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On Common Fixed Point Theorem of Four Mappings

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Abstract. In this paper we shall prove common fixed point theorems for four mappings in complete metric space. Our theorems generalize results of Banach [1], Kannan [5], Fisher [4] and Chatterjee [2].

1. Definitions

Definition 1. *A sequence* $\{x_n\}$ *in a metric space* (X, d) *is said to be convergent to a point x in X if*

$$\lim_{n\to\infty} d(x_n, x) = 0 \quad \textit{for all x in } X.$$

Then x is called the limit of the sequence $\{x_n\}$ *in X.*

Definition 2. *A sequence* $\{x_n\}$ *in a metric space* (X, d) *is said to be Cauchy sequence if*

$$\lim_{m,n\to\infty} d(x_m, x_n) = 0 \quad \textit{for all x in } X.$$

Then x is called the limit of the sequence $\{x_n\}$ *in X.*

Definition 3. *A metric space* (X, d) *is said to be complete if every Cauchy sequence in X is convergent.*

Definition 4. [3] *Let A and S be mappings from a metric space* (X, d) *into itself. Then A and S are said to be compatible of type (A) if*

$$\lim_{n\to\infty} d(ASx_n, SSx_n) = 0$$

and

$$\lim_{n\to\infty} d(SAx_n, AAx_n) = 0$$

whenever $\{x_n\}$ *is a sequence in X such that*

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ax_n = z \quad \text{for some } z \,\varepsilon\, X .$$

The object of this paper is to prove following theorems:

Theorem 1. *Let A, B, S and T be four mappings of complete metric space X into itself satisfying:*

$$d(Ax, By) \le \alpha_1\left[\frac{d(Ty, By)d(Sx, Ty)}{d(Tx, Ax) + d(By, Tx)}\right] + \alpha_2[d(Ax, Tx) + d(Sx, Bx) + d(Ay, Sy)] + \alpha_3\left[d(Tx, Bx) + d(Sy, Tx) + d(By, Ty)\right] + \alpha_4\left[d(Sx, Ty) + d(Tx, By)\right], \tag{1.1}$$

$$A(X) \subset T(X) \quad \text{and} \quad B(X) \subset S(X), \tag{1.2}$$

the pairs A, S and B, T are compatible of type (A), (1.3)

one of A, B, S and T, is continuous, (1.4)

for all x, y *in X, where* $\alpha_i \ge 0$ *and* $\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 < 1$. *Then A, B, S and T have a unique common fixed point in X.*

Proof. Let x_0 be an arbitrary point of X. We define

$$Ax_{2n+1} = y_{2n+2},\ Tx_{2n} = y_{2n},$$
$$Bx_{2n} = y_{2n+1},\ Sx_{2n+1} = y_{2n+1}, \quad n = 1, 2, \cdots$$

By putting $x = x_{2n}$ and $y = x_{2n+1}$ in (1.1), we write

$$d(Ax_{2n}, Bx_{2n+1}) \le \alpha_1\left[\frac{d(Tx_{2n+1}, Bx_{2n+1})\, d(Sx_{2n}, Tx_{2n+1})}{d(Tx_{2n}, Ax_{2n}) + d(Bx_{2n+1}, Tx_{2n})}\right] + \alpha_2[\, d(Ax_{2n}, Tx_{2n})$$
$$+ d(Sx_{2n}, Bx_{2n}) + d(Ax_{2n+1}, Sx_{2n+1})] + \alpha_3[d(Tx_{2n}, Bx_{2n})$$
$$+ d(Sx_{2n+1}, Tx_{2n}) + d(Bx_{2n+1}, Tx_{2n+1})]$$
$$+ \alpha_4[\, d(Sx_{2n}, Tx_{2n+1}) + d(Tx_{2n}, Bx_{2n+1})]$$

$$= \alpha_1\left[\frac{d(y_{2n+1}, y_{2n+2}) d(y_{2n}, y_{2n+1})}{d(y_{2n}, y_{2n+1}) + d(y_{2n+2}, y_{2n})}\right] + \alpha_2[\, d(y_{2n+1}, y_{2n})$$
$$+ d(y_{2n}, y_{2n+1}) + d(y_{2n+2}, y_{2n+1})] + \alpha_3[\, d(y_{2n}, y_{2n+1})$$
$$+ d(y_{2n+1}, y_{2n}) + d(y_{2n+2}, y_{2n+1})]$$
$$+ \alpha_4\left[d(y_{2n}, y_{2n+1}) + d(y_{2n}, y_{2n+2})\right]$$

$$d(y_{2n+1}, y_{2n+2}) \le (\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4)\, d(y_{2n}, y_{2n+1}) + (\alpha_2 + \alpha_3 + \alpha_4)\, d(y_{2n+1}, y_{2n+2})$$

$$d(y_{2n+1}, y_{2n+2}) \le \frac{(\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4)}{(1 - \alpha_2 - \alpha_3 - \alpha_4)}\; d(y_{2n}, y_{2n+1})$$

Putting $h = \frac{(\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4)}{(1 - \alpha_2 - \alpha_3 - \alpha_4)}$, we find $h < 1$, since $\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 < 1$. Hence

$$d(y_{2n+1}, y_{2n+2}) \le h\, d(y_{2n}, y_{2n+1}).$$

Similarly by putting $x = x_{2n-1}$ and $y = x_{2n}$ in (1.1), we have

$$d(Ax_{2n-1}, Bx_{2n}) \le \alpha_1\left[\frac{d(Tx_{2n}, Bx_{2n})\, d(Sx_{2n-1}, Tx_{2n})}{d(Tx_{2n-1}, Ax_{2n-1}) + d(Bx_{2n}, Tx_{2n-1})}\right] + \alpha_2[d(Ax_{2n-1}, Tx_{2n-1})$$
$$+ d(Sx_{2n-1}, Bx_{2n-1}) + d(Ax_{2n}, Sx_{2n})] + \alpha_3[d(Tx_{2n-1}, Bx_{2n-1})$$
$$+ d(Sx_{2n}, Tx_{2n-1}) + d(Bx_{2n}, Tx_{2n})]$$
$$+ \alpha_4\left[d(Sx_{2n-1}, Tx_{2n}) + d(Tx_{2n-1}, Bx_{2n})\right]$$

$$= \alpha_1\left[\frac{d(y_{2n}, y_{2n+1})\, d(y_{2n-1}, y_{2n})}{d(y_{2n-1}, y_{2n}) + d(y_{2n+1}, y_{2n-1})}\right] + \alpha_2\,[d(y_{2n}, y_{2n-1})$$
$$+ d(y_{2n-1}, y_{2n}) + d(y_{2n+1}, y_{2n})] + \alpha_3[d(y_{2n-1}, y_{2n})$$
$$+ d(y_{2n}, y_{2n-1}) + d(y_{2n+1}, y_{2n})]$$
$$+ \alpha_4\left[d(y_{2n-1}, y_{2n}) + d(y_{2n-1}, y_{2n+1})\right]$$

$$d(y_{2n}, y_{2n+1}) \le (\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4)\, d(y_{2n-1}, y_{2n}) + (\alpha_2 + \alpha_3 + \alpha_4)\, d(y_{2n}, y_{2n+1})$$

$$\le \frac{(\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4)}{(1 - \alpha_2 - \alpha_3 - \alpha_4)}\, d(y_{2n-1}, y_{2n})$$

$$d(y_{2n}, y_{2n+1}) \le h\, d(y_{2n-1}, y_{2n}), \quad \text{as} \quad h = \frac{(\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4)}{(1 - \alpha_2 - \alpha_3 + \alpha_4)}.$$

We find $h < 1$, since $(\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4) < 1$. Proceeding in this way, we have

$$d(y_{2n}, y_{2n+1}) \le h^{2n-1} d(y_0, y_1)$$

By routine calculations the following inequaities hold for $k > n$

$$d(y_n, y_{n+k}) \le \sum_{i=1}^{k} d(y_{n+i-1}, y_{n+i})$$

$$\le \sum_{i=1}^{k} h^{n+i-1} d(y_0, y_1)$$

$$\le \frac{h^n}{1-h}\, d(y_0, y_1) \to 0 \quad \text{as} \quad n \to \infty$$

Here $h < 1$. Hence $\{y_n\}$ is a Cauchy sequence and by completeness of X we see that $\{y_n\}$ is converges to a point z in X. Since $\{y_n\}$ is a Cauchy sequence and taking $n \to \infty$, we write

$$Ax_{2n} = Tx_{2n+1} \to z \quad \text{and} \quad Bx_{2n+1} = Sx_{2n+2} \to z$$

Now, suppose A is continuous. Since A and S are compatible mappings of type (A), then

$$AAx_{2n} \quad \text{and} \quad SAx_{2n} \to Az \quad \text{as} \quad n \to \infty.$$

Now putting $x = Ax_{2n}$ and $y = x_{2n+1}$ in (1.1), we write

$$d(AAx_{2n}, Bx_{2n+1}) \leq \alpha_1 \left[\frac{d(Tx_{2n+1}, Bx_{2n+1})\, d(SAx_{2n}, Tx_{2n+1})}{d(TTx_{2n+1}, AAx_{2n}) + d(Bx_{2n+1}, TTx_{2n+1})} \right]$$
$$+ \alpha_2 [d(AAx_{2n}, TTx_{2n+1}) + d(SAx_{2n}, BTx_{2n+1})$$
$$+ d(Ax_{2n+1}, Sx_{2n+1}] + \alpha_3 [d(TTx_{2n+1}, BTx_{2n+1})$$
$$+ d(Sx_{2n+1}, TTx_{2n+1}) + d(Bx_{2n+1}, Tx_{2n+1})]$$
$$+ \alpha_4 [d(SAx_{2n}, Tx_{2n+1}) + d(TT_{2n+1}, Bx_{2n+1})]$$

Taking the limit $n \to \infty$, we write

$$d(Az, z) \leq \alpha_2 d(Az, z)$$

giving a contradiction as $\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 < 1.$

Therefore $Az = z$.

Similarly by putting $x = Sx_{2n}$ and $y = x_{2n+1}$ in (1.1), we write

$$d(ASx_{2n}, Bx_{2n+1}) \leq \alpha_1 \left[\frac{d(Tx_{2n+1}, Bx_{2n+1})\ d(SSx_{2n}, Tx_{2n+1})}{d(TBx_{2n-1}, ASx_{2n}) + d(Bx_{2n+1}, TBx_{2n-1})} \right]$$
$$+ \alpha_2 [d(ASx_{2n}, TBx_{2n-1}) + d(SSx_{2n}, BBx_{2n-1})$$
$$+ d(Ax_{2n+1}, Sx_{2n+1})] + \alpha_3 [d(TBx_{2n-1}, BBx_{2n-1})$$
$$+ d(Sx_{2n+1}, TBx_{2n-1}) + d(Bx_{2n+1}, Tx_{2n+1})]$$
$$+ \alpha_4 [d(SSx_{2n}, Tx_{2n+1}) + d(TBx_{2n-1}, Bx_{2n+1})]$$

Taking the limit $n \to \infty$, we write

$$d(Sz, z) \leq (\alpha_2 + \alpha_4)\, d(Sz, z)$$

giving a contradiction as $\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 < 1$, therefore $Sz = z$.

Similarly $Bz = Tz = z$. Thus z is a common fixed point of A, B, S and T.

For uniqueness let z and $w\,(z \neq w)$ be two fixed points in X such that

$$Az = Bz = Sz = Tz = z \quad \text{and} \quad Aw = Bw = Sw = Tw = w,$$

then by (1.1), we have

$$d(Az, Bw) \leq \alpha_1 \left[\frac{d(Tw, Bw)\, d(Sz, Tw)}{d(Tz, Az) + d(Bw, Tz)} \right] + \alpha_2 [d(Az, Tz) + d(Sz, Bz) + d(Aw, Sw)] + \alpha_3 [d(Tz, Bz) + d(Sw, Tz) + d(Bw, Tw)] + \alpha_4 [d(Sz, Tw) + d(Tz, Bw)]$$

$$d(z, w) \leq (\alpha_3 + 2\alpha_4)\ d(z, w),$$

which is a contradiction , since $\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 < 1$. Hence $z = w$.

This implies the uniqueness of common fixed point of *A, B, S* and *T*.

Theorem 2. *Let A, B, S and T be four mappings of complete metric space X into itself and satisfying* (1.2), (1.3), (1.4) *and*

$$d(Ax, By) \leq \alpha_1 \left[\frac{d(Tx, B^2 y)\, d(Ty, Sx)}{d(Sx, A^2 y)} \right] + \alpha_2 [d(Tx, Ax) + d(Ty, By) + d(Ax, Sx)] + \alpha_3 [d(Tx, By) + d(Ty, Sx) + d(Ty, Bx)] + \alpha_4 [d(Tx, Ty) + d(Tx, Bx)]. \qquad (1.1)$$

Theorem 2 can be proved in the similar manner as Theorem 1.

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