BULLETIN of the
MALAYSIAN
MATHEMATICAL
SCIENCES
SOCIETY

Bull. Malaysian Math. Sc. Soc. (Second Series) **24** (2001) 69-79

On Semi-invariant Submanifolds of *LP*-cosymplectic Manifolds

MUKUT MANI TRIPATHI
Department of Mathematics and Astronomy, Lucknow University, Lucknow, 226 007, India
e-mail: mm_tripathi@hotmail.com

Abstract. Semi-invariant submanifolds of *LP*-cosymplectic manifolds are studied. Integrability of certain distributions on the submanifold are investigated. Totally umbilical and totally geodesic submanifolds are also studied.

1. Introduction

K. Matsumoto introduced [4] the notion of a Lorentzian almost paracontact manifold. Later on several authors studied Lorentzian almost paracontact manifolds including those of [3, 5, 6, 7] and submanifolds of Lorentzian almost paracontact manifolds including those of [8, 11, 12]. In [11, 12], is has been proved that a *LP*-Sasakian manifold does not admit proper almost semi-invariant or semi-invariant submanifolds. In [8], a class of Lorentzian almost paracontact manifold is defined as a *LP*-cosymplectic manifold.

In this paper, we study semi-invariant submanifolds of *LP*-cosymplectic manifolds. The paper is organized as follows. Section 2 is devoted to preliminaries. In section 3, some basic results for submanifolds of a Lorentzian almost paracontact manifold and a *LP*-cosymplectic manifold are given. Section 4 deals with semi-invariant submanifolds of *LP*-cosymplectic manifolds. In section 5 some necessary and sufficient conditions for integrability of certain distributions on semi-invariant submanifolds of a *LP*-cosymplectic manifold are obtained. In last section, totally umbilical and totally geodesic submanifolds are discussed.

2. Preliminaries

Let an n-dimensional smooth connected paracompact Hausdorff manifold $\overline{M}$ be equipped with a Lorentzian metric g, that is, g is a smooth symmetric tensor field of type $(0, 2)$ such that at every point $p \in \overline{M}$, the tensor $g_p : T_p\overline{M} \times T_p\overline{M} \to R$ is a non-degenerate innerproduct of signature, $(-,+,\cdots,+)$ where $T_p\overline{M}$ is the tangent space of $\overline{M}$ at p and R is the real line. In other words, a matrix representation of g_p has one eigenvalue negative and all other eigenvalues positive. Then $\overline{M}$ is *Lorentzian manifold*. A non-zero vector $X_p \in T_p\overline{M}$ is known to be *spacelike, null, non-spacelike* or *timelike* according as

$$g_p(X_p, X_p) > 0, \ = 0, \ \leq 0 \ \text{ or } \ < 0$$

respectively.

Let $\overline{M}$ be an n-dimensional differentiable manifold equipped with a triple (ϕ, ξ, η), where φ is a $(1,1)$ tensor field, ξ is a vector field, η is a 1-form on $\overline{M}$ such that

$$\eta(\xi) = -1 \tag{1}$$

$$\phi^2 = I + \eta \otimes \xi \tag{2}$$

These two equations imply that

$$\eta \circ \phi = 0, \tag{3}$$

$$\phi\xi = 0, \tag{4}$$

$$\text{rank } (\phi) = n-1. \tag{5}$$

Then $\overline{M}$ admits a *Lorentzian metric* g, such that

$$g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y), \tag{6}$$

and $\overline{M}$ is said to admit a *Lorentzian almost paracontact structure* (ϕ, ξ, η, g). In this case, we get

$$g(\xi, X) = \eta(X), \tag{7}$$

$$\Phi(X,Y) \equiv g(\phi X, Y) = g(X, \phi Y) = \Phi(Y,X), \tag{8}$$

$$(\overline{\nabla}_X \Phi)(Y,Z) \equiv g((\overline{\nabla}_X \phi)Y, Z) = (\overline{\nabla}_X \Phi)(Z,X), \tag{9}$$

where $\overline{\nabla}$ is the covariant differentiation with respect to g. The Lorentzian metric g makes ξ a timelike unit vector field, that is, $g(\xi, \xi) = -1$ (see [4, 5]).

A Lorentzian almost paracontact manifold is called a *LP-cosymplectic manifold* [8] if

$$\overline{\nabla}\phi = 0.$$

3. Some basic results

Let M be a submanifold of a Lorentzian manifold $\overline{M}$ with a Lorentzian metric g. Let the induced metric on M also be denoted by g. Then Gauss and Weingarten formulae are given respectively by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y) \quad , \quad X,Y \in TM, \tag{11}$$

$$\overline{\nabla}_X Y = -A_N X + \nabla^{\perp}_X N \quad , \quad N \in T^{\perp}M, \tag{12}$$

where ∇ is the induced connection on M, h is the second fundamental form of the immersion, and $-A_N X$ and $\nabla^{\perp}_X N$ are the tangential and normal parts of $\overline{\nabla}_X N$. From (11) and (12) one gets

$$g(h(X,Y), N) = g(A_N X, Y). \tag{13}$$

Let M be a submanifold of a Lorentzian almost paracontact manifold $\overline{M}$ with Lorentzian almost paracontact structure (φ, ξ, η, g). For $X \in TM$ and $N \in T^{\perp}M$ we put

$$\phi X \equiv PX + FX, \quad PX \in TM, \; FX \in T^{\perp}M, \tag{14}$$

$$\phi N \equiv tN + fN, \quad tN \in TM, \; fN \in T^{\perp}M, \tag{15}$$

$$(\nabla_X P)Y \equiv \nabla_X PY - P\nabla_X Y, \tag{16}$$

$$(\nabla_X F)Y \equiv \nabla^{\perp}_X FY - F\nabla_X Y, \tag{17}$$

$$(\nabla_X t) \equiv \nabla_X tN - t\nabla^{\perp}_X N, \tag{18}$$

$$(\nabla_X f)N \equiv \nabla^{\perp}_X fN - f\nabla^{\perp}_X N. \tag{19}$$

Moreover, if $\xi \in TM$, we write $TM = \{\xi\} \oplus \{\xi\}^{\perp}$, where $\{\xi\}$ is the distribution spanned by ξ and $\{\xi\}^{\perp}$ is the complementary orthogonal distribution of $\{\xi\}$ in M.

We state the following two lemmas whose proofs are straightforward and hence omitted.

Lemma 3.1. *For a submanifold M of a Lorentzian almost paracontact manifold and for all $X \in TM$; $N, V \in T^{\perp}M$ we have*

$$g(X, PY) = g(PX, Y) \tag{20}$$
$$g(X, tN) = g(FX, N) \tag{21}$$
$$g(N, fV) = g(fN, V) \tag{22}$$
$$(\overline{\nabla}_X \phi)Y = \big((\nabla_X P)Y - A_{FY}X - th(X,Y)\big) + \big((\nabla_X F)Y + h(X, PY) - fh(X,Y)\big) \tag{23}$$
$$\big(\overline{\nabla}_X \phi\big)N = \big((\nabla_X t)N - A_{fN}X - PA_N X\big) + \big((\nabla_X f)N + h(X, tN) - FA_N X\big). \tag{24}$$

Lemma 3.2. *For a submanifold M of a Lorentzian almost paracontact manifold with $\xi \in TM$, we have*

$$P\xi = 0 = F\xi, \tag{25}$$
$$\eta \circ P = 0 = \eta \circ F, \tag{26}$$
$$P^2 + tF = I + \eta \otimes \xi, \tag{27}$$
$$FP + fF = 0, \tag{28}$$
$$f^2 + Ft = I, \tag{29}$$
$$tf + Pt = 0. \tag{30}$$

The above two lemmas lead to the following proposition.

Proposition 3.3. *If M is a submanifold of a Lorentzian almost paracontact manifold with $\xi \in TM$, then for every $x \in M$ we have*

$$\ker(P)_x = \ker(P^2)_x = \ker(tF - I - \eta \otimes \xi)_x, \tag{31}$$
$$\ker(F)_x = \ker(tF)_x = \ker(P^2 - I - \eta \otimes \xi)_x, \tag{32}$$
$$\ker(t)_x = \ker(Ft)_x = \ker(f^2 - I)_x, \tag{33}$$
$$\ker(f)_x = \ker(f^2)_x = \ker(Ft - I)_x, \tag{34}$$
$$\ker(f|_{\{\xi\}^\perp})_x = \ker(P^2|_{\{\xi\}^\perp})_x = \ker(tF|_{\{\xi\}^\perp} - I)_x, \tag{35}$$
$$\ker(f|_{\{\xi\}^\perp})_x = \ker(tF|_{\{\xi\}^\perp})_x = \ker(P^2|_{\{\xi\}^\perp} - I)_x, \tag{36}$$

Proof. The relations (31) – (34) follow from relations (20) – (22), (27) and (29). Since $\eta(X) = 0$ for $X \in \{\xi\}^\perp$, the relations (35) and (36) are implied by (31) and (32) respectively.

The following two propositions are for submanifolds of *LP*-cosymplectic manifolds, tangent to ξ.

Proposition 3.4. *For a submanifold M of a LP-cosymplectic manifold such that* $\xi \in TM$, *we have*

$$\nabla_X \xi = 0, \tag{37}$$

$$h(X,\xi) = 0, \tag{38}$$

$$A_N X \in \{\xi\}^{\perp} \tag{39}$$

$$A_N \xi = 0. \tag{40}$$

Proof. We have

$$\nabla_X \xi + h(X,\xi) = \overline{\nabla}_X \xi = 0,$$

which implies (37) and (38). In view of (38) and (13) we get

$$0 = g(h(X,\xi), N) = g(A_N X, \xi) = g(A_N \xi, X),$$

which gives (39) and (40).

Proposition 3.5. *For a submanifold M of a LP-cosymplectic manifold such that* $\xi \in TM$, *we have*

$$(\nabla_X P)Y - A_{FY} X - th(X,Y) = 0, \tag{41}$$

$$(\nabla_X F)Y + h(X,PY) - fh(X,Y) = 0, \tag{42}$$

$$(\nabla_X t)N - A_{fN} X - PA_N X = 0, \tag{43}$$

$$(\nabla_X f)N + h(X,tN) - FA_N X = 0. \tag{44}$$

Consequently,

$$(\nabla_X P)\xi = 0, \tag{45}$$

$$(\nabla_X F)\xi = 0, \tag{46}$$

$$\nabla_\xi P = 0, \tag{47}$$

$$\nabla_\xi F = 0, \tag{48}$$

$$\nabla_\xi t = 0, \tag{49}$$

$$\nabla_\xi f = 0, \tag{50}$$

$$P[X,Y] = \nabla_X PY - \nabla_Y PX + A_{FX} Y - A_{FY} X, \tag{51}$$

$$F[X,Y] = \nabla^{\perp}_X FY - \nabla^{\perp}_Y FX + h(X,PY) - h(PX,Y) \tag{52}$$

Proof. Using (10) in (23) we get (41) and (42). Similarly, using (10) in (24) we get (43) and (44). Putting $Y=\xi$ in (41) and using (25) and (38), we get (45). Putting $Y=\xi$ in (42) and using (25) and (38), we get (46). Putting $X=\xi$ in (41) – (44) and using (38) and (40) we get (47) – (50) respectively. The relations (51) and (52) follow from (41) and (42) respectively.

4. Semi-invariant submanifolds

A submanifold M of a Lorentzian almost paracontact manifold $\overline{M}$ with $\xi \in TM$ is a *semi-invariant submanifold* [11, 12] of $\overline{M}$ if TM can be decomposed as a direct sum of mutually orthogonal differentiable distributions

$$TM = \mathcal{D}^1 \oplus \mathcal{D}^0 \oplus \{\xi\},$$

where

$$\mathcal{D}^1 = \ker(F|_{\{\xi\}^\perp}) = \left\{X \in \{\xi\}^\perp : \| X \| = \| PX \|\right\} = TM \cap \phi(TM),$$

$$\mathcal{D}^0 = \ker(P|_{\{\xi\}^\perp}) = \left\{X \in \{\xi\}^\perp : \| X \| = \| FX \|\right\} = TM \cap \phi(T^\perp M),$$

Here, the distribution $\mathcal{D}^1$ is invariant and the distribution $\mathcal{D}^0$ is anti-invariant by φ. Moreover, we have

$$T^\perp M \equiv \overline{\mathcal{D}}^1 \oplus \overline{\mathcal{D}}^0,$$

where

$$\overline{\mathcal{D}}^1 = \ker(t) = T^\perp M \cap \phi(T^\perp M), \ \overline{\mathcal{D}}^0 = \ker(f) = T^\perp M \cap \phi(TM),$$

$$F\mathcal{D}^0 = \overline{\mathcal{D}}^0, \ t\overline{\mathcal{D}}^0 = \mathcal{D}^0.$$

A submanifold M of a Lorentzian almost paracontact manifold $\overline{M}$ is an *invariant* (resp. *anti-invariant*) *submanifold* of $\overline{M}$ if $\phi(TM) \subset TM$ (resp. $\phi(TM) \subset T^\perp M$). A semi-invariant submanifold of a Lorentzian almost paracontact manifold becomes an invariant submanifold (resp. anti-invariant submanifold) if $\mathcal{D}^0 = \{0\}$ (resp. $\mathcal{D}^1 = \{0\}$).

Proposition 4.1. *Let M be a semi-invariant submanifold of a LP-cosymplectic manifold $\overline{M}$. Then $D \in \{D^1, D^0, D^1 \oplus D^0\}$ is ξ-parallel, that is, $\nabla_\xi X \in D$ for all $X \in D$.*

Proof. For $X \in D^1$ and $Y \in D^0$, we get

$$g(X, \nabla_\xi Y) = g(P^2 X, \nabla_\xi Y) = g(PX, P\nabla_\xi Y)$$
$$= g(PX, \nabla_\xi PY) = -g(\nabla_\xi PX, PY)$$
$$= -g(P\nabla_\xi X, PY) = -g(\nabla_\xi X, P^2 Y) = 0.$$

Also, for $X \in D^1$ or $X \in D^0$ we have

$$g(\nabla_\xi X, \xi) = -g(X, \nabla_\xi \xi) = 0.$$

Thus, the result follows.

Proposition 4.2. *Let M be a semi-invariant submanifold of a LP-cosymplectic manifold $\overline{M}$. Then*

$$A_{FX} Y + A_{FY} X = 0, \quad X,\ Y \in D^0 \oplus \{\xi\}, \tag{53}$$

$$g(\phi h(X, Y), N) = g(h(\phi X, Y), N), \quad X \in D^1 \oplus \{\xi\},\ Y \in TM,\ N \in \overline{D}^1. \tag{54}$$

Proof. For $X, Y \in D^0 \oplus \{\xi\}$ and $Z \in TM$, using (41) we get

$$g(A_{FX} X, Z = g(h(Y, Z), FX) = g(th(Y, Z), X)$$
$$= g(-P\nabla_Z Y + \nabla_Z PY - A_{FY} Z, X)$$
$$= -g(\nabla_Z Y, PX) - g(A_{FY} Z, X) = -g(A_{FY} X, Z),$$

which implies (53). Using (42), for $X \in D^1 \oplus \{\xi\}$, $Y \in TM$, $N \in \overline{D}^1$,

$$g(fh(X, Y), N) = g(\nabla_Y^\perp FX + h(Y, PX) - F\nabla_Y X, N)$$
$$= g(h(PX, Y), N) - g(F\nabla_Y X, N) = g(h(\phi X, Y), N).$$

5. Integrability conditions

In view of Proposition 3.4 we can state the following.

Theorem 5.1. *Let M be a submanifold of a LP-cosymplectic manifold such that $\xi \in TM$. Then*

1. $\{\xi\}$ *and* $\{\xi\}^{\perp}$ *are parallel,*
2. $\{\xi\}$ *and* $\{\xi\}^{\perp}$ *are integrable and their leaves are totally geodesic in M,*
3. *M is locally product of leaves of* $\{\xi\}$ *and* $\{\xi\}^{\perp}$,
4. *M is* $(\{\xi\},\{\xi\}^{\perp})$-*mixed totally geodesic.*

Lemma 5.2. *Let M be a semi-invariant submanifold of a LP-cosymplectic manifold* $\overline{M}$. *Then* $\mathsf{D} \in \{\mathsf{D}^1, \mathsf{D}^0\}$ *is integrable if and only if* $\mathsf{D} \oplus \{\xi\}$ *is integrable.*

Proof. Let D' be a distribution on M orthogonal to $\{\xi\}$ and let $\mathsf{D}' \oplus \{\xi\}$ be integrable. Then, for $X, Y \in \mathsf{D}' \oplus \{\xi\}, [X,Y] \in \mathsf{D}'$. Since $\{\xi\}^{\perp}$ is parallel,

$$g([X,Y],\xi) = g(\nabla_X Y - \nabla_Y X, \xi) = 0.$$

Hence, $[X,Y] \in \mathsf{D}'$ and D' is integrable, which proves if part. Conversely, if $\mathsf{D} \in \{\mathsf{D}^1, \mathsf{D}^0\}$ is integrable, then for $X,\ Y \in \mathsf{D}$, we have

$$[X + \xi, Y + \xi] = [X,Y] + [X,\xi] + [\xi,Y],$$

which in view of (37) and Proposition 4.1, shows that $\mathsf{D} \oplus \{\xi\}$ is integrable.

Theorem 5.3. *Let M be a semi-invariant submanifold of a LP-cosymplectic manifold* $\overline{M}$. *Then the following statements are equivalent*:

(a) D^0 *is integrable,*

(b) $\mathsf{D}^0 \oplus \{\xi\}$ *is integrable,*

(c) $A_{FX} Y = 0,\ X, Y \in \mathsf{D}^0$,

(d) $h(Y,Z) \in \overline{\mathsf{D}}^1,\ X \in \mathsf{D}^0,\ Z \in TM.$

Proof. Statements (a) and (b) are equivalent by Lemma 5.2. For $X,\ Y \in D^0 \oplus \{\xi\}$, in view of (53) and (51), we have

$$2A_{FX}Y = P[X,Y] \in D^1, \tag{55}$$

which shows equivalence of (b) and (c). Lastly, from

$$g(A_{FX}X,Z) = g(h(Y,Z),FX),\ \ X,Y \in D^0,\ \ Z \in TM$$

(c) and (d) are equivalent.

Theorem 5.4. *Let M be a semi-invariant submanifold of a LP-cosymplectic manifold* $\overline{M}$. *If M is* (D^0, D^1)- *mixed totally geodesic then* D^0 *is integrable.*

Proof. For $Y \in D^0$ and $Z \in D^1$, we have $h(Y,Z) = 0$. For $X,Y,Z \in D^0$ in view of (13) and (55), we have

$$2g(h(Y,Z),FX) = 2g(A_{FX}X,Z) = g(P[X,Y],Z) = g([X,Y],PZ) = 0,$$

which in view of the statement (d) of Theorem 5.3 gives the proof.

Theorem 5.5. *Let M be a semi-invariant submanifold of a LP-cosymplectic manifold* $\overline{M}$. *Then the following statements are equivalent*:

(a) D^1 *is integrable,*

(b) $D^1 \oplus \{\xi\}$ *is integrable,*

(c) $h(X,PY) = h(PX,Y),\ \ X,Y \in D^1$,

(d) $g(h(X,PY),FZ) = g(h(PX,Y),FZ),\ \ X,Y \in D^1,\ \ Z \in TM$.

Proof. Statements (a) and (b) are equivalent by Lemma 5.2. In view of (52), we have

$$F[X,Y] = h(X,PY) - h(PX,Y),\ \ X,Y \in D^1 \oplus \{\xi\},$$

which implies the equivalence of (b) and (c). $(c) \Rightarrow (d)$ is obvious. Lastly, let us assume (d). In view of (54), for $X,\ Y \in D^1$, $N \in \overline{D}^1$, we get

$$g(h(X,PY) - h(PX,Y),N) = g(\phi h(Y,X) - \phi h(X,Y),N) = 0,$$

that is, $h(X,PY)-h(PX,Y)$ is perpendicular to $\overline{D}^1$. Therefore, replacing FZ by $h(X,PY)-h(PX,Y)$ in (d), from the above equation we get

$$\| h(X,PY) - h(PX,Y) \| = 0,$$

and (c) follows.

Theorem 5.5 leads to the following.

Corollary 5.6. *Let M be a semi-invariant submanifold of a LP-cosymplectic manifold $\overline{M}$. If M is D^1-totally geodesic then D^1 is integrable .*

6. Totally umbilical and totally geodesic submanifolds

First, we prove a lemma.

Lemma 6.1. *Let D be a distribution on a submanifold M of a LP-cosymplectic manifold such that $\xi \in D$. If M is* D*-totally umbilical then M is* D*-totally geodesic.*

Proof. If *M is* D-totally umbilical then by definition for all $X, Y \in D$ we have

$$h(X,Y) = g(X,Y)\boldsymbol{K}$$

for some $\boldsymbol{K} \in T^{\perp}M$. But in view of (38), we have

$$\boldsymbol{K} = g(\xi,\xi)\boldsymbol{K} = h(\xi,\xi) = 0$$

and therefore *M* is D-totally geodesic.

The Lemma 6.1 implies the following two theorems.

Theorem 6.2. *Each totally umbilical submanifold M of a LP-cosymplectic manifold such that $\xi \in TM$, is totally geodesic.*

Theorem 6.3. *Each totally umbilical semi-invariant submanifold of a LP-cosymplectic manifold is totally geodesic.*

In view of the above theorem and Theorem 5.4 and Corollary 5.6, we have the following:

Theorem 6.4. *If M is a totally umbilical semi-invariant submanifold of a LP-cosymplectic manifold, then* D^0 *and* D^1 *are integrable.*

In last, we propose the following:

Exercise 6.5. To study semi-invariant and almost semi-invariant submanifolds of *LP*-nearly cosymplectic manifolds [8].

References

1. J.K. Beem and P.E. Ehrlich, *Global Lorentzian Geometry*, Marcel Dekker, 1981.
2. A. Bejancu, *Geometry of CR-submanifolds*, D. Reidel Publ. Co., 1986.
3. U.C. De, K. Matsumoto and A.A. Sheikh, On Lorentzian para-Sasakian manifolds, *Rendinconti del Seminario Matematico di Messina Serie II Suplemento al n.* **3** (1999).
4. K. Matsumoto, On Lorentzian paracontact manifolds, *Bull. Yamagata Univ. Nat. Sci.* **12** (1989), 151-156.
5. K. Matsumoto and I. Mihai, On a certain transformation in a Lorentzian para-Sasakian manifold, *Tensor* (*N.S.*) **47** (1988), 189-197.
6. I. Mihai and R. Rosca, On Lorentzian *P*-Sasakian manifolds, Classical Analysis, *World Scientific, Singapore* (1992), 155-169.
7. I. Mihai, A.A. Sheikh and U.C. De, On Lorentzian para-Sasakian manifolds, *Korean J. Math. Sci.* **6** (1999), 1-13.
8. S. Prasad and R.H. Ojha, Lorentzian paracontact submanifolds, *Publ. Math. Debrecen*, **44** (1994), 215-223.
9. I. Satō, On a structure similar to almost contact structure, I, *Tensor* (*N.S.*) **30** (1976), 219-224.
10. I. Satō, On a structure similar to almost contact structure, II, *Tensor* (*N.S.*) **31** (1977), 199-205.
11. M.M. Tripathi, On semi-invariant submanifolds of Lorentzian almost paracontact manifolds, *J. Korea Soc. Math. Ed. Ser. B: Pure Appl. Math.* **8** (2001), no. 1 (*To appear*).
12. M.M. Tripathi and S.S. Shukla, On submanifolds of Lorentzian almost paracontact manifolds, *Publ. Math. Debrecen* (*To appear*).
13. K. Yano and M. Kon, *CR* submanifolds of Kaehlerian and Sasakian manifolds, *Birkhäuser, Boston,* 1983.

Keywords and phrases: Semi-invariant submanifold, Lorentzian almost paracontact manifold, *LP*-cosymplectic manifold.

Mathematics Subject Classification: 53C25, 53C40