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A Unicity Theorem for Meromorphic Functions

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Abstract. In this paper, we study the uniqueness of meromorphic functions and prove the following result: Let $n \geq 3$ be a positive integer, $S = \{z : z^n - z^{n-1} - 1 = 0\}$, and let f and g be two nonconstant meromorphic functions whose poles are of multiplicities at least 2. If $E(0, f) = E(0, g)$, $E(S, f) = E(S, g)$, and $E(\infty, f) = E(\infty, g)$, then $f(z) \equiv g(z)$. This result also answer a question of Gross [4] and improve some results of Fang and Xu [1], Yi [14] and Yi [15].

1. Introduction

In this paper, by a meromorphic function we always mean a function which is meromorphic in the whole complex plane. Let $f(z)$ be a nonconstant meromorphic function. We use the following standard notations of value distribution theory,

$$T(r, f),\ m(r, f),\ N(r, f),\ \overline{N}(r, f),\ N(r, 1/f),\ \cdots$$

(see Hayman [6], Yang [12]). We denote by $S(r, f)$ any function satisfying

$$S(r, f) = o\{T(r, f)\},$$

as $r \to +\infty$, possibly outside of a set E with finite measure. In this paper, E may be different at different places.

Let S be a set of complex numbers. Set

$$E(S, f) = \bigcup_{a \in S} \{z : f(z) - a = 0\},$$

where the zeros point with multiple m is counted m times in the set.

Let $N_{(2}(r, \frac{1}{f-a})$ be the counting function which only includes multiple zeros of $f(z)-a$ and

$$N_2\left(r, \frac{1}{f-a}\right) = \overline{N}\left(r, \frac{1}{f-a}\right) + \overline{N}_{(2}\left(r, \frac{1}{f-a}\right).$$

In 1977, Gross [4] posed the following question.

Question A. Can one find two finite sets S_1 and S_2 such that any two nonconstant entire functions f and g satisfying $E(S_i, f) = E(S_i, g)$ for $i = 1, 2$ must be idential? And if such two sets exist, it would be interesting to know how large the two sets would have to be.

Yi [14] proved such two sets exist.

Theorem B. *Let $n \geq 5$ be a positive integer, $S_1 = \{c\}$ and $S_2 = \{z : z^n = 1\}$ such that $c^{2n} \neq 1$, and let f and g be two nonconstant entire functions. If $E(S_i, f) = E(S_i, g)$ for $i = 1, 2$, then $f(z) \equiv g(z)$.*

Fang and Xu [1], Yi [15] completely solved Question A. They proved the following theorem:

Theorem C. *Let $n \geq 3$ be a positive integer, $S_1 = \{0\}$ and $S_2 = \{z : z^n - z^{n-1} - 1 = 0\}$, and let f and g be two nonconstant entire functions. If $E(S_i, f) = E(S_i, g)$ for $i = 1, 2$, then $f(z) \equiv g(z)$.*

They give examples to show that if both S_1 and S_2 have at most two elements, then $E(S_i, f) = E(S_i, g)$ for $i = 1, 2$ cannot imply $f(z) \equiv g(z)$.

In this note, we extend and improve Theorem C as follows.

Theorem 1. *Let $n \geq 3$ be a positive integer, $S_1 = \{0\}$ and $S_2 = \{z : z^n - z^{n-1} - 1 = 0\}$, and let f and g be two nonconstant meromorphic functions whose poles are of multiplicities at least 2. If $E(\infty, f) = E(\infty, g)$ and $E(S_i, f) = E(S_i, g)$ for $i = 1, 2$, then $f(z) \equiv g(z)$.*

Remark. The condition that the poles of $f(z)$ and $g(z)$ are of multiplicities at least 2 can not be removed in Theorem 1.

Let $f(z) = \dfrac{e^z + e^{2z} + \cdots + e^{(n-1)z}}{1 + e^z + e^{2z} + \cdots + e^{(n-1)z}}$, $g(z) = \dfrac{1 + e^z + \cdots + e^{(n-2)z}}{1 + e^z + e^{2z} + \cdots + e^{(n-1)z}}$,

and let $S_1 = \{0\}$, $S_2 = \{z : z^n - z^{n-1} - 1 = 0\}$. Obviously, $E(\infty, f) = E(\infty, g)$ and $E(S_i, f) = E(S_i, g)$ for $i = 1, 2$, but $f(z) not \equiv g(z)$.

Corollary 2. *Let* $n \geq 3$, k *be two positive integers,* $S_1 = \{0\}$ *and* $S_2 = \{z : z^n - z^{n-1} - 1 = 0\}$, *and let* f *and* g *be two nonconstant meromorphic functions. If* $E(\infty, f) = E(\infty, g)$, $E(S_i, f^{(k)}) = E(S_i, g^{(k)})$ *for* $i = 1, 2$, *then* $f^{(k)}(z) \equiv g^{(k)}(z)$.

2. Some lemmas

In order to prove Theorem 1, we need the following lemmas.

Lemma 1. ([10]) *Let* $a_1, a_2, \cdots, a_n$ *be finite complex numbers,* $a_n \neq 0$, *and let* f *be a nonconstant meromorphic function. Then*

$$T(r, a_n f^n + a_{n-1} f^{n-1} + \cdots + a_1 f) = nT(r, f) + S(r, f).$$

Lemma 2. ([17]) *Let* f *and* g *be two nonconstant meromorphic functions. If* $E(1, f) = E(1, g)$, $E(\infty, f) = E(\infty, g)$, *and*

$$\overline{\lim_{\substack{r\to\infty \\ r\notin E}}} \frac{\overline{N}(r, f) + N_2\left(r, \frac{1}{f}\right) + \overline{N}(r, g) + N_2\left(r, \frac{1}{g}\right)}{T(r, f) + T(r, g)} < \frac{1}{2}$$

then either $f(z) \equiv g(z)$ *or* $f(z)g(z) \equiv 1$.

3. Proof of Theorem 1

By Theorem C, we only prove the case that $E(\infty, f) = E(\infty, g) \neq \emptyset$.

Firstly, we prove that

$$f^n - f^{n-1} \equiv g^n - g^{n-1} \tag{3.1}$$

Set

$$H(z) = \frac{[nf-(n-1)]f'}{f(f-1)(f^n - f^{n-1}-1)} - \frac{[ng-(n-1)]g'}{g(g-1)(g^n - g^{n-1}-1)}. \tag{3.2}$$

Next we consider two cases.

Case 1. $H(z) \equiv 0$, that is

$$\frac{[nf-(n-1)]f'}{f(f-1)(f^n - f^{n-1}-1)} \equiv \frac{[ng-(n-1)]g'}{g(g-1)(g^n - g^{n-1}-1)} \tag{3.3}$$

then by solving (3.3) we get

$$\frac{f^n - f^{n-1}-1}{f^n - f^{n-1}} \equiv c\,\frac{g^n - g^{n-1}-1}{g^n - g^{n-1}} \tag{3.4}$$

By (3.4) and $E(\infty, f) = E(\infty, g) \neq \emptyset$, we deduce that $c = 1$. Hence in this case we obtain (3.1).

Case 2. $H(z) not \equiv 0$. Let $z_0 \in E(\infty, f)$, then by the poles of f and g are of multiplicities at least 2, $E(\infty, f) = E(\infty, g)$, (3.2), and simple computing we get z_0 is a zero $H(z)$ with multiplicity at least $2n-1$. Thus we get by $E(S_i, f) = E(S_i, g)$ $(i=1,2)$, $E(\infty, f) = E(\infty, g)$, that

$$\begin{aligned}\overline{N}(r, f) = \overline{N}(r, g) &\le \frac{1}{2n-1}\,N\left(r, \frac{1}{H}\right) \le \frac{1}{2n-1}N(r, H) + S(r, f) + S(r, g)\\ &\le \frac{1}{2n-1}\left(\overline{N}\left(r, \frac{1}{f-1}\right) + \overline{N}\left(r, \frac{1}{g-1}\right)\right) + S(r, f) + S(r, g) \qquad (3.5)\\ &\le \frac{T(r, f) + T(r, g)}{2n-1} + S(r, f) + S(r, g).\end{aligned}$$

Now we consider two subcases.

Case 2.1. $E(0, f) = E(0, g) \neq \emptyset$. Set

$$\varphi = \frac{(f^n - f^{n-1}-1)'}{f^n - f^{n-1}-1} - \frac{(g^n - g^{n-1}-1)'}{g^n - g^{n-1}-1} \tag{3.6}$$

Next we divide two subcases.

Case 2.1.1. $\varphi(z) \equiv 0$. By solving this, we deduce that

$$f^n - f^{n-1} - 1 \equiv c(g^n - g^{n-1} - 1),$$

where c is a nonzero constant. Since $E(0, f) = E(0, g) \neq \emptyset$, we can easily obtain $c = 1$. Hence, we obtain (3.1).

Case 2.1.2. $\varphi(z) not \equiv 0$. Since $E(0, f) = E(0, g) \neq \emptyset$, we deduce from (3.6) that

$$N\left(r, \frac{1}{f}\right) = N\left(r, \frac{1}{g}\right) \leq N\left(r, \frac{1}{\varphi}\right)$$
$$\leq T(r, \varphi) + O(1) \leq N(r, \varphi) + S(r, f) \leq S(r, f).$$

Hence, we obtain that

$$\overline{N}\left(r, \frac{1}{f^n - f^{n-1}}\right) = \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f-1}\right) \leq \overline{N}\left(r, \frac{1}{f-1}\right) + S(r, f) \quad (3.7)$$

and

$$\overline{N}_{(2}\left(r, \frac{1}{f^n - f^{n-1}}\right) \leq \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}_{(2}\left(r, \frac{1}{f-1}\right) \leq \overline{N}_{(2}\left(r, \frac{1}{f-1}\right) + S(r, f). \quad (3.8)$$

Therefore,

$$\overline{N}(r, f^n - f^{n-1}) + N_2\left(r, \frac{1}{f^n - f^{n-1}}\right)$$

. $$\leq \overline{N}(r, f) + N\left(r, \frac{1}{f-1}\right) + S(r, f) \leq \overline{N}(r, f) + T(r, f) + S(r, f)$$

$$\leq \frac{1}{2n-1}\,(T(r, f) + T(r, g)) + T(r, f) + S(r, f) + S(r, g). \quad (3.9)$$

Likewise, we have

$$\overline{N}(r, g^n - g^{n-1}) + N_2\left(r, \frac{1}{g^n - g^{n-1}}\right)$$

$$\leq \frac{1}{2n-1}(T(r,f) + T(r,g)) + T(r,g) + S(r,f) + S(r,g) \qquad (3.10)$$

By Lemma 1, we have

$$T(r, f^n - f^{n-1}) = nT(r,f) + S(r,f),\ T(r, g^n - g^{n-1}) = nT(r,g) + S(r,g), \qquad (3.11)$$

Set $F = f^n - f^{n-1}$, and $G = g^n - g^{n-1}$. Then by (3.9)-(3.11) and $n \geq 3$, we have

$$\overline{\lim_{\substack{r\to\infty \\ r\notin E}}} \frac{\overline{N}(r,F) + N_2\left(r,\frac{1}{F}\right) + \overline{N}(r,G) + N_2\left(r,\frac{1}{G}\right)}{T(r,F) + T(r,G)} < \frac{1}{2} \qquad (3.12)$$

Since $f(z)$ and $g(z)$ satisfy $E(S_2, f) = E(S_2, g)$ and $E(\infty, f) = E(\infty, g)$, we deduce that $E(1,F) = E(1,G)$ and $E(\infty,F) = E(\infty,G)$. Thus by Lemma 2 we get that either $F \equiv G$ or $FG \equiv 1$.

If $FG = (f^n - f^{n-1})(g^n - g^{n-1}) \equiv 1$, then by $E(\infty, f) = E(\infty, g)$, we deduce that $f \neq 0, 1, \infty$. Thus f is a constant, a contradiction. Hence we prove $F \equiv G$, that is (3.1) holds.

Case 2.2. $E(0, f) = E(0, g) = \emptyset$. Then

$$\overline{N}(r, f^n - f^{n-1}) + \overline{N}_2\left(r, \frac{1}{f^n - f^{n-1}}\right) \leq \overline{N}(r,f) + \overline{N}_2\left(r, \frac{1}{f-1}\right)$$

$$\leq \overline{N}(r,f) + N\left(r, \frac{1}{f-1}\right) + S(r,f)$$

$$\leq \overline{N}(r,f) + T(r,f) + S(r,f)$$

Hence, we get (3.9), (3.10) and (3.12). Next using the similar argument to Case 2.1.2, we obtain (3.1).

Now we prove $f(z) \equiv g(z)$. Suppose on the contrary that $f(z) not \equiv g(z)$. Set $f/g = h$, then by $f^n - f^{n-1} \equiv g^n - g^{n-1}$, we deduce that

$$g = \frac{1 + h + \cdots + h^{n-2}}{1 + h + \cdots h^{n-1}}. \tag{3.13}$$

By $E(0,f) = E(0,g)$ and $E(\infty,f) = E(\infty,g)$ we get $h(z) = e^{\alpha(z)}$, where $\alpha(z)$ is an entire function. If α is not a constant, then by the Nevanlinna second fundamental theorem, for any $a \in C$, $a \neq 0$, we have $N\left(r, \frac{1}{h-a}\right) = T(r,h) + S(r,h)$ and $N\left(r, \frac{1}{h'}\right) = S(r,h)$. Hence the zeros of $h-a$ are almost simple. By (3.13), the poles of $g(z)$ are of almost simple, a contradiction. Thus α is a constant, that is h is a constant. Hence by (3.13), g is a constant, a contradiction. Therefore, we prove that $f(z) \equiv g(z)$. The proof of the theorem is complete.

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