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Arithmetic Funtions Over Rings with Zero Divisors

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Abstract. It is well known that the complex-valued arithmetic functions form a unique factorization domain under addition and convolution, and so do those functions with values in other suitable rings. Here we consider instead such functions with values in a unique factorization ring with zero divisors and prove that under certain yet similar conditions, they also form a unique factorization ring with zero divisors.

1. Introduction

The set of complex-valued arithmetic functions, denoted by A_C, is known to be an integral domain under addition and convolution (or Dirichlet multiplication), see e.g. Apostol [1]. Cashwell and Everett [5] proved in 1959 that A_C is in fact a *unique factorization domain or factorial ring.* The proof of factoriality due to Cashwell and Everett runs briefly as follows: first, it is shown that A_C is isomorphic to $C[[x_1, x_2, \cdots]]$, the ring of formal power series in countably many indeterminates. Since the rings of formal power series (over C) in a finite number of indeterminates are factorial, and C is a field, this induces $C[[x_1, x_2, \cdots]]$ to be factorial which in turn establishes factoriality for A_C. That C is a field plays significant role in the proof, for changing it to other integral domains the situations become much more complicated. This sparked off, especially in the sixties, a number of investigations related to problems about unique factorization, see e.g. Auslander and Buchsbaum [2], Buchsbaum [4], Cashwell and Everett [6], Cohn [7], Lu [12], Samuel [13], [14]. All afore-mentioned investigations are done over rings without zero divisors. In this work, we consider instead the case where the complex field is replaced by a unique factorization ring with zero divisors. We take essentially the same definition of unique factorization rings with zero divisors as in Galovich [9]. Our proof of the main theorem is a combination of the ones due to Cashwell and Everett [5] and Lu [12]. There are, however, additional difficulties due firstly to the fact that the usual cancellation law in the case of domain must be replaced by a weak cancellation law, which amounts to multiplying by units, and secondly the presence of units makes it impossible to take appropriate limits. This is overcome by invoking upon the idea of compactness.

2. Unique factorization rings with zero divisors

Let R be a commutative ring with unity. An element $u \in R$ is a unit if there is $v \in R$ such that $uv = 1$. If $r, s \in R$, then r divides s, written $r \mid s$, if there exists $t \in R$ such that $rt = s$. Two elements r and s are associates, written $r \sim s$, if there exists a unit u such that $ru = s$. An element $r \in R - \{0\}$ is a zero divisor if there exists $s \in R - \{0\}$ such that $rs = 0$. Let $r \in R - \{0\}$; r is prime if, whenever $r \mid ab$, where $a, b \in R - \{0\}$, then $r \mid a$ or $r \mid b$; r is reducible if there exist non-units a, b such that $r = ab$; r is irreducible if r is not reducible. It is easily shown that if r and s are irreducible, then $r \mid s$ if and only if $r \sim s$. An element $d \in R - \{0\}$ is called a greatest common divisor of $a, b \in R$, not both 0, if $d \mid a$ and $d \mid b$, and if $c \in R - \{0\}$ is such that $c \mid a$ and $c \mid b$, then $c \mid d$.

A commutative ring R, with unity and zero divisors, is a *unique factorization ring,* UFR for short, if for each non-zero non-unit $r \in R$,

(i) there exist irreducible elements $r_1, \cdots, r_n$ such that $r = r_1 \cdots r_n$ and

(ii) whenever $r = r_1 \cdots r_n = s_1 \cdots s_m$ where $r_1, \cdots, r_n, s_1, \cdots, s_m$ are irreducible, then $n = m$ and the s_j can be renumbered so that $r_i \sim s_i \, (i = 1, \cdots, n)$.

A typical example of UFR is $Z/p^m Z$, where p is a rational prime and m is a positive integer ≥ 2, see Billis [3].

A very useful fact which will be repeatedly used is that any UFR satisfies a *weak cancellation law*, i.e. whenever $ax = ay \neq 0$, then $x \sim y$. This is easily shown as follows: from the unique factorization into irreducible elements $a = r_1 \cdots r_n, x = s_1 \cdots s_m, y = t_1 \cdots t_k$, the equation $ax = ay \neq 0$ together with uniqueness implies that $m = k$ and after some renumbering $s_j \sim t_j \, (j = 1, \cdots, m)$ and so $x \sim y$.

Denote the set of arithmetic functions over the UFR R by A_R, i.e.

$$A_R := \{ f : N \to R;\ R \text{ is a UFR} \}.$$

It is easily checked that $(A_R, +, *)$ is a ring with respect to addition $(f + g)(n) = f(n) + g(n)$, and convolution $f * g(n) := \sum_{ij=n} f(i) g(j)$.

That A_R has zero divisors is directly inherited from R as seen by the following example. Let $x, y \in R - \{0\}$ be zero divisors in R such that $xy = 0$. Take $f(1) = x$, $f(n) = 0$ for all $n > 1$, and $g(1) = y$, $g(n) = 0$ for all $n > 1$. Then $f * g = 0 \in A_R$, while $f,\ g \in A_R - \{0\}$.

Our first theorem gives a characterization of UFR.

Theorem 1. *Let R be a commutative ring with unity and with zero divisors. Assume that every non-zero non-unit element of R can be written as a product of finitely many irreducible elements of R. Then the following assertions are equivalent:*

(i) R is a UFR.
(ii) Any two elements of R have a greatest common divisor and R satisfies the weak cancellation law.

Proof. Assume the truth of assertion (i). That *R* satisfies the weak cancellation law has already been observed. Let a, b be two elements of *R*, which we may assume to be non-zero and non-unit, for otherwise the proof is trivial. Since *R* is a UFR, we can write $a = r_1^{n_1} \cdots r_k^{n_k}$, $b = r_1^{m_1} \cdots r_k^{m_k}$, where r_i are distinct irreducible elements of *R* and n_i, m_i are non-negative integers. Taking $d = r_1^{\max(n_1,m_1)} \cdots r_k^{\max(n_k,m_k)}$, we easily check that d is a greatest common divisor of a, b.

Assume the truth of assertion (ii). It follows easily by induction that the existence of a greatest common divisor of any two elements of *R* induces the existence of a greatest common divisor of any finite number of elements of *R*. Denote by (a,b) a greatest common divisor of $a, b \in R$. Then, with the aid of the weak cancellation law, we easily deduce the following properties (see e.g. pp.139-140 of Jacobson [10]):

$$((a,b),c) \sim (a,(b,c)),\ c(a,b) = (ca,cb),\ (a,b) \sim 1 \text{ and } (a,c) \sim 1 \Rightarrow (a,bc) \sim 1.$$

Let *p* be an irreducible element of *R* such that $p \mid ab$ with $a, b \in R$. If *p* does not divide *a* and *b*, then by the existence of greatest common divisors, $(p,a) = 1$. For otherwise there would be a non-zero non-unit $c \in R$ such that $c = (p,a)$ and so $cd = p$ for some $d \in R$. Since p is irreducible, d must be a unit, yielding $p \sim c$, and so $p \mid a$, a contradiction. Similarly $(p,b) = 1$. Now by the third property above, $(p,ab) = 1$, contradicting $p \mid ab$. Therefore, $p \mid a$ or $p \mid b$, indicating that an irreducible element must be a prime in *R*. In order to conclude that *R* is a UFR, we must show the uniqueness of factorization. Let *a* be a non-zero non-unit element of *R* having two factorizations into irreducible elements $a = r_1 \cdots r_k = s_1 \cdots s_n$ where r_j and s_j are irreducible. Since irreducible elements are primes, $r_1 \mid \text{some } s_j$, say s_1.

Since s_1 is irreducible, then $r_1 \sim s_1$. Cancelling out the factor r_1, which is permissible by the weak cancellation law, and continue the arguments. Should $k \neq n$, we would end up having a product of irreducible elements equal to a unit, which is impossible. Thus $k = n$ and simultaneously, after appropriate renumbering $r_i \sim s_i$ for all *i*.

3. Power series

Denote by $R_w := R[[x_1, x_2, \cdots]]$, $R_m := R[[x_1, \cdots, x_m]]$ the rings of formal power series in countably many indeterminates $x_1, x_2, \cdots$, respectively, finitely many indeterminates $x_1, \cdots, x_m$, over a UFR R. Since $R \subset R_w$, it follows that $(R_w, +, \bullet)$ and $(R_m, +, \bullet)$ are rings with zero divisors, with respect to addition and multiplication of formal power series. As shown in Section 14 of Cashwell and Everett [5], for $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots$, where $p_1 < p_2 < \cdots$ denotes the (ascending) set of rational primes, the correspondence $f \leftrightarrow \sum f(n) x_1^{\alpha_1} x_2^{\alpha_2} \cdots$, establishes the fact that $(A_R, +, \bullet)$ is isomorphic to $(R_w, +, \bullet)$.

Recall that a commutative ring with unity, R, is said to satisfy the *ascending chain condition for principal ideals,* ACCP for short, provided that every strictly increasing sequence of principal ideals of R admits a maximal element. It is easy to check that any UFR satisfies the ACCP and for any two isomorphic commutative rings with unity R_1 and R_2, if R_1 satisfies the ACCP, so does R_2. Before we proceed to consider the power series ring, we recall another definition which will be used in the next lemma. The *order* of $f \in A_R$ is defined to be

$$v(f) := \begin{cases} \min\{n;\ f(n) \neq 0\} & \text{if} \quad f \neq 0 \\ \infty & \text{if and only if} \quad f = 0 \end{cases}.$$

It is easily checked that $v(f * g) \geq v(f)v(g)$, and $v(f + g) \geq \min\{v(f), v(g)\}$.

Lemma 2.1. *Let R be a commutative ring with unity satisfying the weak cancellation law. If R satisfies the ACCP, so does* R_w.

Proof. Since R_w is isomorphic to A_R, it suffices to show that A_R satisfies the ACCP. Let $(f_1) \subset (f_2) \subset \cdots$ be an ascending chain of principal ideals in A_R. Without loss of generality, assume that $f_1 \neq 0$ and so $f_i \neq 0$ for all $i > 1$. Ideal inclusions imply that there exists $g_i \in A_R - \{0\}$ such that $f_i = f_{i+1} * g_i$. Considering orders, we get $v(f_i) = v(f_{i+1} * g_i) \geq v(f_{i+1})\, v(g_i)$. This yields a non-increasing sequence of positive integers $v(f_1) \geq v(f_2) \geq \cdots$ which must then terminate, i.e. there are two positive integers r and k for which $v(f_{r+j}) = v(f_r) = k$, say, for each integer $j \geq 0$. Now $0 \neq f_r(k) = (f_{r+1} * g_r)(k) = f_{r+1}(k) g_r(1)$ and so we obtain an ascending chain of non-zero principal ideals in R of the form $(f_r(k)) \subset (f_{r+1}(k)) \subset \cdots$. As R satisfies the ACCP, there exists an integer $m \geq r$ such that $(f_m(k)) = (f_{m+j}(k))$ of all $j \geq 0$, which implies, using the weak cancellation law of R, that $f_m(k) = u_j f_{m+j}(k)$, where u_j is a unit in *R*. But the chain inclusions in A_R give $f_m = h_j * f_{m+j}$ for some $h_j \in A_R - \{0\}$. By the weak cancellation law of *R*, $u_j \sim h_j(1)$ and so h_j is a unit in A_R, which means that the chain in A_R terminates, and we are done.

Lemma 2.2. *Let R be a commutative ring with unity satisfying the weak cancellation law. If R satisfies the ACCP, then every non-zero non-unit element of* R_w *is a product of a finite number of irreducible factors.*

Proof. From the last lemma, R_w satisfies the ACCP. Take any non-zero, non-unit *F* in R_w. If *F* is irreducible, we have nothing to prove. Assume that *F* is reducible. Then there exists non-zero, non-units F_1, G_1 in R_w such that $F = F_1G_1$. This yields an inclusion of principal ideals $(F) \subset (F_1)$. If both F_1 and G_1 are irreducible, we are done; otherwise at least one of them, say F_1 is reducible. Thus there are non-zero, non-units F_2, G_2 in R_w such that $F_1 = F_2G_2$ yielding another inclusion of principal ideals $(F_1) \subset (F_2)$. Should *F* not be written as a product of finitely many irreducible factors, we would get an infinite ascending chain of principal ideals generated by proper factors of *F*. But the ACCP insists that the chain must be finite, a desired contradiction.

Let $j \in N$ and $F(x_1, x_2, \cdots) \in R_w$ or R_m with $m \geq j$. By $(F)_j$, the *projection* of *F* on R_j, we mean the series $F(x_1, \cdots, x_j, 0, 0, \cdots)$ obtained from *F* by putting equal to 0 all terms of *F* actually involving any x_i with $i > j$. The map $F \to (F)_j$ is a ring homomorphism of R_w or R_m onto R_j and $(FG)_j = (F)_j(G)_j$.

Lemma 2.3. *Let F be a non-zero, non-unit element of* R_w. *Then there is a least positive integer* $J = J(F)$, *hereby called the index of F, for which* $(F)_j$ *is a non-zero, non-unit element of* R_j *for all* $j \geq J$.

Proof. Since *F* is a non-zero non-unit element of R_w, then *F* contains a non-zero coefficient of some monomial term $x_1^{n_1} x_2^{n_2} \cdots$ with n_i not all zero. If x_k is the last variable in this term with positive n_k, then $(F)_k \neq 0$. Among all such *k*, the least one is our desired *J*.

Lemma 2.4. *Let F be a non-zero, non-unit element of* R_w. *Let J be the index of F. If* $(F)_j$ *is irreducible in* R_j *for some* $j \geq J$, *then* $(F)_m$ *is irreducible in* R_m *for all* $m \geq j$ *and F is also irreducible in* R_w.

Proof. Take any $m \geq j \geq J$ and suppose we have a factorization $(F)_m = G^{(m)}H^{(m)}$, for some $G^{(m)}, H^{(m)} \in R_m$. Observe that $(F)_j = ((F)_m)_j = (G^{(m)}H^{(m)})_j = (G^{(m)})_j(H^{(m)})_j$. By the irreducibility of $(F)_j$, either $(G^{(m)})_j$ or $(H^{(m)})_j$ is a unit of R_j and so $G^{(m)}$ or $H^{(m)}$ is a unit of R_m. Hence, $(F)_m$ is irreducible. The proof of the last assertion is similar.

The proof of the next lemma requires a number of new concepts and definitions which we now elaborate. Let F be a non-zero non-unit in R_w and let j be a positive integer. A proper divisor of $(F)_j \neq 0$ in R_j is an element $\alpha \in R_j$ such that $(F)_j = \alpha\beta$, where β is a non-unit in R_j. By a *true factor* of $(F)_j \neq 0$, $j \in N$, we mean a non-unit proper divisor of $(F)_j$ in R_j, and call such factorization of $(F)_j$ a true factorization. F is said to be *finitely irreducible* if there is a least integer $I \geq$ the index J of F, for which $(F)_j$ is irreducible in R_j for all $j \geq I$. We call a chain $[G^{(1)}, G^{(2)}, \cdots]$, with each $G^{(i)} \in R_i$, *telescopic*, respectively *pseudo-telescopic*, if $G^{(i)} = (G^{(i+1)})_i$, respectively $G^{(i)} \sim (G^{(i+1)})_i$ in R_i for all i. Following Lu [12], we introduce a topology which eases the discussion considerably. We say that a non-zero monomial $cx_1^{k_1} x_2^{k_2} \cdots x_j^{k_j} \in R_j$ is of weight r when $r = 1k_1 + 2k_2 + \cdots + jk_j$. Clearly any $F \in R_j$ can be written as $F = F_0 + F_1 + \cdots$ where each F_i is a sum of all monomials of weight i. Define an order function ord on $R_j \subset R_w$ as follows:

$$\mathrm{ord}(0) = \infty,\ \mathrm{ord}(F) = \min\{n; F_n \neq 0\} \text{ if } F \neq 0 .$$

For $F \in R_w$, k a non-negative integer, define $\{B_k(F) := \{G \in R_w ; \mathrm{ord}(G - F) \geq k\}$. Observe that $R_w = B_0(0)$ and that if $B_s(F) \cap B_t(G) \neq \phi$ with $s \leq t$, then $B_t(G) \subseteq B_s(F)$. By Theorem 3.2, p.67 of Dugundji [8], it follows that $\{ B_k(F); F \in R_w,\ k$ a non-negative integer$\}$ is a basis for a topology, called the *weight topology*, of R_w. Indeed, by the same arguments as used in the proof of Theorem 1 in Lu[12], R_w is the completion of $\bigcup_{j=1}^{\infty} R_j$ with respect to this topology. By the same proof as in Lemma 2 of Lu [12], any infinite telescopic chain is a Cauchy sequence and by completeness has a limit in R_w. The case of pseudo-telescopic chain, which will be encountered, is more complicated and requires some compactness arguments. For convenience, we adopt the following definition. The ring R_w is said to be *u-sequentially compact* (with respect to the weight topology) if every sequence of units in R_w has a convergent subsequence. Clearly, the limit of such a subsequence must then be a unit in R_w.

Lemma 2.5. *If R_w is u-sequentially compact, then any pseudo-telescopic chain has a convergent subchain.*

Proof. Take a pseudo-telescopic chain $[G^{(1)}, G^{(2)}, \cdots]$. Thus

$$G^{(i)} = (u^{(i)} G^{(i+1)})_i, \text{ where } u^{(i)} \text{ is a unit in } R_i.$$

Put $F^{(1)} = G^{(1)}$, $F^{(2)} = u^{(1)} G^{(2)}, \cdots$, $F^{(j)} = u^{(1)} u^{(2)} \cdots u^{(j-1)} G^{(j)}$. Clearly, $G^{(j)} = v^{(j)} F^{(j)}$, where $v^{(j)} = (u^{(1)} \cdots u^{(j-1)})^{-1}$ is a unit in R_j and more importantly, $[F^{(1)}, F^{(2)}, \cdots]$ is a telescopic chain, which must then converge to a limit *F*, say, in R_w. Since R_w is *u*-sequentially compact, there is a subsequence $v^{(j_k)}$ of $v^{(j)}$ which converges to a unit v, say, in R_w. Hence, the subchain $[G^{(j_k)} = v^{(j_k)} F^{(j_k)}]$ converges to the limit vF.

Lemma 2.6. *Let R be a UFR. If R_j is a UFR for every positive integer j and R_w is u-sequentially compact, then all irreducible elements of R_w are finitely irreducible.*

Proof. We claim that for non-zero non-unit $F \in R_w$, if $(F)_j$ is reducible in R_j for all $j \geq$ the index *J* of *F*, then *F* is reducible in R_w. To prove the claim, take any non-zero non-unit *F* in R_w. Assume that for each $j \geq J$,

$$(F)_j = G^{(j)} H^{(j)} \text{ where } G^{(j)} \text{ and } H^{(j)} \text{ are true factors of } (F)_j \text{ in } R_j.$$

Any true factorization $(F)_m = G^{(m)} H^{(m)}$, with $m > J$, induces a true factorization of $(F)_{m-1} = ((F)_m)_{m-1} = (G^{(m)})_{m-1} (H^{(m)})_{m-1} = G^{(m-1)} H^{(m-1)}$. Continuing down to $(F)_J = G^{(J)} H^{(J)}$, we get a telescopic chain of true factors $[G^{(J)}, G^{(J+1)}, \cdots G^{(m)}]$. From the original assumption that for each $j \geq J$, $(F)_j$ is expressible as a product of two true factors in R_j, we have the existence of a sequence

$$k_0 = [G_{00}],\ k_1 = [G_{10} = (G_{11})_J, G_{11}],\ k_2 = [G_{20} = (G_{21})_J, G_{21} = (G_{22})_{J+1}, G_{22}], \cdots,$$

each k_i being a telescopic chain of true factors G_{ij}, $j = 0, 1, \cdots, i$, of $(F)_{J+j}$. By the unique factorization of R_j, the number of true factors of $(F)_j$ is finite for each *j*. Thus there is a true factor T_0 of $(F)_J$ such that there is an infinite set of the chains k_i having their first entry being an associate of T_0. Choose one of these and call it k_0'. Of this infinite set, there is an infinite subset of k_i whose second entry is an associate of some one true factor T_1 of $(F)_{J+1}$. Choose one and call it k_1'. Continuing in this way we get a sequence of telescopic chains

$$k_0' = [g_{00}', \cdots], \; k_1' = [g_{10}', \; g_{11}', \cdots], \; k_2' = [g_{20}', \; g_{21}', \; g_{22}', \; \cdots], \cdots$$

each of which extends, at least to the main diagonal, such that the entries on the diagonal and below have the property that, for each $j \geq 0$, $g_{ij}' \sim T_j$ for all $i \geq j$. Now construct the telescopic infinite chain $k*$ by working only with the main diagonal and the diagonal next below it, as follows: define $g_J^* = g_{00}'$. Since $g_{10}' \sim T_0 \sim g_{00}'$ in R_J, there is a unit $u^{(J)} \in R_J$ such that $g_J^* = g_{10}' u^{(J)} = (g_{11}' u^{(J)})_J$. Define $g_{J+1}^* = g_{11}' u^{(J)}$ in R_{J+1}. Then $g_J^* = (g_{J+1}^*)_J$. Note that g_{J+1}^* is a true factor of $(F)_{J+1}$ and $g_{J+1}^* \sim T_1$ in R_{J+1}. Since $g_{21}' \sim T_1 \sim g_{J+1}^*$ in R_{J+1}, there is a unit $u^{(J+1)} \in R_{J+1}$ such that $g_{J+1}^* = (g_{21}' \, u^{(J+1)} = g_{22}' \, u^{(J+1)})_{J+1}$. Define $g_{J+2}^* = g_{22}' u^{(J+1)}$ in R_{J+2}. Then $g_{J+1}^* = (g_{J+2}^*)_{J+1}$. Note that g_{J+2}^* is a true factor of $(F)_{J+2}$ and $g_{J+2}^* \sim T_2$ in R_{J+2}. Continuing in the same manner, we get an infinite telescopic chain of true factors

$$k* = \left[g_J^*, \; g_{J+1}^*, \; g_{J+2}^*, \; \cdots \right],$$

where $g_J^* = g_{00}' = g_{10}' \, u^{(J)} = (g_{11}' \, u^{(J)})_J$, $g_{J+1}^* = g_{11}' \, u^{(J)} = g_{21}' \, u^{(J+1)} = (g_{22}' \, u^{(J+1)})_{J+1}$, $g_{J+2}^* = g_{22}' \; u^{(J+1)} = g_{32}' \; u^{(J+2)} = (g_{33}' u^{(J+2)})_{J+2}, \cdots,$ which must converge to a limit *g*, say, in R_w. Now for all $j \geq 0$, we have $(F)_{J+j} = g_{J+j}^* \; h_{J+j}$, where h_{J+j} are non-zero, non-unit in R_{J+j}, and so $g_{J+j}^* \; h_{J+j} = (F)_{J+j} = ((F)_{J+j+1})_{J+j}$ $= (g_{J+j+1}^*)_{J+j} (h_{J+j+1})_{J+j} = g_{J+j}^* (h_{J+j+1})_{J+j}$, which by the weak cancellation law gives $h_{J+j} \sim (h_{J+j+1})_{J+j}$ in R_{J+j}. Thus we get a pseudo-telescopic chain $[h_J, h_{J+1}, \cdots]$, which by *u*-sequential compactness of R_w has a subchain converging to a limit *h*, say, in R_w. Now $(F)_{J+j} = g_{J+j}^* \; h_{J+j} = (g)_{J+j} h_{J+j}$ and so by passing to a subsequence, we arrive at $F = gh$. Clearly, both *g* and *h* are non-units in R_w implying that *F* is reducible in R_w and the claim is established. It follows from the claim that if *F* is irreducible in R_w, then there is a least integer $I \geq J$ for which $(F)_I$ is irreducible in R_I and by Lemma 2.4 for which $(F)_j$ is irreducible in R_j for all $j \geq I$. Consequently, *F* is finitely irreducible.

Lemma 2.7. *Let R be a UFR, F and* $G \in R_w$, $D^{(j)}$ *a greatest common divisor in* R_j *of* $(F)_j$ *and* $(G)_j$, $j \in N$. *If* R_j *is a UFR for every* $j \in N$, *then* $(D^{(j+1)})_j \sim D^{(j)}$ *for all* $j \geq L(F,G)$, *where* $L(F,G)$ *is a certain non-negative integer.*

Proof. If *F* or *G* is zero, then the assertion is trivial. Assume that both *F* and *G* are non-zero. Let *n* be the smallest integer such that $(F)_n$ and $(G)_n$ are both non-zero and let i be an integer $\geq n$. Since R_i is a UFR, we can represent $D^{(i)}$ as a finite product of irreducible elements of R_i. Denote by N_i the number of irreducible factors counted with multiplicity of $D^{(i)}$. Since $D^{(i)}$ is a greatest common divisor of $(F)_i$ and $(G)_i$ in R_i, $i \geq n$, $(D^{(i+1)})_i \neq 0$, and the number of irreducible factors of $(D^{(i+1)})_i$ is not less than the number of irreducible factors of $D^{(i+1)}$, then $(D^{(i+1)})_i \mid D^{(i)}$, which thus gives $N_i \geq N_{i+1}$. Note that the projection of each irreducible factor of $(D^{(i+1)})$ on R_i may not be irreducible in R_i. Thus we have the following descending chain of non-negative integers $N_n \geq N_{n+1} \geq \cdots$ and so there are integers *j* and *k* such that $k = N_{n+j+r}$ for every integer $r \geq 0$. Thus for every $m \geq n+j$, the projection of each irreducible factor of $D^{(m+1)}$ on R_m is also irreducible, yielding $(D^{(m+1)})_m \sim D^{(m)}$. Taking $L(F,G) = n+j$, we are done.

4. Main results

Theorem 2. *Let R be a UFR. If* R_j *is a UFR for each positive integer j and* R_w *is u-sequentially compact, then* R_w *is a UFR.*

Proof. Since *R* is a UFR, it satisfies the ACCP and so by Lemma 2.2 any non-zero non-unit element of R_w can be written as a finite product of irreducible elements of R_w. We shall use the characterization in Theorem 1. First, we show that any two elements *F* and *G* of R_w have a greatest common divisor. Since this is trivial when *F* or *G* is zero, we assume that both are non-zero. Let $D^{(i)}$ be a greatest common divisor of $(F)_i$ and $(G)_i$. We construct an infinite telescopic chain $[E^{(L)}, E^{(L+1)}, \cdots,]$ with the initial term in R_L, where $L = L(F,G)$ is as in Lemma 2.7, as follows: put $E^{(L)} = D^{(L)}$. Assume $E^{(j)}$, $j \geq L$, has been defined and let $D^{(j+1)}$ be any greatest common divisor of $(F)_{j+1}$ and $(G)_{j+1}$. Then by Lemma 2.7 there is a unit $u^{(j)}$ in R_j such that $E^{(j)} = (u^{(j)}D^{(j+1)})_j = (u^{(j)}(D^{(j+1)}))_j$. Taking $E^{(j+1)} = u^{(j)}D^{(j+1)}$ we get a telescopic chain $[E^{(L)}, E^{(L+1)}, \cdots]$. This chain has a limit $E \in R_w$. Note that $(E)_j = E^{(j)}$ or $(E^{(L)})_j$ according as $j \geq L$ or $0 \leq j < L$. Let $F^{(j)}$ and $G^{(j)}$ be two elements of R_j such that $(F)_j = F^{(j)}(E)_j$, $(G)_j = G^{(j)}(E)_j$ for each $j \geq L$. Then $(F^{(j+1)})_j \sim F^{(j)}$ and $(G^{(j+1)})_j \sim G^{(j)}$ by the weak cancellation law. Hence we have two pseudo-telescopic chains $[F^{(L)}, F^{(L+1)}, \cdots]$ and $[G^{(L)}, G^{(L+1)}, \cdots]$ with the initial terms in R_L. By Lemma 2.5, a subchain of $[F^{(L)}, F^{(L+1)}, \cdots]$, respectively a subchain of $[G^{(L)}, G^{(L+1)}, \cdots]$ converge to limits *f*, respectively *g* in R_w. Passing to subchain in $(F)_j = F^{(j)}(E)_j$, we deduce that $F = fE$. In the same manner, we get $G = gE$ for the weight topology, i.e. *E* is a common divisor of *F* and *G*. To show that *E* is a greatest divisor of *F* and *G*, we let E^* be any common divisor of *F* and *G* in R_w. Then $(E^*)_j$ is also a common divisor of $(F)_j$ and $(G)_j$ in R_j for each $j \geq L$. Since $(E)_j$ is a greatest common divisor of $(F)_j$ and $(G)_j$ for such *j*, then $(E^*)_j \mid (E)_j$. Thus there is an element $a^{(j)} \in R_j$ such that $(E)_j = a^{(j)}(E^*)_j$, $j \geq L$. Thus for $j \geq L$, we have

$$a^{(j)}(E^*)_j = (E)_j = ((E)_{j+1})_j = (a^{(j+1)}(E^*)_{j+1})_j = (a^{(j+1)})_j(E^*)_j$$

and so by the weak cancellation law $a^{(j)} \sim (a^{(j+1)})_j$ which yields a pseudo-telescopic chain $[a^{(L)}, a^{(L+1)}, \cdots]$. By Lemma 2.5, let its subchain converge to a limit *a* in R_w. Passing to subchain in $(E)_j = (a)_j(E^*)_j$, we get $E = aE^*$. Hence, *E* is a greatest common divisor of *F* and *G*. Lastly, we show that R_w satisfies the weak cancellation law. Let F, G and $H \in R_w$ be such that $FG = FH \neq 0$. Then there are integers *n* and *k* for which $(FG)_j = (FH)_j \neq 0$ for all $j \geq k$ and $(F)_i \neq 0$ for all $i \geq n$. Put $m = \max(k, n)$. Then for all $j \geq m$, we get $(F)_j(G)_j = (FG)_j = (FH)_j = (F)_j(H)_j \neq 0$. By the weak cancellation in R_j, we deduce that $(G)_j = u^{(j)}(H)_j$ for some unit $u^{(j)} \in R_j$. Now the definition of *u*-sequential compactness implies that $(u^{j)})$ has a subsequence converging to a unit *u* in R_w. Passing to subchain in $(G)_j = u^{(j)}(H)_j$ we get $G = uH$. The result now follows from Theorem 1.

As an immediate consequence, we mention

Corollary. *If R is a UFR such that R_j is a UFR for each positive integer j and R_w is u-sequentially compact, then A_R is a UFR.*

Final remarks. The main results in this paper can be extended by replacing the domain N of arithmetic function with an arithmetical semigroup *G*. This is done in the same manner as that in the proof of Proposition 1.3 in Knopfmacher [11] as follows: if *G* has infinitely many primes, then *G* is algebraically isomorphic to N and we are done. If *G* has only finitely many primes, say *j*, then the ring $A_R(G) := \{ f : G \to R\}$ is isomorphic to $R[[x_1, x_2, \cdots, x_j]] = R_j$ and the condition of R_j being a UFR thus implies that $A_R(G)$ is a UFR. Note that *u*-sequential compactness does not come into consideration in this case.

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