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On Convergence in *n*-Inner Product Spaces

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Abstract. We discuss the notions of strong convergence and weak convergence in *n*-inner product spaces and study the relation between them. In particular, we show that the strong convergence implies the weak convergence and disprove the converse through a counter-example, by invoking an analogue of Parseval's identity in *n*-inner product spaces.

1. Introduction

Let $n \geq 2$ be an integer and X be a real vector space of dimension $\geq n$. A real-valued function $<\cdot,\cdot\,|\,\cdot,\cdots,\cdot>$ on X^{n+1} satisfying the following five properties:

$\langle z_1, z_1 | z_2, \cdots, z_n \rangle \geq 0$; $\langle z_1, z_1 | z_2, \cdots, z_n \rangle = 0$ if and only if $z_1, z_2, \cdots, z_n$ are linearly dependent; (I1)

$\langle z_1, z_1 | z_2, \cdots, z_n \rangle = \langle z_{i_1}, z_{i_1} | z_{i_2}, \cdots, z_{i_n} \rangle$ for every permutation $(i_1, \cdots, i_n)$ of $(1, \cdots, n)$; (I2)

$\langle x, y | z_2, \cdots, z_n \rangle = \langle y, x | z_2, \cdots, z_n \rangle$; (I3)

$\langle \alpha x, y | z_2, \cdots, z_n \rangle = \alpha \langle x, y | z_2, \cdots, z_n \rangle, \ \alpha \in R$; (I4)

$\langle x + x', y | z_2, \cdots, z_n \rangle = \langle x, y | z_2, \cdots, z_n \rangle + \langle x', y | z_2, \cdots, z_n \rangle$; (I5)

is called an *n*-inner product on *X*, and the pair $(X, <\cdot,\cdot\,|\,\cdot,\cdots,\cdot>)$ is called an *n*-inner product space.

On an *n*-inner product space $(X, <\cdot,\cdot\,|\,\cdot,\cdots,\cdot>)$, the following function

$$\| z_1, z_2, \cdots, z_n \| := \langle z_1, z_1 | z_2, \cdots, z_n \rangle^{1/2}$$

defines an *n*-norm, which enjoys the following four properties:

$\| z_1, \cdots, z_n \| \geq 0$, $\| z_1, \cdots, z_n \| = 0$ if and only if

$z_1, \cdots, z_n$ are linearly dependent; (N1)

$\| z_1, \cdots, z_n \|$ is invariant under permutation; (N2)

$\| \alpha z_1, z_2, \cdots, z_n \| = |\alpha| \cdot \| z_1, z_2, \cdots, z_n \|$, $\alpha \in R$; (N3)

$\| x + y, z_2, \cdots, z_n \| \leq \| x, z_2, \cdots, z_n \| + \| y, z_2, \cdots, z_n \|$. (N4)

For example, any inner product space $(X, < \cdot, \cdot >)$ can be equipped with the standard *n*-inner product

$$\langle x, y | z_2, \cdots, z_n \rangle := \begin{vmatrix} < x, y > & < x, z_2 > & \cdots & < x, z_n > \\ < z_2, y > & < z_2, z_2 > & \cdots & < z_2, z_n > \\ \vdots & \vdots & \ddots & \vdots \\ < z_n, y > & < z_n, z_2 > & \cdots & < z_n, z_n > \end{vmatrix}.$$

Observe here that the induced *n*-norm

$$\| z_1, \cdots, z_n \| = \begin{vmatrix} < z_1, z_1 > & < z_1, z_2 > & \cdots & < z_1, z_n > \\ < z_2, z_1 > & < z_2, z_2 > & \cdots & < z_2, z_n > \\ \vdots & \vdots & \ddots & \vdots \\ < z_n, z_1 > & (z_n, z_2 > & \cdots & < z_n, z_n > \end{vmatrix}^{1/2}$$

represents the volume of the *n*-dimensional parallelepiped spanned by $z_1, \cdots, z_n$.

An *n*-inner product enjoys many properties analogous to those of an inner product. For instance, one may verify that the Cauchy-Schwarz inequality

$$\langle x, y | z_2, \cdots, z_n \rangle^2 \leq \| x, z_2, \cdots, z_n \|^2 \; \| y, z_2, \cdots, z_n \|^2$$

holds for every $x, y, z_2, \cdots, z_n \in X$.

The concept of 2-normed spaces was first introduced by Gähler [3], while that of 2-inner product spaces was developed by Diminnie, Gähler and White [1,2]. Their generalization for $n \geq 2$ may be found in Misiak's works [9,10]. For recent results on *n*-normed spaces and *n*-inner product spaces, see, for example, [4,5,7,8].

In this paper, we shall discuss the notions of strong convergence and weak convergence in *n*-inner product spaces and study the relation between them. In particular, we show that the strong convergence implies the weak convergence and disprove the converse through a counter-example, by invoking an analogue of Parseval's identity in *n*-inner product spaces.

2. Main results

Let $(X, <\cdot,\cdot|\cdot,\cdots,\cdot>)$ be an *n*-inner product space and $\|\cdot,\cdots,\cdot\|$ be the induced *n*-norm. A sequence (x_k) in X is said to *converge strongly* to a point $x \in X$ whenever $\|x_k - x, z_2, \cdots, z_n\| \to 0$ for every $z_2, \cdots, z_n \in X$. In such a case, we write $x_k \to x$. Meanwhile, (x_k) is said to *converge weakly* to x whenever $< x_k - x, y | z_2, \cdots, z_n > \to 0$ for every $y, z_2, \cdots, z_n \in X$.

Clearly if (x_k) and (y_k) converge strongly/weakly to *x* and *y* respectively, then, for any $\alpha, \beta \in \boldsymbol{R}$, $(\alpha x_k + \beta y_k)$ converges strongly/weakly to $\alpha x + \beta y$.

Moreover, one may observe that if $x_k \to x$, then $\|x_k, z_2, \cdots, z_n\| \to \|x, z_2, \cdots, z_n\|$ for every $z_2, \cdots, z_n \in X$. This tells us that $\|\cdot,\cdots,\cdot\|$ is continuous in the first variable. By Property (N2) of *n*-norms, $\|\cdot,\cdots,\cdot\|$ is continuous in each variable.

Next, if $x_k \to x$ and $y_k \to y$, then by the triangle inequality for real numbers and the Cauchy-Schwarz inequality for the *n*-inner product we have

$$\begin{aligned}
&\left| \langle x_k, y_k | z_2, \cdots, z_n \rangle - \langle x, y | z_2, \cdots, z_n \rangle \right| \\
&\quad \le \left| \langle x_k - x, y | z_2, \cdots, z_n \rangle \right| + \left| \langle x_k - x, y_k - y | z_2, \cdots, z_n \rangle \right| \\
&\qquad + \left| \langle x, y_k - y | z_2, \cdots, z_n \rangle \right| \\
&\quad \le \|x_k - x, z_2, \cdots, z_n\| \cdot \|y, z_2, \cdots, z_n\| \\
&\qquad + \|x_k - x, z_2, \cdots, z_n\| \cdot \|y_k - y, z_2, \cdots, z_n\| \\
&\qquad + \|x, z_2, \cdots, z_n\| \cdot \|y_k - y, z_2, \cdots, z_n\|
\end{aligned}$$

whence $< x_k, y_k | z_2, \cdots, z_n > \to < x, y | z_2, \cdots, z_n >$. This shows that $<\cdot,\cdot|\cdot,\cdots,\cdot>$ is continuous in the first two variables.

Now we come to our main results. The first proposition below tells us that a sequence cannot converge weakly to two distinct points.

Proposition 2.1. *If* (x_k) *converges weakly to x and* x' *simultaneously, then* $x = x'$.

Proof. By hypothesis and Property (I5) of *n*-inner products, we have $< x_k, y | z_2, \cdots, z_n > \to < x, y | z_2, \cdots, z_n >$ and at the same time $< x_k, y | z_2, \cdots, z_n > \to < x', y | z_2, \cdots, z_n >$ for every $y, z_2, \cdots, z_n \in X$. By the uniqueness of the limit of a sequence of real numbers, we must have $< x, y | z_2, \cdots, z_n > = < x', y | z_2, \cdots, z_n >$ or

$$\langle x-x', y \,|\, z_2, \cdots, z_n \rangle = 0$$

for every y, $z_2, \cdots, z_n \in X$. In particular, by taking $y = x - x'$, we obtain

$$\| x-x', z_2, \cdots, z_n \| = 0$$

for every y, $z_2, \cdots, z_n \in X$. By Property (N1) of 2-norms and elementary linear algebra, this can only happens if $x - x' = 0$ or $x = x'$.

The next proposition says that the strong convergence implies the weak convergence.

Proposition 2.2. *If (x_k) converges strongly to x, then it converges weakly to x.*

Proof. By the Cauchy-Schwarz inequality, we have

$$\left| \langle x_k - x, y \,|\, z_2, \cdots, z_n \rangle \right| \le \| x_k - x, z_2, \cdots, z_n \| \cdot \| y, z_2, \cdots, z_n \|$$

for every y, $z_2, \cdots, z_n \in X$. Since by hypothesis the right-hand side tends to 0 for every y, $z_2, \cdots, z_n \in X$, so does the left-hand side.

Corollary 2.3. *A sequence cannot converge strongly to two distinct points.*

The terminology that we use suggests that there are sequences that converge weakly but do not converge strongly. Here is one example that invokes an analogue of Parseval's identity.

Example 2.4. Let $(X, <\cdot,\cdot>)$ be a separable Hilbert space of infinite dimension and (e_k), indexed by $\boldsymbol{N}$, be an orthonormal basis for *X*. Then, for each *x* and $z \in X$, we have $\sum_k < x, e_k >< z, e_k > \; = \; < x, z >$. In particular, if $x = z$, then we have Parseval's identity $\sum_k < x, e_k >^2 = \| x \|^2$, where $\| \cdot \| = \; < \cdot,\cdot >^{1/2}$ denotes the induced norm.

Now equip *X* with the standard *n*-inner product $<\cdot,\cdot \,|\, \cdot, \cdots, \cdot>$, as given previously in the introduction. Then, for each *x*, $z_2, \cdots, z_n \in X$, we have the following analogue of Parseval's identity

$$\sum_k \langle x, e_k \,|\, z_2, \cdots, z_n \rangle^2 = \| x, z_2, \cdots, z_n \|^2 \| z_2, \cdots, z_n \|_{n-1}^2$$

where $\| \cdot, \cdots, \cdot \|_{n-1}$ denotes the standard $(n-1)$-norm on *X* (see [6]). For $n = 2$, the identity can be verified easily as follows

$$\sum_k \langle x, e_k \mid z \rangle^2 = \sum_k \left[\langle x, e_k \rangle \| z \|^2 - \langle x, z \rangle \langle z, e_k \rangle \right]^2$$
$$= \sum_k \left[\langle x, e_k \rangle^2 \| z \|^4 - 2\langle x, e_k \rangle \langle z, e_k \rangle \langle x, z \rangle \| z \|^2 + \langle x, z \rangle^2 \langle z, e_k \rangle^2 \right]$$
$$= \| x \|^2 \| z \|^4 - 2\langle x, z \rangle^2 \| z \|^2 + \langle x, z \rangle^2 \| z \|^2$$
$$= \left[\| x \|^2 \| z \|^2 - \langle x, z \rangle^2 \right] \cdot \| z \|^2$$
$$= \| x, z \|^2 \ \| z \|^2.$$

As the reader would have already expected by now, our counter-example is (e_k). Because of Parseval's identity, we must have $< x, e_k \mid z_2, \cdots, z_n > \to 0$ for every $x, z_2, \cdots, z_n \in X$, that is, (e_k) converges weakly to 0. Now, for each $k \in \boldsymbol{N}$ and $z_2, \cdots, z_n \in X$, denote by e_k^* the orthogonal projection of e_k on the subspace spanned by $z_2, \cdots, z_n$. Then one may observe that $\left\| e_k - e_k^* \right\| \to 1$, whence

$$\| e_k, z_2, \cdots, z_n \| = \left\| e_k - e_k^* \right\| \cdot \| z_2, \cdots, z_n \|_{n-1} \to \| z_2, \cdots, z_n \|_{n-1} \neq 0.$$

whenever $z_2, \cdots, z_n$ are linearly independent. This shows that (e_k) does not converge strongly to 0 in *X*.

3. Special cases

As shown in [5], on any *n*-inner product space $(X, <\cdot,\cdot \mid \cdot, \cdots, \cdot>)$, we can define an inner product $<\cdot, \cdot>$ with respect to a linearly independent set $\{a_1, \cdots, a_n\} \subseteq X$ by

$$\langle x, y \rangle := \sum_{\{i_2, \cdots, i_n\} \subseteq \{1, \cdots, n\}} \left\langle x, y \middle| a_{i_2, \cdots, a_{i_n}} \right\rangle$$

and put $\| \cdot \| = <\cdot,\cdot>^{1/2}$ as the induced norm. Then, given a sequence (x_k) in *X*, we can also define the strong convergence with respect to $\| \cdot \|$ and the weak convergence with respect to $<\cdot,\cdot>$. These types of convergence are in general weaker than the previous ones, defined with respect to $\| \cdot, \cdots, \cdot \|$ and $<\cdot,\cdot \mid \cdot, \cdots, \cdot>$ respectively.

In the standard case, however, one may observe that they are as strong as the previous ones, respectively, so that we have the following relation between the four types of convergence:

$$\begin{array}{ccc} \text{strong convergence w.r.t. } \|\cdot,\cdots,\cdot\| & \Rightarrow & \text{weak convergence w.r.t. } <\cdot,\cdot|\cdot,\cdots,\cdot> \\ \Updownarrow & & \Updownarrow \\ \text{strong convergence w.r.t. } \|\cdot\| & \Rightarrow & \text{weak convergence w.r.t. } <\cdot,\cdot> \end{array}$$

(see [8] for basic ideas). This gives us another explanation why our counter-example in the previous section works.

Finally, in the finite-dimensional case, we know that any sequence that converges weakly with respect to $<\cdot,\cdot>$ will converge strongly with respect to $\|\cdot\|$, and that any sequence that converges strongly with respect to $\|\cdot\|$ will converge strongly with respect to $\|\cdot,\cdots,\cdot\|$. Therefore, the four types of convergence are all equivalent.

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