BULLETIN *of the*
MALAYSIAN
MATHEMATICAL
SCIENCES
SOCIETY

Bull. Malaysian Math. Sc. Soc. (Second Series) **25** (2002) 23-29

On Generalized Gamma Near-Fields

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Abstract. In this paper, we introduce the concept of generalized gamma near-fields and obtain some characterization of generalized gamma near-fields and gamma near-fields.

1. Introduction

Throughout this paper M stands for a Γ-near-ring. Γ-near-rings were defined by Satyanarayana [6]. For basic terminology in near-ring we refer to Pilz [5] and in Γ-near-ring we refer to Satyanarayana [6]. By analogy with the concept of inverse semi–group in semi-group theory, Murty [4] introduced the concept of generalized near–field in near-rings. In this paper we introduce the notion of generalized gamma near–field and obtain equivalent conditions for generalized gamma near–field using regularity conditions. Also we derive some equivalent conditions for gamma near-field. A Γ-near-ring is a triple $(M, +, \Gamma)$ where

(i) $(M, +)$ is a (not necessarily abelian) group.

(ii) Γ is a non-empty set of binary operators on M such that for each $\gamma \in \Gamma$, $(M, +, \gamma)$ is a right near–ring.

(iii) $x\gamma(y\mu z) = (x\gamma y)\mu z$ for all $x, y, z \in M$ and $\gamma, \mu \in \Gamma$.

A Γ-near-ring M is said to be zero-symmetric if $a\gamma 0 = 0$ for all $a \in M$ and for all $\gamma \in \Gamma$. An element $e \in M$ is called an identity in M if $n\gamma e = e\gamma n = n$ for all $n \in M$ and for all $\gamma \in \Gamma$. A Γ-near-ring M is said to be a gamma near-field if M contains an identity and if every non-zero element has multiplicative inverse, that is if for $0 \neq a \in M$, there exists a unique $a' \in M$ such that $a\gamma a' = a'\gamma a = e$ for all $\gamma \in \Gamma$, where e is the identity in M. A Γ-near-ring M is called a generalized gamma near field (GGNF) if for each $a \in M$, there exists a unique $a' \in M$ such that $a\gamma_1 a'\gamma_2 a = a$ and $a'\gamma_1 a\gamma_2 a' = a'$ for every pair of non-zero elements γ_1, γ_2 of Γ. An element $a \in M$ is called idempotent if $a\gamma a = a$ for all $\gamma \in \Gamma$.

An element $d \in M$ is called left distributive if $d\gamma(a + b) = d\gamma a + d\gamma b$ for all $a, b \in M$ and for all $\gamma \in \Gamma$.

An element $0 \neq a \in M$ is called nilpotent if there exists a positive integer $n \geq 1$ such that $(a\gamma)^n a = 0$ for each $\gamma \in \Gamma$ [1]. A Γ-near-ring M is said to be sub-commutative if $a\gamma M = M\gamma a$ for all $a \in M$ and for all $\gamma \in \Gamma$. A Γ-near-ring M is called regular if for each $a \in M$ and for every pair of non-zero elements γ_1, γ_2 of Γ, $a = a\gamma_1 b\gamma_2 a$ for some $b \in M$. A Γ-near-ring M is said to fulfill the insertion-of-factors property (IFP) provided that for all $a, b, n \in M$, $a\gamma b = 0$ for all $\gamma \in \Gamma$ implies $a\gamma_1 n\gamma_2 b = 0$ for every pair of non-zero elements γ_1, γ_2 of Γ. A Γ-near-ring M is said to be without zero divisors [6], if $a\gamma b = 0$ for some $\gamma \in \Gamma$, then either $a = 0$ or $b = 0$.

2. Preliminary results on GGNF

Proposition 2.1. *If M is a* GGNF, *then M is zero-symmetric.*

Proof. Since M is a GGNF, for every $a \in M$, there exists a unique a' such that $a\gamma_1 a'\gamma_2 a = a$ and $a'\gamma_1 a\gamma_2 a' = a'$ for every pair of non-zero elements γ_1, γ_2 of Γ. Now for each $\gamma \in \Gamma$ and $0 \neq a \in M$, $a\gamma 0 \in M$ and so by hypothesis for every pair of non-zero elements γ_1, γ_2 of Γ there exists a unique $x \in M$ such that $(a\gamma 0)\gamma_1 x\gamma_2 (a\gamma 0) = a\gamma 0$ and $x\gamma_1 (a\gamma 0)\gamma_2 x = x$. Both 0 and $a\gamma 0$ satisfy the above equations. So by uniqueness $0 = a\gamma 0$.

Theorem 2.2. *A Γ-near-ring M is a* GGNF *if and only if M is regular and idempotents commute.*

Proof. Let M be a GGNF. By definition of GGNF, M is certainly regular. Let e and f be two idempotents and let $x = (e\gamma f)'$ for $\gamma \in \Gamma$. Then $(e\gamma f)\gamma_1 x\gamma_2 (e\gamma f) = (e\gamma f)$ and $x\gamma_1 (e\gamma f)\gamma_2 x = x$. Since $(f\gamma_2 x\gamma_1 e)\gamma(f\gamma_2 x\gamma_1 e)$ $= f\gamma_2 (x\gamma_1 (e\gamma f)\gamma_2 x)\gamma_1 e = f\gamma_2 x\gamma_1 e$, the element $f\gamma_2 x\gamma_1 e$ is idempotent. Also $f\gamma_2 x\gamma_1 e$ satisfies the above two equations and hence $(e\gamma f)$ is an inverse of $f\gamma_2 x\gamma_1 e$. But $f\gamma_2 x\gamma_1 e$, being an idempotent, is its own unique inverse and so $f\gamma_2 x\gamma_1 e = e\gamma f$. It follows that $(e\gamma f)$ is an idempotent. Similarly one can show that $(f\gamma e)$ is an idempotent. Hence $(e\gamma f)\gamma_1 (f\gamma e)\gamma_2 (e\gamma f) = (e\gamma f)\gamma(e\gamma f) = e\gamma f$ and $(f\gamma e)\gamma_1 (e\gamma f)\gamma_2 (f\gamma e) = (f\gamma e)\gamma(f\gamma e) = f\gamma e$ and so $(f\gamma e)$ is an inverse of $(e\gamma f)$. But $(e\gamma f)$, being an idempotent, is its own unique inverse and so $e\gamma f = f\gamma e$ for $\gamma \in \Gamma$. Thus idempotents commute.

Conversely assume that M is regular and idempotents commute. Since M is regular, for each $a \in M$ and for every pair of non-zero elements γ_1, γ_2 of Γ, $a = a\gamma_1 x\gamma_2 a$ for some $x \in M$. Let $a' = x\gamma_2 a\gamma_1 x$. Then $a\gamma_1 a'\gamma_2 a = a\gamma_1(x\gamma_2 a\gamma_1 x)\gamma_2 a = a\gamma_1 x\gamma_2 a = a$ and $a'\gamma_1 a\gamma_2 a' = x\gamma_2(a\gamma_2 x\gamma_2 a)\gamma_1 x$ $= x\gamma_2 a\gamma_1 x = a'$. Thus a' is an inverse of a. If possible let us assume that there exists $a'' \in M$ such that $a\gamma_1 a''\gamma_2 a = a$, $a''\gamma_1 a\gamma_2 a'' = a''$. Since $(a'\gamma_1 a)$ and $(a''\gamma_1 a)$ are idempotents, for all $\gamma_2 \in \Gamma$, $(a'\gamma_1 a)\gamma_2(a''\gamma_1 a) = a'\gamma_1(a\gamma_2 a''\gamma_1 a)$ $= a'\gamma_1 a$ and $(a''\gamma_1 a)\gamma_2 (a'\gamma_1 a) = a''\gamma_1(a\gamma_2 a'\gamma_1 a) = a''\gamma_1 a$ implies that $a'\gamma_1 a = a''\gamma_1 a$. Similarly we can prove that $a\gamma_2 a' = a\gamma_2 a''$ for all $\gamma_2 \in \Gamma$. So $a' = a'\gamma_1 a\gamma_2 a' = a''\gamma_1 a\gamma_2 a' = a''\gamma_1 a\gamma_2 a'' = a''$. Thus M is a GGNF.

Proposition 2.3. *If M is a* GGNF, *then M has no non-zero nilpotent elements*.

Proof. Let $a \in M$ and $a\gamma a = 0$ for all $\gamma \in \Gamma$ and let a has the inverse b. Since $a\gamma_2 b$ and $b\gamma_1 a$ are idempotents and hence commute (by Theorem 2.2). Therefore $b^2 = b\gamma b = (b\gamma_1 a\gamma_2 b)\gamma(b\gamma_1 a\gamma_2 b) = b\gamma_1(a\gamma_2 b)\gamma(b\gamma_1 a)\gamma_2 b$ $= b\gamma_1(b\gamma_1 a)\gamma(a\gamma_2 b)\gamma_2 b = (b\gamma_1 b)\gamma_1 0 = 0$. Also $a\gamma_1(b\gamma_2 a\gamma_1(b\gamma_2 a + b))\gamma_2 a$ $= a\gamma_1(b\gamma_2 a + b)\gamma_2 a = a\gamma_1(b\gamma_2 a\gamma_2 a + b\gamma_2 a) = a\gamma_1(0 + b\gamma_2 a) = a\gamma_1 b\gamma_2 a = a$ and $(b\gamma_2 a\gamma_1(b\gamma_2 a + b))\gamma_2 a\gamma_1(b\gamma_2 a\gamma_1(b\gamma_2 a + b)) = b\gamma_2 a\gamma_{1_1}(b\gamma_2(a\gamma_2 a)\gamma_1 b\gamma_2 a\gamma_1$ $(b\gamma_2 a + b) + b\gamma_2 a\gamma_1 b\gamma_2 a\gamma_1(b\gamma_2 a + b)) = b\gamma_2 a\gamma_1(b\gamma_2 a\gamma_1 b\gamma_2 a\gamma_1(b\gamma_2 a + b))$ $= (b\gamma_2 a\gamma_1 b)\gamma_2(a\gamma_1 b\gamma_2 a)\gamma_1(b\gamma_2 a + b) = b\gamma_2 a\gamma_1(b\gamma_2 a + b)$. By uniqueness of b, $b\gamma_2 a\gamma_1(b\gamma_2 a + b) = b$. Thus $0 = b\gamma_1 b = b\gamma_2 a\gamma_1(b\gamma_2 a + b)\gamma_1 b = b\gamma_2 a\gamma_1$ $(b\gamma_2 a\gamma_1 b + 0) = b\gamma_2 a\gamma_1 b = b$. So a must be 0. Hence M contains no non-zero nilpotent elements.

Proposition 2.4. *A Γ-near-ring M which is zero-symmetric and without non-zero nilpotent elements is an* IFP *Γ-near-ring*.

Proof. If $x\gamma y = 0$ for $x, y \in M$ and for all $\gamma \in \Gamma$, then $(y\gamma x)^2 = y\gamma 0 = 0$. This implies that $y\gamma x = 0$. Now for $\gamma_1, \gamma_2 \in \Gamma$, $n \in M$, $(x\gamma_1 n\gamma_2 y)^2$ $= x\gamma_1 n\gamma_2(y\gamma x)\gamma_1 n\gamma_2 = (x\gamma_1 n)\gamma_2 0\gamma_1(n\gamma_2 y) = (x\gamma_1 n)\gamma_2 0 = 0$ and hence $x\gamma_1 n\gamma_2 y = 0$. Therefore M is a IFP Γ-near-ring .

Corollary 2.5. *Every* GGNF *is an* IFP *Γ-near-ring.*

3. Characterization of GGNF by regularity

In the following theorem, we obtain equivalent conditions for generalized gamma near-field.

Theorem 3.1. *The following are equivalent:*

(i) M *is a* GGNF
(ii) M *is regular and each idempotent is central*
(iii) M *is regular and sub-commutative*.

Proof.

(i) ⇒ (ii): Let M be GGNF. By Theorem 2.2, M is regular and idempotents commute. Let $e \in M$ be an idempotent. For $a \in M$ and $\gamma \in \Gamma$, we have $(a - a\gamma e)\gamma e = a\gamma e - a\gamma(e\gamma e) = 0$. Since M is an IFP (Corollary 2.5), $(a - a\gamma e)\gamma b\gamma e = 0$ for $b \in M$ and so $a\gamma b\gamma e = a\gamma(e\gamma b\gamma e)$. Since $(e\gamma b - e\gamma b\gamma e)\gamma e = 0$, $e\gamma(e\gamma b - e\gamma b\gamma e) = 0$ and so $e\gamma b\gamma(e\gamma b - e\gamma b\gamma e) = 0$. Consider $(e\gamma b - e\gamma b\gamma e)^2 = (e\gamma b - e\gamma b\gamma e)\gamma(e\gamma b - e\gamma b\gamma e) = (e\gamma b)\gamma(e\gamma b - e\gamma b\gamma e) - (e\gamma b\gamma e)\gamma(e\gamma b - e\gamma b\gamma e) = -(e\gamma b)\gamma e\gamma(e\gamma b - e\gamma b\gamma e) = -(e\gamma b)\gamma 0 = 0$. This implies that $e\gamma b - e\gamma b\gamma e = 0$ which implies that $e\gamma b = e\gamma b\gamma e$. Therefore $a\gamma b\gamma e = a\gamma(e\gamma b\gamma e) = a\gamma(e\gamma b)$. Since M is regular, $a = f\gamma a$ where f is a suitable idempotent and for all $\gamma \in \Gamma$. So $a\gamma e = (f\gamma a)\gamma e = f\gamma e\gamma a = e\gamma f\gamma a = e\gamma a$. Therefore e is central.

(ii) ⇒ (iii): Since M is regular for each $0 \neq a \in M$, $a = a\gamma_1 x\gamma_2 a$ for some $x \in M$ and for every pair of non-zero elements γ_1, γ_2 of Γ. Now $(a\gamma x)^2 = (a\gamma x)\gamma(a\gamma x) = (a\gamma x\gamma a)\gamma x = a\gamma x$ and $(x\gamma a)^2 = (x\gamma a)\gamma(x\gamma a) = x\gamma(a\gamma x\gamma a) = x\gamma a$. Therefore $a\gamma x$ and $x\gamma a$ are idempotents. Hence $a\gamma M = (a\gamma x\gamma a)\gamma M = a\gamma(x\gamma a)\gamma M = a\gamma M\gamma(x\gamma a) \subseteq M\gamma a$. Similarly $M\gamma a = M\gamma(a\gamma x\gamma a) = M\gamma(a\gamma x)\gamma a = (a\gamma x)\gamma M\gamma a = a\gamma(x\gamma M\gamma a) \subseteq a\gamma M$. Thus $a\gamma M = M\gamma a$ i.e., M is sub-commutative.

(iii) ⇒ (i): Let e and f be idempotents in M. Since M is sub-commutative, $M\gamma e = e\gamma M$ for all $\gamma \in \Gamma$. Therefore there exists x and $y \in M$ such that $f\gamma e = e\gamma x$ and $e\gamma f = y\gamma e$. This gives that $e\gamma f\gamma e = e\gamma(e\gamma x) = e\gamma x = f\gamma e$. Also $e\gamma f\gamma e = (y\gamma e)\gamma e = y\gamma e = e\gamma f$. This implies that idempotents commute. By Theorem 2.2, M is a GGNF.

4. Characterization of gamma near-filed

Proposition 4.1. *If for each $0 \neq a \in M$ and for every pair of non-zero elements γ_1, γ_2 of Γ, there exists a unique $a' \in M$ such that $a\gamma_1 a'\gamma_2 a = a$, then M has no zero divisors.*

Proof. Suppose $c \neq 0$ and $a \neq 0$, then $c\gamma a \neq 0$ for all $\gamma \in \Gamma$. For, if $c\gamma_2 a = 0$ for $\gamma_2 \in \Gamma$, we have that $a\gamma_1(a' + c)\gamma_2 a = a\gamma_1(a'\gamma_2 a + c\gamma_2 a)$ $= a\gamma_1 a'\gamma_2 a = a$. By uniqueness of a', $a' + c = a'$ which implies $c = 0$. This is a contradiction. Hence M has no zero divisors.

Proposition 4.2. *If for each $0 \neq a \in M$ and for every pair of non-zero elements γ_1, γ_2 of Γ there exists a unique $a' \in M$ such that $a\gamma_1 a'\gamma_2 b = b$, $b\gamma_1 a'\gamma_2 a = b$ for all $b \in M$, then M has no zero divisors.*

Proof. Suppose $b \neq 0$ and $a \neq 0$, then $b\gamma a \neq 0$ for all $\gamma \in \Gamma$. For, if $b\gamma_2 a = 0$ for $\gamma_2 \in \Gamma$ and for some $a \neq 0 \in M$, $b\gamma_1(a' + b)\gamma_2 a = b\gamma_1(a'\gamma_2 a + b\gamma_2 a)$ $= b\gamma_1 a'\gamma_2 a = b$. Now $b\gamma_2 b = b\gamma_2(a\gamma_1 a'\gamma_2 b) = (b\gamma_2 a)\gamma_1 a'\gamma_2 b = 0$. Therefore $a\gamma_1(a' + b)\gamma_2 b = a\gamma_1(a'\gamma_2 b + b\gamma_2 b) = a\gamma_1 a'\gamma_2 b = b$. By uniqueness of a', $a' + b = a'$ which implies $b = 0$. This is a contradiction. Hence M has no zero divisors.

Theorem 4.3. *Let M be a Γ-near-ring. Then the following are equivalent:*

(i) *Every non-zero element of M has multiplicative inverse i.e., for all $0 \neq a \in M$ there exists a unique $a' \in M$ such that $a\gamma a' = a'\gamma a = e$ for all $\gamma \in \Gamma$, where e is the identity in M.*

(ii) *For $a \neq 0$ in M and for every pair of non-zero elements γ_1, γ_2 of Γ there exists a unique $a' \in M$ such that $a\gamma_1 a'\gamma_2 b = b$ and $b\gamma_1 a'\gamma_2 a = b$ for all $b \in M$.*

(iii) *M is a* GGNF *and has no zero divisors.*

(iv) *M contains a non-zero left distributive element and for each $0 \neq a \in M$ and for every pair of non-zero elements γ_1, γ_2 of Γ there exists a unique $a' \in M$ such that $a\gamma_1 a'\gamma_2 a = a$.*

Proof.

(i) ⇒ (ii): Let $0 \neq a \in M$. For $\gamma_1 \neq 0$ and $\gamma_2 \neq 0$ in Γ and for $b \in M$, $a\gamma_1 a'\gamma_2 b = (a\gamma_1 a')\gamma_2 b = e\gamma_2 b = b$. Similarly $b\gamma_1 a'\gamma_2 a = b\gamma_1(a'\gamma_2 a)$ $= b\gamma_1 e = b$.

(ii) ⇒ (iii): By Proposition 4.2, *M* contains no zero divisors. By taking $b = a$ we get $a\gamma_1 a'\gamma_2 a = a$ where a' is unique. Now $a\gamma_1(a'\gamma_1 a\gamma_2 a')\gamma_2 a = (a\gamma_1 a'\gamma_1 a)\gamma_2 a'\gamma_2 a$ $= a\gamma_2 a'\gamma_2 a = a$. By uniqueness of a', $a'\gamma_1 a\gamma_2 a' = a'$.

(iii) ⇒ (iv): For $0 \neq a \in M$, $a\gamma_2(a\gamma_1 a'\gamma_2 a) = a\gamma_2 a$ which implies that $(a\gamma_2 a\gamma_1 a')\gamma_2 a - a\gamma_2 a = 0$. That is $(a\gamma_2 a\gamma_1 a' - a)\gamma_2 a = 0$ and since *M* contains no zero divisors, $a\gamma_2 a\gamma_1 a' = a$. Thus $a\gamma_1 a' = e$ becomes two-sided identity in *M*. Therefore *M* contains a left distributive element $e = a\gamma_1 a'$. By hypothesis, it is clear that for each $0 \neq a \in M$ and for every pair of non-zero elements γ_1, γ_2 of Γ there exists a unique $a' \in M$ such that $a\gamma_1 a'\gamma_2 a = a$.

(iv) ⇒ (i): For each $0 \neq a \in M$, there exists a unique $x \in M$ such that $(a\gamma 0)\gamma_1 x\gamma_2(a\gamma 0) = a\gamma 0$. Both 0 and $(a\gamma 0)$ satisfy the above equation. So by uniqueness $0 = a\gamma 0$. Also $(a'\gamma_1 a\gamma_2 a' - a')\gamma_1 a = a'\gamma_1(a\gamma_2 a'\gamma_1 a) - a'\gamma_1 a$ $= a'\gamma_1 a - a'\gamma_1 a = 0$. By Proposition 4.1 we get $a'\gamma_1 a\gamma_2 a' = a'$. Let $n \neq 0$ be a left distributive element in *M*. By hypothesis there exists a unique $n' \in M$ such that $n\gamma_1 n'\gamma_2 n = n$. Now $n\gamma_2(n\gamma_1 n'\gamma_2 n) = n\gamma_2 n$ and so $(n\gamma_2 n\gamma_1 n')\gamma_2 n - n\gamma_2 n = 0$. Since $n \neq 0$ and $(n\gamma_2 n\gamma_1 n' - n)\gamma_2 n = 0$, we get $n\gamma_2 n\gamma_1 n' - n = 0$. That is $n\gamma_2 n\gamma_1 n' = n$. Therefore $n\gamma_1 n' = e$ becomes two-sided identity for *n*. Let $m \in M$. Then $(m\gamma_2 e - m)\gamma_2 n = m\gamma_2 e\gamma_2 n - m\gamma_2 n = m\gamma_2 n - m\gamma_2 n = 0$. By Proposition 4.1, $m\gamma_2 e - m = 0$ and so $m\gamma_2 e = m$. Since n is left distributive, $n\gamma_2(e\gamma_2 m - m) = n\gamma_2(e\gamma_2 m) - n\gamma_2 m = (n\gamma_2 e)\gamma_2 m - n\gamma_2 m$ $= n\gamma_2 m - n\gamma_2 m = 0$. By Proposition 4.1, we get $e\gamma_2 m - m = 0$. That is $e\gamma_2 m = m$. Thus e becomes two-sided the identity for *M*. Let $0 \neq m \in M$. Then there exists $m' \in M$ such that $m\gamma_1 m'\gamma_2 m = m = e\gamma_2 m$, i.e., $m\gamma_1 m'\gamma_2 m - e\gamma_2 m = 0$. This gives that $(m\gamma_1 m' - e)\gamma_2 m = 0$ and so $m\gamma_1 m' = e$ and $m'\gamma_1 m\gamma_2 m' = m' = e\gamma_2 m'$. Similarly this implies that $m'\gamma_1 m = e$, that is m' is the inverse of *m*.

Acknowledgement. The authors would like to thank the referee for valuable suggestions for filling certain gaps in the proof of a vital result in the paper.

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