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Certain Meromorphic Harmonic Functions

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Abstract. We introduce and study new classes of meromorphic harmonic functions. In addition to finding certain coefficient characterizations, we obtain several inclusion relations, convexity conditions, and extreme points for these classes.

1. Introduction

A continuous function $f = u + iv$ is a complex valued harmonic function in a complex domain $D \subseteq C$ if both u and v are real harmonic in D. Hengartener and Schober [2], among other things, investigated the family $\sum_H$ of functions $f = h + \overline{g}$ which are harmonic, meromorphic, orientation preserving, and univalent in $\tilde{U} = \{z : |z| > 1\}$ where

$$h(z) = z + \sum_{n=1}^{\infty} a_n z^{-n} \quad , \quad g(z) = \sum_{n=1}^{\infty} b_n z^{-n} \quad z \in \tilde{U} \tag{1}$$

Motivated by the results of [2], Jahangiri and Silverman [4] and Jahangiri [3] studied the classes of functions in $\sum_H$ which are starlike or convex in $\tilde{U}$. In particular, they investigated starlike and convex functions in the class $\sum_{\overline{H}}$ consisting of functions $f = h + \overline{g}$ where h and g are of the form

$$h(z) = z + \sum_{n=1}^{\infty} |a_n| z^{-n} \quad , \quad g(z) = \sum_{n=1}^{\infty} |b_n| z^{-n} \quad z \in \tilde{U}. \tag{2}$$

In this paper we look at the following six classes of functions. For these classes we provide coefficient characterizations, inclusion and convexity conditions, and extreme points.

Definition 1. *For* $0 \le \alpha < 1$, *we define*

$$f \in \Sigma_H P(\alpha) \Leftrightarrow \mathrm{R}\left(\frac{f(z)}{z}\right) > \alpha, \ z \in \tilde{U} \tag{1.1}$$

$$f \in \Sigma_H Q(\alpha) \Leftrightarrow \mathrm{R}\left(\frac{\frac{\partial}{\partial\theta} f(z)}{\frac{\partial}{\partial\theta} z}\right) > \alpha \quad , \quad z = re^{i\theta} \in \tilde{U} \tag{1.2}$$

$$f \in \Sigma_H R(\alpha,\lambda) \Rightarrow \mathrm{R}\left\{(1-\lambda)\frac{f(z)}{z} + \lambda\frac{\frac{\partial}{\partial\theta} f(z)}{\frac{\partial}{\partial\theta} z}\right\} > \alpha \ , \ \lambda \ge 0 \ , \ z \in \tilde{U} \tag{1.3}$$

$$\Sigma_{\overline{H}} P(\alpha) \equiv \Sigma_H P(\alpha) \cap \Sigma_{\overline{H}} \tag{1.4}$$

$$\Sigma_{\overline{H}} Q(\alpha) \equiv \Sigma_H Q(\alpha) \cap \Sigma_{\overline{H}} \tag{1.5}$$

$$\Sigma_{\overline{H}} R(\alpha,\lambda) \equiv \Sigma_H R(\alpha,\lambda) \cap \Sigma_{\overline{H}} . \tag{1.6}$$

Note that if the co-analytic part of $f = h + \overline{g}$ is zero, i.e. if $g \equiv 0$, then the conditions 1.1, 1.2, and 1.3 reduce to the respective analytic cases $\mathrm{R}(h(z)/z) > \alpha$, $\mathrm{R}(zh'(z))' > \alpha$, and $\mathrm{R}\{(1-\lambda)(h(z)/z) + \lambda h'(z)\} > \alpha$. See, for example, [4, 5, 6]. We also note that $\Sigma_H R(\alpha, 0) \equiv \Sigma_H P(\alpha)$ and $\Sigma_H R(\alpha,1) \equiv \Sigma_H Q(\alpha)$.

2. Coefficient inequalities

First we give a sufficient coefficient bound for function to be in $\Sigma_H R(\alpha,\lambda)$.

Theorem 1. *Let* $f = h + \overline{g}$ *where h and g are given by* (1). *If*

$$\sum_{n=1}^{\infty}\left[\left|(n+1)\lambda - 1\right|\left|a_n\right| + \left|(n-1)\lambda + 1\right|\left|b_n\right|\right] \le 1 - \alpha$$

for some $\lambda \ge 0$ *and* $\alpha\,(0 \le \alpha < 1)$, *then* $f \in \Sigma_H R(\alpha,\lambda)$.

Proof. Letting $w(z) = (1-\lambda)[f(z)/z] + \lambda[\frac{\partial}{\partial\theta} f(z)]/[\frac{\partial}{\partial\theta} z]$, it suffices to show that $|1-\alpha+w| \ge |1+\alpha-w|$. This is equivalent to show that if the condition (3) holds then

$$\left|(1-\alpha)z + (1-\lambda)(h(z) + \overline{g(z)}) + \lambda(zh'(z) - \overline{zg'(z)})\right| -$$

$$\left|(1+\alpha)z - (1-\lambda)(h(z) + \overline{g(z)}) - \lambda(zh'(z) - \overline{zg'(z)})\right| := M(\alpha,\lambda) \ge 0.$$

Substituting for h and g in $M(\alpha,\lambda)$ yields

$$\begin{aligned} M(\alpha,\lambda) &= \left|(2-\alpha)z - \sum_{n=1}^{\infty}(\lambda(n+1)-1)a_n z^{-n} + \sum_{n=1}^{\infty}(\lambda(n-1)+1)\overline{b_n z^{-n}}\right| \\ &\quad - \left|\alpha z + \sum_{n=1}^{\infty}(\lambda(n+1)-1)a_n z^{-n} - \sum_{n=1}^{\infty}(\lambda(n-1)+1)\overline{b_n z^{-n}}\right| \\ &\ge 2|z|\left[1-\alpha-\left(\sum_{n=1}^{\infty}|\lambda(n+1)-1||a_n| + \sum_{n=1}^{\infty}(\lambda(n-1)+1)|b_n|\right)|z|^{-(n+1)}\right] \\ &\ge 2|z|\left\{1-\alpha-\sum_{n=1}^{\infty}\left[|\lambda(n+1)-1||a_n| + (\lambda(n-1)+1)|b_n|\right]\right\}. \end{aligned}$$

Now this last expression is non-negative by the hypothesis of Theorem 1 and so the proof is complete.

Remark. For $f = h + \overline{g}$ as given by (1), Jahangiri and Silverman [4] prove that if $\sum_{n=1}^{\infty} n(|a_n| + |b_n|) \le 1$ then f is orientation preserving and univalent in $\tilde{U}$. Since $n \le (n-1)\lambda + 1 \le (n+1)\lambda - 1$, we conclude that the hypothesis of Theorem 1 is sufficient for the harmonic function $f = h + \overline{g}$ to be orientation preserving and univalent in $\tilde{U}$.

Corollary 1. *Let* $f = h + \overline{g}$ *where* h *and* g *are given by* (1). *Then* $f \in \sum_H P(\alpha)$ *if* $\sum_{n=1}^{\infty}(|a_n| + |b_n|) \le 1-\alpha$, $0 \le \alpha < 1$.

Corollary 2. *Let* $f = h + \overline{g}$ *where* h *and* g *are given by* (1). *Then* $f \in \sum_H Q(\alpha)$ *if* $\sum_{n=1}^{\infty} n(|a_n| + |b_n|) \le 1-\alpha$, $0 \le \alpha < 1$.

In the following theorem we show that the sufficient coefficient condition given by (3) is also necessary for the family $\sum_{\overline{H}} R(\alpha, \lambda)$.

Theorem 2. *Let* $f = h + \overline{g}$ *where h and g are given by* (2). *Then* $f \in \sum_{\overline{H}} R(\alpha, \lambda)$ *if and only if*

$$\sum_{n=1}^{\infty} \left[\left| (n+1)\lambda - 1 \right| \left| a_n \right| + \left| (n-1)\lambda + 1 \right| \left| b_n \right| \right] \leq 1 - \alpha . \tag{4}$$

Proof. In view of the fact that $\sum_{\overline{H}} R(\alpha, \lambda) \subset \sum_H R(\alpha, \lambda)$, the "if" part follows from Theorem 1. For the "only if" part, assume that $f \in \sum_{\overline{H}} R(\alpha, \lambda)$.

If $\lambda > 0$, then for $z = re^{i\theta}, r > 1$, and θ real we have

$$\begin{aligned}
\mathrm{R}\left\{ (1-\lambda)\frac{f(z)}{z} + \lambda \frac{\frac{\theta}{\partial\theta} f(z)}{\frac{\partial}{\partial\theta} z} \right\} &= \mathrm{R}\left\{ (1-\lambda)\left(\frac{h(z)}{z} + \frac{\overline{g(z)}}{z} \right) + \lambda \left(h'(z) + \frac{\overline{zg'(z)}}{z} \right) \right\} \\
&= \mathrm{R}\left\{ 1 - \sum_{n=1}^{\infty} (\lambda(n+1)-1) \left| a_n \right| r^{-(n+1)} e^{-i(n+1)\theta} \right. \\
&\qquad \left. - \sum_{n=1}^{\infty} (\lambda(n-1)+1) \left| b_n \right| r^{-(n+1)} e^{i(n-1)\theta} \right\} \\
&= \left\{ 1 - \sum_{n=1}^{\infty} (\lambda(n+1)-1) \left| a_n \right| r^{-(n+1)} \cos(n+1)\theta \right. \\
&\qquad \left. - \sum_{n=1}^{\infty} (\lambda(n-1)+1) \left| b_n \right| r^{-(n+1)} \cos(n-1)\theta \right\} \geq \alpha .
\end{aligned}$$

The above inequality must hold for all $z = re^{i\theta} \in \tilde{U}$. In particular, it must hold for $\theta = 0$ and $r \to 1^+$, which gives the required condition (4).

If $\lambda = 0$, then by the Definition 1.3, we must have

$$\begin{aligned}
\mathrm{R}\left\{ \frac{f(z)}{z} \right\} &= \mathrm{R}\left\{ 1 + \sum_{n=1}^{\infty} \left| a_n \right| z^{-(n+1)} + \frac{\bar{z}}{z} \sum_{n=1}^{\infty} \left| b_n \right| (\bar{z})^{-(n+1)} \right\} \\
&= \mathrm{R}\left\{ 1 + \sum_{n=1}^{\infty} \left| a_n \right| r^{-(n+1)} e^{-i(n+1)\theta} + \sum_{n=1}^{\infty} \left| b_n \right| r^{-(n+1)} e^{-i(n-1)\theta} \right\} \\
&= 1 + \sum_{n=1}^{\infty} \left| a_n \right| r^{-(n+1)} \cos(n+1)\theta + \sum_{n=1}^{\infty} \left| b_n \right| r^{-(n+1)} \cos(n-1)\theta \geq \alpha .
\end{aligned}$$

Similarly, we let $f = h + \overline{g}$ and $r \to 1^+$ to obtain $\sum_{n=1}^{\infty}(|a_n| + |b_n|) \leq 1 - \alpha$ and so the proof is complete.

Corollary 3. *Let* $f = h + \overline{g}$ *where* h *and* g *are given by* (2). *If* $0 \leq \alpha < 1$, *then* $f \in \sum_{\overline{H}} P(\alpha)$ *if and only if* $\sum_{n=1}^{\infty}(|a_n| + |b_n| \leq 1 - \alpha$, *and* $f \in \sum_{\overline{H}} Q(\alpha)$ *if and only if* $\sum_{n=1}^{\infty} n(|a_n| + |b_n|) \leq 1 - \alpha$.

3. Inclusion relations and extreme points

The inclusion relations between the classes $\sum_{\overline{H}} R(\alpha, \lambda)$, $\sum_{\overline{H}} P(\alpha)$, and $\sum_{\overline{H}} Q(\alpha)$ are given in the following.

Theorem 3. *Let* $0 \leq \alpha < 1$. *Then*

(i) $\sum_{\overline{H}} Q(\alpha) \subset \sum_{\overline{H}} P(\alpha)$

(ii) $\sum_{\overline{H}} Q(\alpha) \subset \sum_{\overline{H}} R(\alpha, \lambda)$, $0 < \lambda \leq 1$

(iii) $\sum_{\overline{H}} R(\alpha, \lambda) \subset \sum_{\overline{H}} Q(\alpha)$, $\lambda \geq 1$.

Proof. Part (i) follows from Corollary 3 upon noting that

$$\sum_{n=1}^{\infty}\left(|a_n| + |b_n|\right) \leq \sum_{n=1}^{\infty} n(|a_n| + |b_n|) \leq 1 - \alpha.$$

(ii) For $0 < \lambda \leq 1$, we observe that

$$\sum_{n=1}^{\infty}\left(\lambda(n+1) - 1\right)|a_n| + \sum_{n=1}^{\infty}\left(\lambda(n-1+1)\right)|b_n| \leq \sum_{n=1}^{\infty} n|a_n| + \sum_{n=1}^{\infty} n|b_n| \leq 1 - \alpha,$$

by the second part of Corollary 3. And so (ii) follows fromTheorem 2.

For (iii) we note that if $\lambda \geq 1$ then by Theorem 2

$$\sum_{n=1}^{\infty} n\left(|a_n| + |b_n|\right) \leq \sum_{n=1}^{\infty}\left(\lambda(n+1) - 1\right)|a_n| + \sum_{n=1}^{\infty}\left(\lambda(n-1) + 1\right)|b_n| \leq 1 - \alpha.$$

Therefore the result follows from the second part of Corollary 3.

Note that for $\lambda = 1$, $\Sigma_{\overline{H}}\, Q(\alpha) \equiv \Sigma_{\overline{H}}\, R(\alpha,\lambda)$. Also note that the containment in (i) is proper since $z + (1-\alpha)z^{-2} \in \Sigma_{\overline{H}}\, P(\alpha) - \Sigma_{\overline{H}}\, Q(\alpha)$ for $0 \le \alpha < 1$.

We now examine certain inclusion properties related to the starlikeness and convexity of the functions in $\Sigma_{\overline{H}}\, R(\alpha,\lambda)$.

Denote by $\Sigma_{\overline{H}}\, S^*(\alpha)$, the subclass of $\Sigma_{\overline{H}}$, consisting of functions $f = h + \overline{g}$ which are starlike of order $\alpha\,(0 \le \alpha < 1)$ in $\tilde{U}$, that is, f satisfy the condition

$$\frac{\partial}{\partial\theta}\arg\left(f(re^{i\theta})\right) \ge \alpha,\ 0 \le \alpha < 1,\ z = re^{i\theta},\ 0 \le \theta < 2\pi,\ r > 1.$$

Also, denote by $\Sigma_{\overline{H}}\, K(\alpha)$, the subclass of $\Sigma_{\overline{H}}$, consisting of functions $f = h + \overline{g}$ which are convex of order $\alpha\,(0 \le \alpha < 1)$ in $\tilde{U}$, that is, $f \in \Sigma_{\overline{H}}\, K(\alpha)$ if and only if

$$\frac{\partial}{\partial\theta}\left(\arg\left\{\frac{\partial}{\partial\theta} f(re^{i\theta})\right\}\right) \ge \alpha,\ 0 \le \alpha < 1,\ z = re^{i\theta},\ 0 \le \theta < 2\pi,\ r > 1.$$

For such functions, the second author [3] proved the following

Theorem A. *Let* $f = h + \overline{g}$ *where h and g are of the form* (2). *Then*

(i) $f \in \Sigma_{\overline{H}}\, S^*(\alpha) \Leftrightarrow \sum_{n=1}^{\infty}\left\{(n+\alpha)|a_n| + (n-\alpha)|b_n|\right\} \le 1-\alpha$

(ii) $f \in \Sigma_{\overline{H}}\, K^*(\alpha) \Leftrightarrow \sum_{n=1}^{\infty}\left\{n(n+\alpha)|a_n| + n(n-\alpha)|b_n|\right\} \le 1-\alpha$.

We are now ready to state and prove the following:

Theorem 4.

(i) $\sum_{\overline{H}} S^*(0) \subset \sum_{\overline{H}} R(0,\lambda)$, *if* $0 \le \lambda < 1$

(ii) $\sum_{\overline{H}} R(\alpha,\lambda) \subset \sum_{\overline{H}} S^*(\beta)$, *if* $1 \le \lambda,\ 0 \le \alpha, \beta < 1,\ 0 \le \beta \le \alpha/(2-\alpha)$

(iii) $\sum_{\overline{H}} K(\beta) \subset \sum_{\overline{H}} Q(\alpha) \subset \Sigma_{\overline{H}}\, P(\alpha)$, *if* $0 \le \alpha \le \beta < 1,\ b_1 = 0$

(iv) $\sum_{\overline{H}} K(\beta) \subset \sum_{\overline{H}} R(\alpha,\lambda)$, *if* $0 < \lambda \le 1,\ 0 \le \alpha \le \beta < 1$.

Proof. (i) If $0 \le \lambda < 1$ and $f \in \Sigma_{\overline{H}} S^*(0)$, then by Theorem A(i) we have

$$\sum_{n=1}^{\infty} |\lambda(n+1)-1| |a_n| + \sum_{n=1}^{\infty} (\lambda(n-1)+1)|b_n| \le \sum_{n=1}^{\infty} n(|a_n| + |b_n|) \le 1.$$

Now in view of Theorem 2, it follows that $f \in \Sigma_{\overline{H}} R(0,\lambda)$.

(ii) Let $f \in \Sigma_{\overline{H}} R(\alpha,\lambda)$. Then by the hypothesis we have

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{n+\beta}{1-\beta}|a_n| + \sum_{n=1}^{\infty} \frac{n-\beta}{1-\beta}|b_n| &\le \sum_{n=1}^{\infty} \frac{n+\beta}{1-\beta}|a_n| + \sum_{n=1}^{\infty} \frac{n}{1-\beta}|b_n| \\ &\le \sum_{n=1}^{\infty} \frac{2n-\alpha(n-1)}{2(1-\alpha)}|a_n| + \sum_{n=1}^{\infty} \frac{2n-n\alpha}{2(1-\alpha)}|b_n| \\ &\le \sum_{n=1}^{\infty} \frac{n}{1-\alpha}|a_n| + \sum_{n=1}^{\infty} \frac{n}{1-\alpha}|b_n| \\ &\le \sum_{n=1}^{\infty} \frac{\lambda(n+1)-1}{1-\alpha}|a_n| + \sum_{n=1}^{\infty} \frac{\lambda(n-1)+1}{1-\alpha}|b_n|.\end{aligned}$$

This last expression is less than or equal to 1, by Theorem 2. Now part (ii) follows from Theorem A (i) above.

(iii) If $f \in \Sigma_{\overline{H}} K(\beta)$, then by Theorem A (ii) and the given conditions, we have

$$\begin{aligned}\sum_{n=1}^{\infty} (|a_n| + |b_n|) &\le \sum_{n=1}^{\infty} n(|a_n| + |b_n|) \\ &\le \sum_{n=1}^{\infty} n(n+\beta)|a_n| + \sum_{n=1}^{\infty} n(n-\beta)|b_n| \\ &\le 1-\beta \le 1-\alpha.\end{aligned}$$

Therefore, $f \in \Sigma_{\overline{H}} Q(\alpha) \subset \Sigma_{\overline{H}} P(\alpha)$ by Corollary 3.

(iv) Assume that $f \in \Sigma_{\overline{H}} R(\alpha,\lambda)$. We need to show that $f \in \Sigma_{\overline{H}} R(\alpha,\lambda)$. In view of Theorem A (ii) and proceeding as in (iii), we have

$$\sum_{n=1}^{\infty} |\lambda(n+1)-1| |a_n| + \sum_{n=1}^{\infty} (\lambda(n-1)+1)|b_n| \le \sum_{n=1}^{\infty} n(|a_n| + |b_n|) \le 1-\alpha.$$

Then by Theorem 2 the proof is complete.

Theorem 5. *The family* $\Sigma_{\overline{H}}\ R(\alpha,\lambda)$ *is convex but not compact.*

Proof. For $i=1,2,3,\cdots$ suppose that $f_i \in \Sigma_{\overline{H}}\ R(\alpha,\lambda)$ where f_i is given by

$$f_i(z) = z + \sum_{n=1}^{\infty} \left|a_{i_n}\right| z^{-n} - \sum_{n=1}^{\infty} \left|b_{i_n}\right| (\overline{z})^{-n}.$$

Then, by Theorem 2,

$$\sum_{n=1}^{\infty} \left[\left|\lambda(n+1)-1\right|\left|a_{i_n}\right| + \left|\lambda(n-1)+1\right|\left|b_{i_n}\right| \right] \le 1-\alpha. \tag{5}$$

For $\sum_{i=1}^{\infty} t_i = 1,\ 0 \le t_i \le 1,$ the convex combinations of f_i may be written as

$$\sum_{i=1}^{\infty} t_i f_i(z) = z + \sum_{n=1}^{\infty} \left(\sum_{i=1}^{\infty} t_i \left|a_{i_n}\right| \right) z^{-n} - \sum_{n=1}^{\infty} \left(\sum_{i=1}^{\infty} t_i \left|b_{i_n}\right| \right) (\overline{z})^{-n}.$$

Now theorem follows by rearranging the above equation and using (5).

To see that $\Sigma_{\overline{H}}\ R(\alpha,\lambda)$ is not compact, we observe that

$$f_n(z) = z - \frac{n}{n+1}\ \overline{z} \in \Sigma_{\overline{H}}\ R(\alpha,\lambda) \quad (n = 1,2,3,\cdots)$$

while $\lim_{n\to\infty} f_n(z) = z - \overline{z}$ which is not even univalent in $\tilde{U}$.

Theorem 6. *Let* $\lambda \ge 1$. *Then each function in the family* $\Sigma_{\overline{H}}\ R(\alpha,\lambda)$ *maps the disk* $|z| = r > 2$ *onto convex domains. The constant 2 is best possible.*

Proof. Let $f \in \Sigma_{\overline{H}}\ R(\alpha,\lambda)$. Then $r^{-1} f(rz) \in \Sigma_{\overline{H}}\ R(\alpha,\lambda)$ for any $r>1$, by Theorem 2. It now suffices to show that $(f(2z))/2 \in \Sigma_{\overline{H}}\ K(0)$. Note that

$$\frac{1}{2} f(2z) = z + \sum_{n=1}^{\infty} \frac{1}{2^{n+1}} \left|a_n\right| z^{-n} - \sum_{n=1}^{\infty} \frac{1}{2^{n+1}} \left|b_n\right| (\overline{z})^{-n}.$$

Also, by Theorem 2,

$$\sum_{n=1}^{\infty}\left[n^2\frac{1}{2^{n+1}}|a_n| + n^2\frac{1}{2^{n+1}}|b_n|\right] = \sum_{n=1}^{\infty} n\big(|a_n| + |b_n|\big)\frac{n}{2^{n+1}}$$
$$\leq \sum_{n=1}^{\infty} n\big(|a_n| + |b_n|\big)$$
$$\leq \sum_{n=1}^{\infty}\big([\lambda(n+1)-1]|a_n| + [\lambda(n-1)+1]|b_n|\big)$$
$$\leq 1-\alpha \leq 1.$$

As a consequence of Theorem A(ii), it now follows that $(f(2z))/2 \in \Sigma_{\overline{H}}\ K(0)$.

Finally, we introduce extreme points for the classes $\Sigma_{\overline{H}}\ R(\alpha,\lambda)$, $\Sigma_{\overline{H}}\ P(\alpha)$, and $\Sigma_{\overline{H}}\ Q(\alpha)$.

Theorem 7. *For fixed* $\lambda \geq 0$ *and* $0 \leq \alpha < 1$, *we have* $f \in clco\Sigma_{\overline{H}}\ R(\alpha,\lambda)$ *if and only if* f *can be expressed as* $f(z) = \sum_{n=0}^{\infty}(X_n h_n(z) + Y_n g_n(z))$, *where* $X_n \geq 0$, $Y_n \geq 0$, $\sum_{n=0}^{\infty}(X_n + Y_n) = 1$, $h_0(z) = z$, $g_0(z) = z$, *and*

$$h_n(Z) = z + \frac{1-\alpha}{|\lambda(n+1)-1|}z^{-n},\ \ g_n(z) = z - \frac{1-\alpha}{|\lambda(n-1)+1|}(\bar{z})^{-n}\ ,\ (n = 1,2,\cdots).$$

In particular, the extreme points of $clco\Sigma_{\overline{H}}\ R(\alpha,\lambda)$ are $\{h_n\}$ and $\{g_n\}$.

Proof. Note that

$$f(z) = \sum_{n=0}^{\infty}\big(X_n h_n(z) + Y_n g_n(z)\big)$$
$$= z + \sum_{n=1}^{\infty}\frac{1-\alpha}{|\lambda(n+1)-1|}X_n z^{-n} - \sum_{n=1}^{\infty}\frac{1-\alpha}{|\lambda(n-1)+1|}Y_n(\bar{z})^{-n}.$$

Then f satisfies the required condition (3) because

$$\sum_{n=1}^{\infty}|\lambda(n+1)-1|\frac{1-\alpha}{|\lambda(n+1)-1|}X_n + \sum_{n=1}^{\infty}\big(\lambda(n-1)+1\big)\frac{1-\alpha}{|\lambda(n-1)+1|}Y_n$$
$$= (1-\alpha)\sum_{n=1}^{\infty}(X_n + Y_n) = (1-\alpha)\big(1-(X_0+Y_0)\big) \leq 1-\alpha.$$

This complete the first part of the proof. Conversely, suppose $f \in clco\Sigma_{\overline{H}}\ R(\alpha,\lambda)$.

Letting

$$X_n = \frac{|\lambda(n+1) - 1|}{1 - \alpha} |a_n| , \; Y_n = \frac{|\lambda(n-1)+1|}{1-\alpha} |b_n|, \quad (n = 1, 2, \cdots)$$

and $Y_0 = 1 - X_0 - \sum_{n=1}^{\infty} \left(X_n + Y_n\right)$, we may write

$$\begin{aligned} f(z) &= z + \sum_{n=1}^{\infty} |a_n| z^{-n} - \sum_{n=1}^{\infty} |b_n| (\bar{z})^{-n} \\ &= z + \sum_{n=1}^{\infty} \frac{1-\alpha}{|\lambda(n+1)-1|} X_n z^{-n} - \sum_{n=1}^{\infty} \frac{1-\alpha}{|\lambda(n-1)+1|} Y_n (\bar{z})^{-n} \\ &= z + \sum_{n=1}^{\infty} (h_n(z) - z) X_n + \sum_{n=1}^{\infty} \left(g_n(z) - z\right) Y_n \\ &= \sum_{n=1}^{\infty} \left(X_n h_n(z) + Y_n g_n(z)\right). \end{aligned}$$

Corollary 4. *The extreme points of* $clco \sum_{\overline{H}} P(\alpha)$ *are* $h_0(z) = z$, $g_0(z) = z$ $h_n(z) = z + (1 - \alpha) z^{-n}$, $g_n(z) = z - (1 - \alpha)(\bar{z})^{-n}$, *for* $n = 1, 2, \cdots$.

Corollary 5. *The extreme points of* $clco \sum_{\overline{H}} Q(\alpha)$ *are* $h_0(z) = z$, $g_0(z) = z$, $h_n(z) = z + \frac{1-\alpha}{n} z^{-n}$, $g_n(z) = z - \frac{1-\alpha}{n} (\bar{z})^{-n}$, *for* $n = 1, 2, \cdots$.

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