A Unicity Theorem for Meromorphic Functions

HUILING QIU AND MINGLIANG FANG
Department of Mathematics, Nanjing Normal University, Nanjing 210097 People's Republic of China
e-mail: qiuhuiling1304@sina.com

Abstract. In this paper, we study the uniqueness of meromorphic functions and prove the following result: Let \( n \geq 3 \) be a positive integer, \( S = \{ z : z^n - z^{-1} - 1 = 0 \} \), and let \( f \) and \( g \) be two nonconstant meromorphic functions whose poles are of multiplicities at least 2. If \( E(0, f) = E(0, g) \), \( E(S, f) = E(S, g) \), and \( E(\infty, f) = E(\infty, g) \), then \( f(z) = g(z) \). This result also answers a question of Gross [4] and improves some results of Fang and Xu [1], Yi [14] and Yi [15].

1. Introduction

In this paper, by a meromorphic function we always mean a function which is meromorphic in the whole complex plane. Let \( f(z) \) be a nonconstant meromorphic function. We use the following standard notations of value distribution theory,

\[ T(r, f), m(r, f), N(r, f), \overline{N}(r, f), N(r, 1/f), \cdots \]

(see Hayman [6], Yang [12]). We denote by \( S(r, f) \) any function satisfying

\[ S(r, f) = o \{ T(r, f) \}, \]

as \( r \to + \infty \), possibly outside of a set \( E \) with finite measure. In this paper, \( E \) may be different at different places.

Let \( S \) be a set of complex numbers. Set

\[ E(S, f) = \bigcup_{a \in S} \{ z : f(z) - a = 0 \}, \]

where the zeros point with multiple \( m \) is counted \( m \) times in the set.
Let $N_2(r, \frac{1}{f-a})$ be the counting function which only includes multiple zeros of $f(z) - a$ and

$$\overline{N_2} \left( r, \frac{1}{f-a} \right) = \overline{N} \left( r, \frac{1}{f-a} \right) + \overline{N}_2 \left( r, \frac{1}{f-a} \right).$$

In 1977, Gross [4] posed the following question.

**Question A.** Can one find two finite sets $S_1$ and $S_2$ such that any two nonconstant entire functions $f$ and $g$ satisfying $(S_i, f) = E(S_i, g)$ for $i = 1, 2$ must be identical? And if such two sets exist, it would be interesting to know how large the two sets would have to be.

Yi [14] proved such two sets exist.

**Theorem B.** Let $n \geq 5$ be a positive integer, $S_1 = \{c\}$ and $S_2 = \{z : z^n = 1\}$ such that $c^{2n} \neq 1$, and let $f$ and $g$ be two nonconstant entire functions. If $E(S_i, f) = E(S_i, g)$ for $i = 1, 2$, then $f(z) = g(z)$.

Fang and Xu [1], Yi [15] completely solved Question A. They proved the following theorem:

**Theorem C.** Let $n \geq 3$ be a positive integer, $S_1 = \{0\}$ and $S_2 = \{z : z^n - z^{n-1} - 1 = 0\}$, and let $f$ and $g$ be two nonconstant entire functions. If $E(S_i, f) = E(S_i, g)$ for $i = 1, 2$, then $f(z) \equiv g(z)$.

They give examples to show that if both $S_1$ and $S_2$ have at most two elements, then $E(S_i, f) = E(S_i, g)$ for $i = 1, 2$ cannot imply $f(z) = g(z)$.

In this note, we extend and improve Theorem C as follows.

**Theorem 1.** Let $n \geq 3$ be a positive integer, $S_1 = \{0\}$ and $S_2 = \{z : z^n - z^{n-1} - 1 = 0\}$, and let $f$ and $g$ be two nonconstant meromorphic functions whose poles are of multiplicities at least 2. If $E(\infty, f) = E(\infty, g)$ and $E(S_i, f) = E(S_i, g)$ for $i = 1, 2$, then $f(z) \equiv g(z)$.

**Remark.** The condition that the poles of $f(z)$ and $g(z)$ are of multiplicities at least 2 cannot be removed in Theorem 1.
Let \( f(z) = \frac{e^z + e^{2z} + \ldots + e^{(n-1)z}}{1 + e^z + e^{2z} + \ldots + e^{(n-1)z}} \), \( g(z) = \frac{1 + e^z + \ldots + e^{(n-2)z}}{1 + e^z + e^{2z} + \ldots + e^{(n-1)z}} \),

and let \( S_1 = \{0\}, S_2 = \{z : z^n - z^{n-1} - 1 = 0\} \). Obviously, \( E(\infty, f) = E(\infty, g) \) and \( E(S_i, f) = E(S_i, g) \) for \( i = 1, 2 \), but \( f(z) \neq g(z) \).

**Corollary 2.** Let \( n \geq 3, \ k \) be two positive integers, \( S_1 = \{0\} \) and \( S_2 = \{z : z^n - z^{n-1} - 1 = 0\} \), and let \( f \) and \( g \) be two nonconstant meromorphic functions. If \( E(\infty, f) = E(\infty, g) \), \( E(S_i, f^{(k)}) = E(S_i, g^{(k)}) \) for \( i = 1, 2 \), then \( f^{(k)}(z) = g^{(k)}(z) \).

2. Some lemmas

In order to prove Theorem 1, we need the following lemmas.

**Lemma 1.** ([10]) Let \( a_1, a_2, \ldots, a_n \) be finite complex numbers, \( a_n \neq 0 \), and let \( f \) be a nonconstant meromorphic function. Then

\[
T(r, a_n f^n + a_{n-1} f^{n-1} + \ldots + a_1 f) = nT(r, f) + S(r, f).
\]

**Lemma 2.** ([17]) Let \( f \) and \( g \) be two nonconstant meromorphic functions. If \( E(1, f) = E(1, g) \), \( E(\infty, f) = E(\infty, g) \), and

\[
\lim_{r \to \infty, r \neq E} \frac{\overline{N}(r, f) + N_2 \left( r, \frac{1}{f} \right) + \overline{N}(r, g) + N_2 \left( r, \frac{1}{g} \right)}{T(r, f) + T(r, g)} < \frac{1}{2}
\]

then either \( f(z) = g(z) \) or \( f(z)g(z) = 1 \).

3. Proof of Theorem 1

By Theorem C, we only prove the case that \( E(\infty, f) = E(\infty, g) \neq \emptyset \).

Firstly, we prove that

\[
f^n - f^{n-1} = g^n - g^{n-1}
\]

(3.1)
Set
\[ H(z) = \frac{[nf-(n-1)]f'}{f(f-1)(f^n-f^{n-1}-1)} - \frac{[ng-(n-1)]g'}{g(g-1)(g^n-g^{n-1}-1)}. \tag{3.2} \]

Next we consider two cases.

**Case 1.** \( H(z) = 0 \), that is
\[ \frac{[nf-(n-1)]f'}{f(f-1)(f^n-f^{n-1}-1)} = \frac{[ng-(n-1)]g'}{g(g-1)(g^n-g^{n-1}-1)} \tag{3.3} \]
then by solving (3.3) we get
\[ \frac{f^n-f^{n-1}-1}{f^n-f^{n-1}} = \frac{g^n-g^{n-1}-1}{g^n-g^{n-1}} \tag{3.4} \]

By (3.4) and \( E(\omega, f) = E(\omega, g) \neq \emptyset \), we deduce that \( c = 1 \). Hence in this case we obtain (3.1).

**Case 2.** \( H(z) \neq 0 \). Let \( z_0 \in E(\omega, f) \), then by the poles of \( f \) and \( g \) are of multiplicities at least 2, \( E(\omega, f) = E(\omega, g) \), (3.2), and simple computing we get \( z_0 \) is a zero \( H(z) \) with multiplicity at least \( 2n-1 \). Thus we get by \( E(S_i, f) = E(S_i, g) \) \((i=1, 2)\), \( E(\omega, f) = E(\omega, g) \), that
\[ \overline{N}(r, f) = \overline{N}(r, g) \leq \frac{1}{2n-1} \left( N(r, 1) - H \right) \leq \frac{1}{2n-1} N(r, H) + S(r, f) + S(r, g) \]
\[ \leq \frac{1}{2n-1} \left( \overline{N}(r, 1) - f' \right) + S(r, f) + S(r, g) \tag{3.5} \]
\[ \leq \frac{T(r, f) + T(r, g)}{2n-1} + S(r, f) + S(r, g). \]

Now we consider two subcases.

**Case 2.1.** \( E(0, f) = E(0, g) \neq \emptyset \). Set
\[ \phi = \frac{(f^n-f^{n-1}-1)'}{f^n-f^{n-1}} - \frac{(g^n-g^{n-1}-1)'}{g^n-g^{n-1}} \tag{3.6} \]
Next we divide two subcases.

**Case 2.1.1.** $\varphi(z) = 0$. By solving this, we deduce that

$$f^n - f^{n-1} - 1 = c(g^n - g^{n-1})$$

where $c$ is a nonzero constant. Since $E(0, f) = E(0, g) \neq \emptyset$, we can easily obtain $c = 1$. Hence, we obtain (3.1).

**Case 2.1.2.** $\varphi(z) \neq 0$. Since $E(0, f) = E(0, g) \neq \emptyset$, we deduce from (3.6) that

$$N \left( r, \frac{1}{f} \right) = N \left( r, \frac{1}{g} \right) \leq N \left( r, \frac{1}{\varphi} \right)$$

\[ \leq T(r, \varphi) + O(1) \leq N(r, \varphi) + S(r, f) \leq S(r, f). \]

Hence, we obtain that

$$\overline{N} \left( r, \frac{1}{f^n - f^{n-1}} \right) = \overline{N} \left( r, \frac{1}{f} \right) + \overline{N} \left( r, \frac{1}{f-1} \right) \leq \overline{N} \left( r, \frac{1}{f-1} \right) + S(r, f) \quad (3.7)$$

and

$$\overline{N} \left( r, \frac{1}{f^n - f^{n-1}} \right) \leq \overline{N} \left( r, \frac{1}{f} \right) + \overline{N} \left( r, \frac{1}{f-1} \right) \leq \overline{N} \left( r, \frac{1}{f-1} \right) + S(r, f). \quad (3.8)$$

Therefore,

$$\overline{N}(r, f^n - f^{n-1}) + N_2 \left( r, \frac{1}{f^n - f^{n-1}} \right)$$

\[ \leq \overline{N}(r, f) + N \left( r, \frac{1}{f-1} \right) + S(r, f) \leq \overline{N}(r, f) + T(r, f) + S(r, f) \]

\[ \leq \frac{1}{2n-1} (T(r, f) + T(r, g)) + T(r, f) + S(r, f) + S(r, g). \quad (3.9) \]
Likewise, we have

\[
\overline{N}(r, g^n - g^{n-1}) + N_2\left(r, \frac{1}{g^n - g^{n-1}}\right)
\leq \frac{1}{2n-1}(T(r, f) + T(r, g)) + T(r, g) + S(r, f) + S(r, g)
\]  

(3.10)

By Lemma 1, we have

\[
T(r, f^n - f^{n-1}) = nT(r, f) + S(r, f), \quad T(r, g^n - g^{n-1}) = nT(r, g) + S(r, g).
\]  

(3.11)

Set \( F = f^n - f^{n-1} \), and \( G = g^n - g^{n-1} \). Then by (3.9)-(3.11) and \( n \geq 3 \), we have

\[
\lim_{r \to \infty} \frac{\overline{N}(r, F) + N_2\left(r, \frac{1}{F}\right) + \overline{N}(r, G) + N_2\left(r, \frac{1}{G}\right)}{T(r, F) + T(r, G)} < \frac{1}{2}
\]  

(3.12)

Since \( f(z) \) and \( g(z) \) satisfy \( E(S_z, f) = E(S_z, g) \) and \( E(\infty, f) = E(\infty, g) \), we deduce that \( E(1, F) = E(1, G) \) and \( E(\infty, F) = E(\infty, G) \). Thus by Lemma 2 we get that either \( F = G \) or \( FG = 1 \).

If \( FG = (f^n - f^{n-1})(g^n - g^{n-1}) = 1 \), then by \( E(\infty, f) = E(\infty, g) \), we deduce that \( f \neq 0, 1, \infty \). Thus \( f \) is a constant, a contradiction. Hence we prove \( F = G \), that is (3.1) holds.

**Case 2.2.** \( E(0, f) = E(0, g) = \emptyset \). Then

\[
\overline{N}(r, f^n - f^{n-1}) + \overline{N}_2\left(r, \frac{1}{f^n - f^{n-1}}\right) \leq \overline{N}(r, f) + \overline{N}_2\left(r, \frac{1}{f-1}\right)
\]

\[
\leq \overline{N}(r, f) + N\left(r, \frac{1}{f-1}\right) + S(r, f)
\]

\[
\leq \overline{N}(r, f) + T(r, f) + S(r, f)
\]

Hence, we get (3.9), (3.10) and (3.12). Next using the similar argument to Case 2.1.2, we obtain (3.1).
Now we prove $f(z) = g(z)$. Suppose on the contrary that $f(z) \not= g(z)$. Set $f/g = h$, then by $f^n - f^{n-1} = g^n - g^{n-1}$, we deduce that

$$g = \frac{1 + h + \ldots + h^{n-2}}{1 + h + \ldots h^{n-1}}.$$  \hspace{1cm} (3.13)

By $E(0, f) = E(0, g)$ and $E(\infty, f) = E(\infty, g)$ we get $h(z) = e^{\alpha(z)}$, where $\alpha(z)$ is an entire function. If $\alpha$ is not a constant, then by the Nevanlinna second fundamental theorem, for any $a \in \mathbb{C}$, $a \neq 0$, we have $N(r, \frac{1}{h-a}) = T(r, h) + S(r, h)$ and $N(r, \frac{1}{h}) = S(r, h)$. Hence the zeros of $h-a$ are almost simple. By (3.13), the poles of $g(z)$ are of almost simple, a contradiction. Thus $\alpha$ is a constant, that is $h$ is a constant. Hence by (3.13), $g$ is a constant, a contradiction. Therefore, we prove that $f(z) = g(z)$. The proof of the theorem is complete.

References


Keywords: Meromorphic function, entire function, unicity, finite sets.

1991 Mathematics Subject Classification: 30D35