4.2) which satisfies the nonlinear free surface conditions (2.1*c*,*d*) is

$$\eta = \sum_{j=0}^{J} \sum_{k=k_1(j)}^{k=k_2(j)} a_{jk} \cos\left\{\frac{k}{k_\circ}\left(x-\frac{t}{2}\right) - j\alpha t\right\} \tag{4.4a}$$

$$\phi = \sum_{j=0}^{J} \sum_{k=k_1(j)}^{k=k_2(j)} b_{jk} \exp\frac{|k|z}{k_\circ} \sin\left\{\frac{k}{k_\circ}\left(x-\frac{t}{2}\right) - j\alpha t\right\} \tag{4.4b}$$

where $\alpha = (\beta + \frac{1}{2})$. The index $j = 0$ for which $k_1(0) = 1$. Since η has a zero mean, describes the surface displacement and water motion which is steady relative to the group structure. The index $j = 1$ refers to the harmonics in (4.2) for the dominant waveband with *k* near to $k_\circ$. Each higher value of *j* describes a set of harmonics in a waveband about $(jk_\circ)$, where $k_1(j),\ j > 0$ may be negative. When the series (4.4*a*,*b*) are substituted into the boundary conditions (2.1*c*,*d*), equations similar in form to equations (3.2) are obtained, which are solved by the same method as is described previously for a_{jk}, b_{jk}, α as a function of ε and $k_\circ$. It was found that the calculated spectral peek in the dominant waveband $(j = 1)$ moved outside the neighbourhood of $k_\circ$ as ε increased to larger values. Such solutions need re-scaling with a more appropriate wavelength $2\pi l'$ so that $k'_\circ = \frac{L}{l'}$ corresponds to the actual centre of the dominant waveband. The non-dimensional group velocity is change from $\frac{1}{2}$ to *v* according to:

$$\frac{1}{2}\sqrt{gl} = v\sqrt{gl'} \tag{4.5}$$

and Equations (4.4) have the same form in the new non-dimensional variables except that $\frac{1}{2}$ is replaced by *v*. If the primes are dropped from the new variables, Equations (4.4) become

$$\eta = \sum_{j=0}^{J} \sum_{k=k_1(j)}^{k=k_2(j)} a_{jk} \cos\left\{\frac{k}{k_\circ}(x - vt) - j\alpha t\right\} \tag{4.6a}$$

$$\phi = \sum_{j=0}^{J} \sum_{k=k_1(j)}^{k=k_2(j)} b_{jk} \exp\frac{|k|z}{k_\circ} \sin\left\{\frac{k}{k_\circ}(x - vt) - j\alpha t\right\} \tag{4.6b}$$

where for any particular solution the choice of $k_\circ$ determines *v* and conversely.

Two examples are presented of two dimensional wave groups with envelopes of permanent shape, both of which lie outside the range of validity of the nonlinear Schroedinger equation. This first is of a wave group containing two wavelengths per group length, $k_\circ = 2$, in the range of larger wave heights of such groups. Its parameter values are $\varepsilon = 0.470$, $v = 0.6$ with $\alpha = 0.4308$ and the wave height (trough to crest) to wavelength ratio for a wave at the centre of the group is 0.100. One group length is sketched in Figure 3, which shows the water surface displacement at an instant, and the envelope of permanent shape. The upper envelope is a height 0.470*l* above the mean level at the centre of the group, while the lower envelope is a depth 0.308*l* below the mean level there. This is an obvious departure from the permanent envelope analytical solutions of the nonlinear Schroedinger equation (4.3) because they are symmetric about the mean level.

A wave at the centre of the group of permanent envelope is composed of the group of permanent envelope is compared in Figure 4 with the wave of permanent shape having the same wave height of its crest above the mean level being $0.470l$ compared with $0.471l$ for a permanent wave. The horizontal particle velocity at the crest is $0.75\sqrt{gl}$ compared with $0.45\sqrt{gl}$ for a permanent wave. The horizontal component of particle acceleration in front of the crest, and deceleration behind the crest, is of maximum magnitude (0.46 g) compared with (0.31 g) for the permanent wave. This comparison indicates that a wave passing through the centre of the group is closer to the point of wave breaking than a permanent wave of the same height wavelength.

The wave group solution contains 302 harmonics (605 variables) in 14 wavebands $0 \le j \le 13$, the wave number range being $-17 \le k \le 40$. The maximum amplitude of F over the (128×32) points used in the final calculations was (0.186) but the root mean square deviation of F from zero was (0.018). The maximum magnitude of G over the same points was 0.022 with a root mean square deviation from zero to 0.002.

The second example is of a wave group containing one wavelength or amplitude, $k_\circ = 1$. As the wave train propagates through the group structure, each wave shape oscillates with an angular frequency $\sqrt{\frac{g}{l}}$ about a symmetric shape. Since the dominant harmonics in (4.6) are now those for which $j = k,\ k = 1,\ 2, \cdots$, the shape of oscillation is modeled better by changing the summation in Equations (4.6) to keep $k \ge 0$ with $c = v + \alpha$ and $m = k - j$. The series then becomes:

$$\eta = \sum_{m=-m_1}^{m_2} \sum_{k=0}^{k_2(m)} A_{mk} \cos\left\{ k\left(x - ct\right) - m\alpha t \right\} \tag{4.7a}$$

$$\phi = \sum_{m=-m_1}^{m_2} \sum_{k=0}^{k_2(m)} B_{mk} e^{kz} \sin\left\{ k\left(x - ct\right) + m\alpha t \right\} \tag{4.7b}$$

The dominant waveband, $m = 0$, describes a steady wave propagating with velocity $c\sqrt{gl}$. Other wavebands $m \ne 0$, describe the cyclic oscillation of this wave space as it propagates in the x-direction, through it where a periodic wave train set into an oscillation about a permanent shape. It can be seen that when a cyclic wave passes through points of maximum wave height (at the centre of the wave group), it is closer to wave breaking then is a steady permanent wave of the same height and wavelength, the comparison being similar to that for the wave group example above. Since ocean waves of large height are never completely steady in shape, it may be more realistic to model these with cyclic waves rather than with steady waves.

5. Discussion

The particular wave geometries described here are only a small selection of those which may be calculated by the Fourier transform method applied to the full nonlinear governing equations. Generalisations to the examples above include standing wave geometries, waves in finite depth, and short waves influenced by surface tension. Most analytic methods and model equations are valid only for linear or weakly nonlinear waves, that is, for waves of small but finite height. The present method as shown is set up for nonlinear waves without restrictions on wave height. Also, the calculation of the irrotational velocity field simultaneously with the water surface displacement provides insight into the physical properties of water wave motion.

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