Regularity and Normality via Ideals

[1]R.A. MAHMOUD AND [2]A.A. NASEF

[1]Department of Mathematics, Faculty of Scicnce, Menoufia University, Shebin El-Koom, Egypt
[2]Department of Mathematics, Faculty of Education, Suez-Canal University, El-Arish, Egypt

Abstract. In this paper, the concepts of *I*-regular, *I*-normal and completely *I*-normal spaces have been presented utilizing the *I*-open notion. Moreover, $I - T$, -spaces, $i = 3, 4, 5$ are also given.

The similarities and dissimilarities between them and some other known corresponding types are discussed. Also, some of their characterizations and several of fundamental properties have been established.

1. Introduction

Recently, the topic of an ideal in topological spaces was comprehensively treated by general topologists, and many of their interesting properties were obtained. *I*-openness is one of these properties, which was introduced in 1991 by Jankovic and Hamlett [7] via the local function concept due to Viadyanathaswamy [15]. In 1992, Abd El-Monsef *et al.* [3] gave and studied both of *I*-closed sets and *I*-continuity. Moreover, many results via ideals which studied in [3,7] are generalized in [4]. While, in 1995, Dontchev [5] defined each of *I*-Hausdorff spaces and *I*-irresolute functions and several of their properties are also offered. Therefore, we devoted this paper to present both classes of *I*-regularity and *I*-normality. The relationships between them and some types, which were defined before are investigated. Also, we studied some equivalent definitions of each class. Moreover several of their properties and both of images and inverse images of these new spaces via some types of continuous functions have been discussed.

2. Preliminaries

Throughout this paper, (X, τ) and (Y, σ) denote topological spaces on which no separation properties are assumed unless explicitly stated. A nonempty collection $I \subseteq P(X)$ is called an ideal if it is closed under the heredity and finite additivity properties. The notation (X, τ, I) means a topological space (X, τ) with an ideal I on X. For any $x \in X, \tau(x) = \{U \subseteq X : U \in \tau \text{ and } x \in U\}$. If $A \subseteq X$, $\tau - cl(A)$ and $\tau - \text{int}(A)$ are the closure and the interior of A with respect to τ, respectively. For a space (X, τ, I)

and $A \subseteq X$, $A^*(I) = \{x \in X : A \cap U \notin I,$ for each $U \in \tau(x)\}$ denotes the local function of A with respect to I and τ [15]. Also, $A \subseteq X$ is I-open [7], semi-open [11], preopen [10], and β-open [1], if $A \subseteq \tau - \text{int}(A^*(I))$, $A \subseteq \tau - cl(\tau - \text{int}(A))$, $A \subseteq \tau - \text{int}(\tau - cl(A))$ and $A \subseteq \tau - cl(\tau - \text{int}(\tau - cl(A)))$, respectively. The class of all I-open (resp. semi-open, preopen, β-open) in (X, τ) will be denoted by $IO(X, \tau)$ (resp. $SO(X, \tau)$, $PO(X, \tau)$, $\beta O(X, \tau)$). While, $IO(X, x) = \{W \subseteq X : W \in IO(X, \tau),\ x \in W\}$, and τ^α mean the class of all α-sets [12], given by $\tau^\alpha = SO(X, \tau) \cap PO(X, \tau)$. In $(X, \tau), A \subseteq X$ is δ-open [16] if it is arbitrary union of regular-open sets. (X, τ, I) is I-Hausdorff [5] if for every distinct x, $y \in X$, there exists $W_x, W_y \in IO(X, \tau)$ such that $x \in W_x$, $y \in W_y$ and $W_x \cap W_y = \varnothing$. While, (X, τ) is s-regular [9], p-regular [14] and β-regular [2], if for each $x \in X$ and each closed set F not containing x, the two disjoint sets which are containing F and x are semi-open, preopen and β-open, respectively. But, (X, τ) is supra-normal denoted by s^*-normal [11], if for each disjoint closed sets F_1, F_2 there exists disjoint supra-open sets W_1, W_2 having $W_i \subseteq F_i$, $i = 1, 2$. Mashhour *et al.* [11] defined the classes of pre-normal space "denoted by p-normal" and β-normal space, also they showed that these types of spaces are considered as special cases of s^*-normal. A function $f : (X, \tau, I) \to (Y, \sigma, J)$ is I-irresolute [5], if $f^{-1}(W) \in IO(X, \tau)$ for each $W \in JO(Y, \sigma)$.

3. On I-regular spaces

Definition 3.1. *A space (X, τ, I) is I-regular if for each closed set $F \subseteq X$ and each $x \in X \setminus F$, there are disjoint H, $W \in IO(X, \tau)$ such that $F \subseteq X$ and $x \in W$.*
The implications between this new class of spaces with some corresponding types are given nextly.
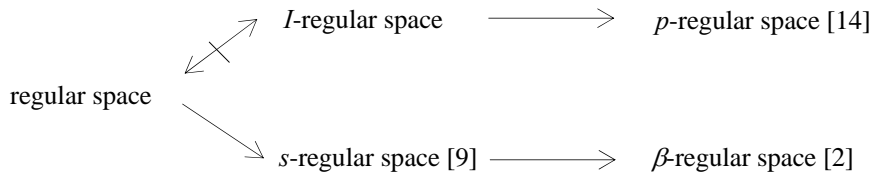
Remark 3.2.
(i)   The reverse between I-regularity and p-regularity is not true in general, as Example (3.3) shows. While the others studied in [2, 9,14].
(ii)   The independence between regular and I-regular spaces is illustrated throughout Examples (3.3) and (3.4).
(iii) I-regularity and p-regularity of $(X, \tau, \varnothing)$ are equivalent, for $IO(X, \tau, \varnothing) = PO(X, \tau)$.

Example 3.3. If $X = \{a,b,c\}, \tau = \{X, \emptyset, \{a\}, \{b,c\}$ and $I = \{\emptyset, \{b\}, \{c\}, \{b,c\}\}$. One can deduce that a space (X, τ, I) is regular and therefore p-regular but it is not I-regular.

Example 3.4. Let $X = \{a,b,c,d\}$ with $\tau = \{X, \emptyset, \{b,c\}\}$ and $I = \{\emptyset, \{d\}\}$. Then (X, τ, I) is I-regular but not regular.

Definition 3.5. In a space (X, τ, I), the I-closure of any $A \subseteq X$ is the smallest I-closed set containing A and denoted by $l - cl(A)$.

The next result is an immediate consequence of an ideal closure operator.

Proposition 3.6. For (X, τ, I) and $A \subseteq X$, $x \in I - cl(A)$ iff $AW \neq \emptyset$ for each $W \in IO(X, x)$.

Proposition 3.7. The following properties are hold, for any (X, τ, I).

(a) $I - cl(A) \subseteq I - cl(B)$, whenever $A \subseteq B$.

(b) $\displaystyle\bigcup_{\alpha \in \nabla} I - cl(A_\alpha) \subseteq I - cl\left(\bigcup_{\alpha \in \nabla} A_\alpha\right)$, while $A_\alpha \in P(X)$, $\alpha \in \nabla$.

Theorem 3.8. For a space (X, τ, I), the following are equivalent:

(a) (X, τ, I) is I-regular.

(b)   Each member of $\tau(x)$ contains the I-closure of member of $IO(X, x)$.

(c)   For each $A \subseteq X$ and each $U \in \tau$ such that $A \cap U \neq \emptyset$ there is $W \in IO(X, \tau)$ having $A \cap W \neq \emptyset$ and $I - cl(W) \subseteq U$.

(d)   For any $\emptyset \neq A \subseteq X$ and each closed set F of X with $A \cap F = \emptyset$, then there are disjoint H, $W \in IO(X, \tau)$ such that $A \cap H \neq \emptyset$ and $F \subseteq W$.

Proof.

(i)→(ii):   Let $U \in \tau(x)$, then $X \setminus U$ is closed not containing x. So, by (i) there are disjoint H, $W \in IO(X, \tau)$ such that $x \in H$ and $X \setminus U \subseteq W$. Hence $x \in H \subseteq I - cl(H) \subseteq U$.

(ii)→(iii):   Assume $A \in P(X)$ having $A \cap U \neq \emptyset$ for some $U \in \tau$, so letting $x \in A \cap U$. By (ii) there is $W \in IO(X, \tau)$ such that $x \in W \subseteq I - cl(W) \subseteq U$. Also, one can deduce that $A \cap W \neq \emptyset$.

(iii)→(iv):   Consider $A \cap F = \emptyset$ for a closed set F and for any $\emptyset \neq A \subseteq X$. This means that $X \setminus F \in \tau$ having $A \cap (X \setminus F) \neq \emptyset$. By (iii) there is $H \in IO(X, \tau)$ such that $A \cap H \neq \emptyset$ and $I - cl(H) \subseteq X\backslash F$. Putting $W = X \setminus I - cl(H)$, then $F \subseteq W \in IO(X, \tau)$.

(iv) → (i):   It is follows by taking $A = \{x\}$.

Definition 3.9. *A space* (X, τ, I) *is called* $I - T_3$ *if it is I-regular and* T_1 *- space.*

Theorem 3.10. *Each* $I - T_3$ *space is I-Hausdorff.*

Proof. Let (X, τ, I) be $I - T_3$ and distinct x_1, $x_2 \in X$. Then each $\{x_i\}$, $i = 1, 2$ is closed and $x_j \in X \setminus \{x_i\}$ for $i, j = 1, 2$ and $i \neq j$. By I-regularity of (X, τ, I) there exist disjoint I-open sets H, W such that $x_i \in \{x_i\} \subseteq H$ and $x_j \in W$ where $i, j \in \{1, 2\}$ and $i \neq j$. This shows that (X, τ, I) is I-Hausdorff.

The reverse of Theorem (3.10) is not true in general, as shown in the following example.

Example 3.11. Let $X = \{a, b, c, d\}$ with the topology $\tau = \{X, \varnothing, \{a, b\}\}$ and $I = \{\varnothing, \{d\}\}$ one can deduce that a space (X, τ, I) is I-Hausdorff but not $I - T_3$.

4. On *I*-normal spaces

Definition 4.1. *A space* (X, τ, I) *is I-normal if for each disjoint closed sets* F_1, F_2 *of X, there exist disjoint I-open sets* W_1, W_2 *such that* $F_i \subseteq W_i$, $i = 1, 2$.
The connection between I-normality with some others are given as follows,
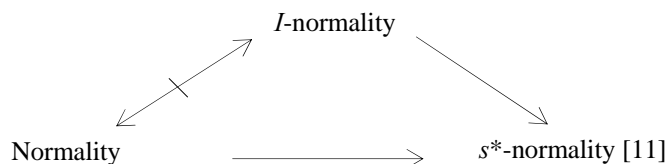
while the converses need not be hold in general as the next examples show.

Example 4.2.

(i) If $X = \{a, b, c, d, e\}$ topologized by $\tau = \{X, \varnothing, \{a, d\}, \{a, c, d\}, \{a, b, d\}, \{a, b, c, d\}, \{a, c, d, e\}\}$. Then (X, τ) is s^*-normal but not normal. While, if we take $I = \{\varnothing\}$, it is clear that (X, τ, I) is I-normal.

(ii) For any X consisting of two distinct elements with a discrete topology D and $I = P(X)$. Clearly (X, D) is normal and therefore s^*-normal but it is not I-normal.

Theorem 4.3. *For a space* (X, τ, I)*, the following statements are equivalent.*

(i) (X, τ, I) *is I-normal.*

(ii)   *For each disjoint closed sets* F_1, F_2*, there exists* $H \in IO(X, \tau)$ *such that* $F_1 \subseteq H$ *and* $I - cl(H)$ *is disjoint of* F_2*.*

(iii)   *For any closed set* $F \subseteq X$ *and any* $U \in \tau$ *containing F, there is* $H \in IO(X, \tau)$ *such that* $F \subseteq H \subseteq I - cl(H) \subseteq U$*.*

Proof.

(i)→(ii):   Let F_1, F_2 be nonempty disjoint closed in an *I*-normal space (X, τ, I). Then there are H, $W \in IO(X, \tau)$ such that $F_1 \subseteq H$, $F_2 \subseteq W$ and $H \cap W = \varnothing$. Thus $X \setminus W \subseteq X \setminus F_2$, this implies $I - cl(X \backslash W) = X \backslash W \subseteq X \backslash F_2$ but *H* and its *I*-closure are in $X \setminus W$. Therefore, $I - cl(H) \subseteq X \backslash F_2$ hence the result.

(ii)→(iii):   Assume *F* is closed and $U \in \tau$ such that $F \subseteq U$, then $X \setminus U$ is closed and disjoint of *F*. By (ii), there is $H \in IO(X, \tau)$ having $F \subseteq H$ and $I - cl(H) \cap (X \setminus U) = \varnothing$. This gives $I - cl(H) \subseteq U$ and so $F \subseteq H \subseteq I - cl(H) \subseteq U$.

(iii)→(i):   Consider any two nonempty closed disjoint sets F_1, F_2 of *X*, then $F_1 \subseteq X \setminus F_2 \in \tau$. Applying (iii), there exists $H \in IO(X, \tau)$ such that $F_1 \subseteq H \subseteq I - cl(H) \subseteq X \backslash F_2$. Therefore $F_2 \subseteq X \backslash I - cl(H) \in IO(X, \tau)$ and *H* is disjoint of $X \backslash I - cl(H)$. This completes the proof.

Theorem 4.4. If (X, τ, I) is $I - T_3$, then it is *I*-normal.

Proof. Let F_1, F_2 be disjoint closed sets, and for every $x \in F_1 \subseteq X \setminus F_2 \in \tau$, there exists $H_x \in IO(X, \tau)$. Since $\cup H_x \in IO(X, \tau)$ [7] and by (ii) of Proposition (3.7), we get $F_1 = \cup \{x\} \subseteq \cup H_x \subseteq \cup (I - cl(H_x)) \subseteq I - cl(\cup H_x) \subseteq X \setminus F_2$. Hence (X, τ, I) is *I*-normal.

Example 4.5. Let $X = \{a, b, c, d\}$ with a topology $\tau = \{X, \varnothing, \{a\}, \{b\}, \{a, b\}\}$ and $I = \{\varnothing, \{c\}\}$. Then (X, τ, I) is *I*-normal but not $I - T_3$.

Definition 4.6.   *An I-normal space which is* T_1 *is* $I - T_4$*.*

Theorem 4.7.   Each $I - T_4$ is $I - T_3$.

Proof. Let (X, τ, I) be $I - T_4$, $F \subseteq X$ be closed and $x \notin F$. Clearly $\{x\}$ is also closed and disjoint of *F*. *I*-normality of (X, τ, I) gives disjoint H, $W \in IO(X, \tau)$ having $x \in \{x\} \subseteq H$ and $F \subseteq W$. Hence the result.

Definition 4.8. *A space* (X, τ, I) *is completely I-normal if for any two separated subsets A, B of X, there are disjoint* $H, W \in IO(X, \tau)$ *such that* $A \subseteq H$ *and* $B \subseteq W$. *While a completely I-normal space, which is a* T_1- *space is an* I - T_5 - *space.*

Theorem 4.9. *Every completely I-normal space is I-normal.*

Proof. This is obviously by the fact that each pair of closed disjoint sets is separated.

5. Properties of *I*-regularity and *I*-normality

Note that if *I* and *J* are ideals on *X* with $I \subseteq J$, then for any $A \subseteq X, A^*(J) \subseteq A^*(I)$ [6]. This fact is useful to present the following result.

Proposition 5.1. *If I and J are ideals on X having* $I \subseteq J$. *Then* (X, τ, I) *is I-regular (I-normal) if* (X, τ, J) *is J-regular (J-normal).*

Proof. It is follows by the fact that $IO(X, \tau, J) \subseteq IO(X, \tau, I)$ when $I \subseteq J$.

Lemma 5.2. [7] *In* (X, τ, I) *if* $U \in \tau$, *then* $U \cap A^*(I) \subseteq (U \cap A)^*(I)$, *for any* $A \subseteq X$.

However, any subspace of *I*-regular (*I*-normal) spaces is *I*-regular (*I*-normal). To show this fact, we first state the next result.

Proposition 5.3. *For a space* (X, τ, I), *if* $U \in \tau$ *and* $W \in IO(X, \tau)$. *Then* $U \cap W \subseteq IO(U, \tau_U)$ [5].

Proof.   Since $U \cap W \subseteq \tau - \mathrm{int}(U) \cap \tau - \mathrm{int}(A^*(I)) \subseteq \tau - \mathrm{int}(U \cap A^*(I))$ $\subseteq \tau - \mathrm{int}(U \cap A)^*(I)$ {see Lemma (5.2)}.   Moreover, $\tau - \mathrm{int}(U \cap W)^*(I)$ $\subseteq \tau_U - \mathrm{int}(U \cap A)^*(I_U)$ for $\tau_U \subseteq \tau$. Thus $U \cap W$ is *I*-open in *U*.

Theorem 5.4. *Every open subspace of I-regular (resp. I-normal, completely I-normal) is I-regular (resp. I-normal, completely I-normal).*

Proof. Let (X, τ, I) be *I*-regular and $Y \in \tau$. To show that (Y, τ_Y, I_Y) is *I*-regular, let $K \subseteq Y$ be closed and $y \in Y \setminus K$. This shows that, there exists closed $F \subseteq X$ with $K = Y \cap F$. *I*-regularity of (X, τ, I) means that there are disjoint $H, W \in IO(X, \tau)$ having $F \subseteq X$ and $y \in W$. Above proposition illustrates that $Y \cap H$, $Y \cap W$ are I_Y - open sets which are containing *K* and *y*, respectively. Hence the result. While, the other cases are similar to *I*-regularity.

Corollary 5.5. *Any open subspace of $I - T_i$ - space, $i = 3, 4, 5$ is $I - T_i$, $i = 3, 4, 5$.*

Corollary 5.6. *Each regular-open and each δ-open subspace of an I-regular, I-normal, completely I-normal and $I - T_i$ - space, $i = 3, 4, 5$ is also.*

Lemma 5.7.   *For any space (X, τ, I), we have*

(a) $IO(X, \tau) \subseteq PO(X, \tau) \subseteq \beta O(X, \tau)$ [3].

(b)   *If $A \in SO(X, \tau)$ and $B \in PO(X, \tau)$, then $A \cap B \in PO(A, \tau_A)$ [13].*

(c)   *If $A \in \tau^\alpha$ and $B \in \beta O(X, \tau)$, then $A \cap B \in \beta O(A, \tau_A)$ [1].*

Theorem 5.8.

(i)   *If (X, τ, I) is I-regular and $A \in SO(X, \tau)$ (resp. $A \in \tau$), then a subspace (A, τ_A, I_A) is p-regular (resp. β-regular).*

(ii)   *Any semi-open (resp. α-set) subspace of I-normal is p-normal (resp. β-normal)*

Proof. This is immediately by Lemma (5.7).

For any function, one can observe that the image of any ideal is also ideal.

Theorem 5.9. *For a bijective continuous $f : (X, \tau, I) \rightarrow (Y, \sigma, f(I))$.   If f^{-1} is I-irresolute and (X, τ, I) is I-regular, (I-normal and completely I-normal) then $(Y, \sigma, f(I))$ is also.*

Proof. Let K be closed in Y and $y \in Y \setminus K$. By hypothesis $F = f^{-1}(K)$ is closed in X not containing $x = f^{-1}(y)$. I-regularity of (X, τ, I) gives there are disjoint $H, W \in IO(X, \tau)$ with $F \subseteq H$ and $x \in W$. But $f(H)$ and $f(W)$ are I-open sets in Y containing K and y, respectively which complete the proof. While the proofs of the other cases are similar.

Theorem 5.10. *The inverse image of I-regular, (I-normal and completely I-normal) space under a closed I-irresolute bijection is I-regular, (I-normal and completely I-normal) space.*

Proof. Let $f : (X, \tau, I) \rightarrow (Y, \sigma, f(I))$ be bijective closed *I*-irresolute and $(Y, \sigma, f(I))$ is *I*-regular. To shows that (X, τ, I) is *I*-regular, let $F \subseteq X$ be closed and $x \in X \setminus F$ and so, $K = f(F)$ is closed in Y and $y = f(x) \in Y \setminus K$. Hence, there exist disjoint $H, W \in IO(Y, \sigma)$ having $K \subseteq H$ and $y \in W$. I-irresoluteness of f means that $f^{-1}(H)$, $f^{-1}(W)$ are disjoint *I*-open sets in X containing F and x, respectively. Hence the result. While, the other parts follow by the above technique.

References

1. M.E Abd El-Monsef, S.N. El-Deeb and R.A. Mahmoud, β-open sets and β-continuous mappings, *Bull. Fac. Sci., Assiut Univ.* **12** (1) (1983), 77-90.

2. M.E. Abd El-Monsef, A.N. Geaisa and R.A. Mahmoud, β-regular spaces, *Proc. Math. Phys. Soc. Egypt* **60** (1985), 47-52.

3. M.E. Abd El-Monsef, E.F. Lashien and A.A. Nasef, On *I*-open sets and *I*-continuous functions, *Kyungpook Math. J.* **32** (1) (1992), 21-30.

4. M.E. Abd El-Monsef, R.A. Mahmoud and A.A. Nasef, On almost *I*-openness and almost *I*-continuity, *To appear in J. Egyptian Math. Soc*. **7** (2) (1999).

5. J. Dontchev, On Hausdorff spaces via topological ideals and *I*-irresolute functions, *General Topology and Appl. Ann. New-York Acad. Sci.* **767** (1995), 28-38.

6. D. Jankovic and T.R. Hamlett, New topologies from old via ideals, *Amer. Math. Monthly* **97** (1990), 295-310.

7. D. Jankovic and T.R. Hamlett, Compatible extensions of ideals, *Boll. U.M.I.* **7** (1992), 453-465.

8. N. Levine, Semi-open sets and semi-continuity in topological spaces, *Amer. Math. Monthly* **70** (1963), 36-41.

9. S.N. Maheshwari and R. Prasad, On *s*-regular spaces, *Glasnik Mat. Ser.* III **10** (30) (1975), 347-350.

10. A.S Mashhour, M.E. Abd El-Monsef and S.N. El-Deeb, On precontinuous and weak-precontinuous mappings, *Proc. Math. Phys. Soc. Egypt* **53** (1982), 47-53.

11. A.S. Mashhour, A.A. Allam and I.A. Hasanein, Some supratopological results, *J. Hahannes Kepler Univ. Linz, Austria* **25** (1983).

12. O. Njåstad, On some classes of nearly open sets, *Pacific J. Math.* **15** (1965), 961-970.

13. T. Noiri, Hyper connectedness and preopen sets, *Rev. Roumaine Math. Pures Appl.* **29** (4) (1984), 329-334.

14. T. Noiri, A.S. Mashhour, I.A. Hasanein and S.N. El-Deeb, On *p*-regular spaces, *Bull. Math. Soc. Sci. Math. R.S. Roumanie (N.S.)* **27** (75) (1983), 311-315.

15. R. Vaidyanathaswamy, The localization theory in set-topology, *Proc. Indian Acad. Sci.* **20** (1945), 51-61.

16. N.V. Velicko, *H*-closed topological spaces, *Amer. Math. Soc. Transl.* **78** (1968), 78-103.