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On Full Hilbert C*- Modules

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Abstract. Let M be both a full Hilbert A-module and a full Hilbert B-module. In this paper we prove that a map $\phi : A \to B$ is an isometrically *-isomorphism iff it satisfies $ax = \phi(a)x$ *and* $\phi(< x, y >_A) = < x, y >_B$ where $a \in A, x, y \in M$. We also show that the fullness condition can not be dropped.

1. Introduction

Hilbert C*-modules are used as a powerful tool in C*-algebraic quantum group theory, K- and KK-theory, induced representations of C*-algebras and Morita equivalence. Some sources of references to the subject are [1] and [2].

The goal of this paper is to show that if M is full Hilbert C*-modules over C*-algebras A and B and $\phi : A \to B$ is a map, then ϕ is *-isomorphism iff $ax = \phi(a)x$ and $\phi(< x, y >_A) = < x, y >_B$ where $a \in A, x, y \in M$. We show that without any one of the assumptions of M being full the result does not in general hold. Our result is interesting in its own.

Definition 1.1. *Suppose A is a C*-algebra. Let M be a complex linear space which is a right A-module and* $\lambda(ax) = (\lambda a)x = a(\lambda x)$ *where* $\lambda \in C, a \in A$ *and* $x \in M$. *M is called a pre-Hilbert A-module if there exists an (A-valued) inner product* $< \cdot , \cdot > : M \times M \to A$ *satisfying:*

(i) $\langle x, x \rangle \geq 0,$

(ii) $\langle x, x \rangle \geq 0,$ *iff* $x = 0,$

(iii) $\langle x + \lambda y , z \rangle = \langle x , z \rangle + \lambda \langle y , z \rangle,$

(iv) $\langle x, y \rangle = \langle y, x \rangle^*,$

(v) $\langle ax, y \rangle = a\langle x, y \rangle.$

A pre-Hilbert A-module is caled a Hilbert A-module or Hilbert C-module over A, if it is complete with respect to the norm* $\| x \| = \| < x, x > \|^{\frac{1}{2}}$. *M is said to be full if the linear span of* $\{ < x, y >;\ x, y \in M \}$ *is dense in A.*

Example 1.2. Let A be a C*-algebra. Then A together with its product as the usual action is a left A-module. In addition it equipped with inner product $< a, b > = ab^*$ is a full Hilbert A-module.

2. Main theorem

Let M be a (full) Hilbert B-module and $\phi : A \to B$ *a* *-isomorphism of C*-algebras. Define the module action by $ax = \phi(a)x$ and A-valued inner product by $\phi(< x, y >_A) = < x, y >_B$. Then it is straightforward to show that M is a (full) Hilbert A-module. We are going to establish a converse statement to the above.

Lemma 2.1. *Let N be a full Hilbert C*-module over C and* $u \in C$. *Then* $ux = 0$ *for all* $x \in N$ *iff* $u = 0$.

Proof. Since N is full, there exists $\{u_n\}$ in $< N, N >$ such that $u = \lim_n u_n$. Each u_n is of the form $u_n = \sum_{i=1}^{k_n} < x_{i,n}, y_{i,n} >$ in which $x_{i,n}, y_{i,n} \in N$. Hence

$$uu^* = u \lim_n u_n^* = \lim_n uu_n^* = \lim_n \left(u \sum_{i=1}^{k_n} \langle y_{i,n}, x_{i,n} \rangle \right) = \lim_n \sum_{i=1}^{k_n} \langle uy_{i,n}, x_{i,n} \rangle = 0 .$$

Hence $u = 0$.

Theorem 2.2. *Let M be both a full Hilbert A-module and a full Hilbert B-module and there exist a map* $\phi : A \to B$ *in such a way that* $ax = \phi(a)x$ *and* $\phi(< x, y >_A) = < x, y >_B$. *Then* ϕ *is an (isometrically) *-isomorphism.*

Proof. If $a_n \to 0$ and $\phi(a_n) \to b$, then $a_n x \to 0$ and $\phi(a_n)x \to bx$, But $\phi(a_n)x \to 0$, Hence $bx = 0$. By Lemma 2.1 $b = 0$. Thus ϕ is continuous. $(\phi(ab) - \phi(a)\phi(b))x = (ab)x - a(bx) = 0$. It follows from Lemma 2.1 that $\phi(ab) = \phi(a)\phi(b)$. Similarly one can show that ϕ is linear.

If $a \in A$, then we may assume that $a = \lim_n u_n, u_n = \sum_{i=1}^{k_n} < x_{i,n} \ , \ y_{i,n} >_A$ where $x_{i,n}, y_{i,n} \in M$. Hence

$$\phi(a^*) = \lim_n \phi(u_n^*) = \lim_n \sum_{i=1}^{k_n} \phi\left(\left\langle y_{i,n}, x_{i,n} \right\rangle_A\right) = \lim_n \sum_{i=1}^{k_n} \left\langle y_{i,n}, x_{in} \right\rangle_B$$
$$= \left(\lim_n \sum_{i=1}^{k_n} \left\langle x_{i,n}, y_{i,n} \right\rangle_B \right)^* = \left(\phi(a)\right)^*$$

If $\phi(a) = 0$, then $ax = \phi(a)x = 0$ for all $x \in M$. Hence $a = 0$. ϕ is therefore one to one.

Given $b \in B$ and $\varepsilon > 0$, there are $\{x_i\}_{1 \le i \le n}, \{y_i\}_{1 \le i \le n}$ in M such that $\left\| b - \sum_{i=1}^{n} < x_i, y_i >_B \right\| < \varepsilon$, Hence $\left\| b - \phi \sum_{i=1}^{n} < x_i, y_i >_A \right\| < \varepsilon$. Therefore ϕ has a dense range. But ϕ is a *-homomorphism from *A* into *B*, so that its range is closed. Thus ϕ is a *-isomorphism.

Remark 2.3. The result may fail, if any one of the conditions of M being full is dropped.

For example, first, take *A* to be a von Neumann algebra acting on a Hilbert space which has a central projection $p \neq 0, I$. Put $B = M = Ap$ and consider M as a Hilbert *B*-module and a Hilbert *A*-module with the usual actions and the inner products $< x, \ y > = xy^*$. Clearly M is not full *A*-module. Then $\phi: A \to B \ \phi(a) = ap$ has evidently the properties $ax = \phi(a)x$ and $\phi(< x, y >_A) = < x, y >_B$, but is not one to one (and hence is not isometry).

Second, let *A* and *B* be arbitrary *C**-algebras and *A* be a proper subset of *B*. Put $M = A$ and consider it as a Hilbert *A*-module and a Hilbert *B*-module such above. Clearly *M* is not full *B*-module. Then the inclusion map $\phi: A \to B$ satisfies obviously ax=$\phi(a)x$ and $\phi(< x, y >_A) = < x, y >_B$, but is not surjecive.

References

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