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A Generalization of the Residue Formula

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Abstract. We show that a suitable variation of the well-known residue formula holds when an analytic function has isolated essential singularities along the contour of integration.

1. Introduction

Let C be a smooth simple curve contained in a complex region Ω. Let F be analytic in the region $\Omega \setminus \mathsf{G}$ where G is a finite set. If no singularities are located on C, that is, if $C \cap \mathsf{G} = \varnothing$, then the well-known residue formula from elementary complex variable courses,

$$\oint_C F(z)dz = 2\pi i \sum_{j=1}^{n} \operatorname{Res}_{z=b_j} F(z), \tag{1.1}$$

holds. Here $b_1, \cdots, b_n$ are the singularities of F inside C and $\operatorname{Res}_{z=b_j} F(z)$ their respective residues. The b_j can be poles or essential singularities of F.

Interestingly, it is true and known, although not so well-known, that a simple generalization of (1.1) holds if some of the singularities, say $a_1, \cdots, a_m$, are *poles* located on the contour C. Indeed, in such a case [6]

$$\text{F.p.} \oint_C F(z)\,dz = 2\pi i \sum_{j=1}^{n} \operatorname{Res}_{z=b_j} F(z) + \pi i \sum_{j=1}^{m} \operatorname{Res}_{z=a_j} F(z). \tag{1.2}$$

Observe that the integral is not an ordinary integral in this case: the presence of poles makes it divergent and thus we consider its finite part, as explained in Section 2. When all the poles are simple the finite part integral reduces to its Cauchy principal value and in such a case the generalized formula (1.2) was already given by Cauchy.

Our aim is to give a generalization of (1.2) to the case when some of the a_j are essential singularities of F. Simple examples, such as

$$\oint_{|z|=1} e^{1/(z-1)}\ dz, \tag{1.3}$$

or

$$\int_{-\infty}^{\infty} \sin\left(\frac{1}{x}\right) \frac{dx}{x-z}\ ,\ z \in \mathbb{C} \setminus \mathbb{R}, \tag{1.4}$$

show that (1.2) does not hold in such cases. Interestingly, many times the integrals, as (1.3) and (1.4) illustrate, can be ordinary *convergent* integrals. As we show, however, when *F* has distributional boundary values from the inside and the residues are redefined appropriately, then (1.2) still holds.

The plan of the article is as follows. In Section 2 we discuss some useful preliminary concepts, such as finite parts and distributional limits. In Section 3 we define a new residue along a contour for a function with an essential singularity; this is exactly the residue needed in our generalized formula. Section 4 treats the relationship between finite part integrals and distributional limits, while in Section 5 we state and prove our main result.

2. Preliminaries

In this section we explain the ideas of the principal value and finite part of divergent integrals. These are fundamental notions in the study of integral equations, since, in fact, the integrals that appear in singular integral equations are usually principal value integrals [6], [7], [11] while those in hyper-singular integral equations are usually finite part integrals [1], [8], [10]. A detailed discussion can be found in [6, Chapter 1]. We also give some ideas from the distributional boundary values of analytic functions [2], [4].

Let *C* be a smooth simple contour in the complex plane and let $\xi_0 \in C$. Suppose that a function g is defined and continuous on $C \setminus \{\xi_0\}$. Then one may define *the improper integral of g over C* as

$$\oint_C g(\xi)\ d\xi = \lim_{\varepsilon,\eta\to 0}\left(\oint_{C_\varepsilon^+} g(\xi)\ d\xi + \oint_{C_\eta^-} g(\xi)\, d\xi\right). \tag{2.1}$$

Here C_ε^- is the part of *C* from the initial point to the first intersection of *C* and a circle of radius ε about ξ_0 while C_ε^+ is the part of contour from the second intersection to the final point. If the curve is closed one just takes as initial and final points any point of *C* different from ξ_0. Observe that since the curve is smooth, the circle does intersect it at exactly two points if ε is small enough. The key for the existence of the improper integral is that ε and η tend to zero *independently*.

It happens many times that the improper integral (2.1) does not exist but the limit exists when ε and η are related in an appropriate way, say, for instance, when $\varepsilon = \eta$. In this case we obtain the Cauchy principal value of the integral,

$$\text{p.v.} \oint_C g(\xi)\,d\xi = \lim_{\varepsilon\to 0} \oint_{C_\varepsilon} g(\xi)\,d\xi, \tag{2.2}$$

where $C_\varepsilon = C_\varepsilon^- \cup C_\varepsilon^+$. Typical principal value integrals are the Cauchy type integrals $\oint_C (f(\xi)/(\xi - \xi_0))\,d\xi$, where f is a function defined on *C*.

It is important to point out the formula

$$\text{p.v.} \oint_C \frac{d\xi}{\xi - \xi_0} = \pi i, \tag{2.3}$$

where C is a smooth simple contour and $\xi_0 \in C$. Formula (2.3) is a sort of average of the Cauchy formula

$$\text{p.v.} \oint_C \frac{d\xi}{\xi - z} = \begin{cases} 2\pi i & , \quad z \text{ inside } C, \\ 0 & , \quad z \text{ outside } C. \end{cases} \tag{2.4}$$

When the limit (2.2) does not exist, one may obtain a finite result by using the *Hadamard finite part* method. Let

$$G(\varepsilon) = \oint_{C_\varepsilon} g(\xi)\,d\xi. \tag{2.5}$$

We want to study the bahavior of the limit of $G(\varepsilon)$ as $\varepsilon \to 0$. Then we write

$$G(\varepsilon) = G_1(\varepsilon) + G_0(\varepsilon), \tag{2.6}$$

where $G_1(\varepsilon)$, the "infinite part", captures what makes the limit divergent while $G_0(\varepsilon)$, the "finite part" satisfies that

$$\lim_{\varepsilon\to 0} G_0(\varepsilon) = L, \tag{2.7}$$

exists. There is not a unique way to perform the split (2.6); to obtain a unique decomposition we require that the infinite part be given as

$$G_1(\varepsilon) = \phi_1(\varepsilon) + \cdots + \phi_n(\varepsilon), \tag{2.8}$$

where $\phi_1, \cdots, \phi_n$ are chosen from a *given* set of functions that become infinite as $\varepsilon \to 0$, usually inverse powers of ε and logarithms. We then write

$$\text{F.p.} \oint_C g(\xi)\, d\xi = L, \tag{2.9}$$

for the finite part of the integral.

If $k \in N$, $k \geq 2$, the integral

$$\int_C \frac{d\xi}{(\xi - \xi_0)^k} \tag{2.10}$$

does not exist as an improper integral nor as a principal value integral.
However, its finite part exists and we have

$$\text{F.p.} \int_C \frac{d\xi}{(\xi - \xi_0)^k} = \frac{1}{k-1}\left(\frac{1}{(a - \xi_0)^{k-1}} - \frac{1}{(b - \xi_0)^{k-1}}\right), \tag{2.11}$$

where a is the initial point and b is the final point of the contour. In particular,

$$\text{F.p.} \int_C \frac{d\xi}{(\xi - \xi_0)^k} = 0, \tag{2.12}$$

if C is a closed contour. Of course,

$$\oint_C \frac{d\xi}{(\xi - z)^k} = 0, \quad z \in \mathbb{C} \setminus C, \tag{2.13}$$

also.

It is useful to generalized the finite part method by requiring the limit (2.7) to hold not in the ordinary sense but in the distributional sense of Lojasiewicz [9]. This one may call the distributional finite part method.

Let us consider distributional boundary values of analytic functions [2], [4]. Let C be a simple smooth contour; it may be open or closed. The space of test functions over C, denoted as $\mathcal{D}(C)$, is formed by those smooth functions ϕ defined on C that vanish in a neighborhood of the endpoints of C (if C is closed, $\mathcal{D}(C)$ is just the space of smooth functions over C). The space of distributions over C is the dual space $\mathcal{D}'(C)$; that is, a distribution f over C is a continuous linear functional on the space of test functions. The evaluation of a distribution $f \in \mathcal{D}'(C)$ at a test function $\phi \in \mathcal{D}(C)$ is denoted as $\langle f, \phi \rangle$.

The curve C determines two sides of $\mathbb{C} \setminus C$. Corresponding to the positive orientation of C, one side is to the left and the other to the right of the curve; these sides are local, but if the curve is closed they become global, the left corresponding to the interior and the right to the exterior of C if the counterclockwise orientation is used. Usually the left side is called the positive side, while the right is the negative side.

Let C_ε be a continuous family contours for $0 \le \varepsilon \le \varepsilon_0$ that approach $C = C_0$ as $\varepsilon \to 0$, say from the positive side. Suppose for instance that the parametrization of C_ε is given by $z(t;\varepsilon)$, $a \le t \le b$. One obtains an isomorphism of $\mathcal{D}(C)$ and $\mathcal{D}(C_\varepsilon)$ by putting $\phi_\varepsilon(z(t;\varepsilon)) = \phi(z(t;0))$ if $\phi \in \mathcal{D}(C)$. Let now $F(z)$ be an analytic function defined on the positive side of C. We then say that the *distributional* limit of $F(z)$ as z approaches C from the positive side exists and equals $f \in \mathcal{D}'(C)$ if

$$\lim_{\varepsilon \to 0} \oint_{C_\varepsilon} F(\xi)\phi_\varepsilon(\xi)\, d\xi = \langle f, \phi \rangle, \tag{2.14}$$

for $\phi \in \mathcal{D}(C)$. Similar considerations apply to limits from the negative side. Notice the cases

$$\lim_{r \to 1^-} \oint_{|\xi|=1} F(r\xi)\phi(\xi)\, d\xi \tag{2.15}$$

for limits from iniside the unit circle and

$$\lim_{y \to 0^+} \int_{-\infty}^{\infty} F(x + iy)\phi(x)\, dx \tag{2.16}$$

for limits from the upper half plane. Actually, most results are the same for any smooth contour, since one may use the Riemann mapping theorem to reduce the situation to (2.15) or (2.16). In particular, in (2.15), F has distributional boundary values if and only if there exist M, $\beta \in \mathbb{R}$ such that

$$|F(z)| \le \frac{M}{(1-|z|)^\beta}, \quad |z| < 1, \tag{2.17}$$

while in (2.16) the distributional limit exists if only if for each $a > 0$ there exist $M(a),\ \beta(a),\ y(a) \in R$ such that

$$\left|F(x + iy)\right| \leq \frac{M(a)}{y^{\beta(a)}}\ ,\ 0 < y < y(a),\ \left|x\right| \leq a. \tag{2.18}$$

3. The residue along a contour

Our first task is to define the residue of an analytic function at a singularity *relative to a given contour*. When the singularity is a pole, we obtain the ordinary residue at a pole, but when the singularity is an essential singularity, we obtain precisely what is needed to generalize the residue formula (1.2).

Suppose first that a is a pole of the analytic function F. Then we can write

$$F(z) = F_0(z) + S(z), \tag{3.1}$$

where $S = S_a$, the singular part of $F(z)$ at $z = a$, is given by

$$S(z) = \frac{\alpha_k}{(z - a)^k} + \cdots + \frac{\alpha_1}{z - a}, \tag{3.2}$$

where k is the order of the pole and where $\alpha_j = \alpha_j(a)$ are constants. In particular, $\alpha_1 = \mathrm{Res}_{z=a}\, F(z)$, is the residue at $z = a$. Observe that in this case $F(z) \sim \alpha_k/(z - a)^k$ as $z \to a$, and that this is true along *any* contour through $z = a$.

Suppose now that a is an isolated essential singularity of F. Then (3.1) still holds, where F_0 is analytic at $z = a$, while $S = S_a$, the singular part is given by a series of the form

$$S(z) = \sum_{j=1}^{\infty} \frac{\alpha_j}{(z - a)^j}\ , \tag{3.3}$$

for z near a. In this case, however, the behavior of $F(z)$ as $z \to a$ can be rather complicated. If C is a contour through $z = a$, the limit $\lim_{z\to a} F(z)$ may even exist despite the presence of the highly singular term (3.3) in (3.1), and this limit may be different from $F_0(a)$.

Example. Let $F(z) = e^{-1/z^2}$. Then $z = 0$ is an essential singularity of *F*, the singular part is $S(z) = e^{-1/z^2} - 1$, and $F_0(z) = 1$. However, on the real axis the function $f(x) = F(x + i0)$ is not singular at $x = 0$, and in fact all the derivatives $f^{(j)}(0)$ exist and equal 0.

We define the residue of F at $z = a$ *along a contour C* as follows. Suppose that

$$F(z) = \frac{\alpha_k}{(z-a)^k} + \cdots + \frac{\alpha_1}{z-a} + F_0(z), \tag{3.4}$$

where $\lim_{z \to a, z \in C} F_0(z)$ exists in the distributional sense. Then we call $S(z) = S_a(z) = \sum_{j=1}^{k} \alpha_j / (z-a)^j$ the *singular part of* $F(z)$ *at* $z = a$ *along C* and, in particular, we call α_1 the residue of $F(z)$ *at* $z = a$ *along C* and write

$$\alpha_1 = \operatorname{Res}_{z=a;C} F(z).$$

Observe that the existence of $\operatorname{Res}_{z=a;C} F(z)$ does not imply the existence of $\operatorname{Res}_{z=a;C_1} F(z)$ for other curves C_1 through $z = a$ and even if the two residues exist, they might be different.

Example. Let

$$F(z) = \frac{1}{z} \int_0^{z^{-2}} e^{-\omega^q} d\omega, \quad z \in \mathbb{C} \setminus \{0\}. \tag{3.6}$$

Then $z = 0$ is an isolated essential singularity and *F* has *q* different residues

$$\operatorname{Res}_{z=0;C_j} F(z) = e^{\frac{2\pi i j}{q}} \int_0^{\infty} e^{-t^q} dt, \tag{3.7}$$

$1 \le j \le q$, along the *q* lines

$$C_j : \arg \omega = -\frac{\pi j}{q}. \tag{3.8}$$

Example. Consider the function $G(z) = e^{1/z}$ along the imaginary axis *I*. If $z = i\varepsilon$, $\varepsilon \in \mathbb{R}$, then $G(i\varepsilon) = e^{-i\varepsilon^{-1}}$ does not have an ordinary limit as $\varepsilon \to 0$. However, $\lim_{\varepsilon \to 0} G(i\varepsilon) = 0$ in the distributional sense of Lojasiewicz and thus

$$\operatorname{Res}_{z=0;I} e^{1/z} = 0. \tag{3.9}$$

4. A lemma

Let F be an analytic function in $\Omega \setminus G$, where G is finite. Suppose C is a curve in Ω and let us try to consider the function F as a distribution on C. If $C \cap G \neq \emptyset$ then F does not define a unique distribution on C. Indeed, we may consider the finite part distribution $\text{F.p.}(F)$ defined as

$$\left\langle \text{F.p.}(F), \phi \right\rangle = \text{F.p.} \oint_C F(\xi)\phi(\xi)\, d\xi , \quad \phi \in D(C).$$

But, one may also consider the distributional limits F_+ and F_- from the positive and negative sides of C, respectively. Not all three $\text{F.p.}(F)$, F_+ and F_- need to exist, perhaps none does. But even if they exist, they are usually different. Recall for instance the well-known dispersion relations [5, (2.4.18)]

$$\frac{1}{(x \pm i0)^n} = \text{F.p.}\left(\frac{1}{x^n}\right) \mp \frac{i\pi(-1)^n}{(n-1)!}\,\delta^{(n-1)}(x) \tag{4.2}$$

that show that the three distributions are different when $F(z) = 1/z^n$, and $C = R$. It follows that, in general, if all the singularities of F on C are poles, then the three distributions $\text{F.p.}(F)$, F_+ and F_- exist and *are different*.

The situation near an isolated essential singularity is of another nature. Let us suppose, to fix the ideas, that there is only one singularity, $G = \{\xi_0\}$, where $\xi_0 \in C$, and that it is an essential singularity. Then at least one of the two distributional limits F_+ of F_- does not exist, although neither may exist. Suppose F_+ and $\text{F.p.}(F)$ both exist (and F_- does not), then in general $F_+ \neq \text{F.p.}(F)$. However, there is one particular and important case when the two distributions have to coincide: when the distributional point value $F(\xi_0)$ exists. We prove this result in the lemma below, but before we do so it is worth to notice some related results.

If we use the Phragmén-Lindelöf theorems [12, section 5.6] we can easily show that if F, initially defined on $C \setminus \{\xi_0\}$, can be extended to a continuous function on C and if F is bounded on the intersection of a neighborhood of ξ_0 with the positive side of C, then actually $F(z) \to F(\xi_0)$ as z approaches ξ_0 from the positive side non-tangentially. Actually, the same argument works if the distributional limit from the positive side, F_+, exists.

Notice, however, the example $F(z) = z \sin(1/z)$. Along the real axis F can be extended from $\mathbb{R} \setminus \{0\}$ to $\mathbb{R}$ by setting $F(0) = 0$. However, $F(z)$ does not approach 0 as z approaches 0 non-tangentially from either the upper or lower half-planes. In this example neither of the distributional limits F_+ nor F_- exist.

The next proof is based on ideas from the theory of Fourier series.

Lemma 1. *Let F be analytic in* $\Omega \setminus \{\xi_0\}$, *where* ξ_0 *is an essential singularity. Let C be a smooth simple curve that goes through* ξ_0. *Suppose the finite part distribution* $f\{\xi) = \text{F.p.}(F(\xi))$ *exists in* $\mathcal{D}'(C)$ *and the distributional point value* $f(\xi_0)$ *exists. Suppose also that the distributional boundary limit of F from the positive side,* F_+, *exists. Then*

$$F_+ = \text{F.p.}(F). \tag{4.3}$$

Proof. It is enough to give the proof when $C = \{z \in \mathbb{C} : |z| = 1\}$ is the unit circle, $\xi_0 = 1$, and Ω is a region that contains the unit disc; the general case can be reduced to this one by conformal mapping. We use the standard notation $\xi = e^{i\theta}$, $\theta \in \mathbb{R}$.

It is clear that F_+ and $\text{F.p.}(F)$ coincide for $\xi \neq 1$. Thus they differ by a distribution concentrated at $\xi = 1$, that is, by a finite sum of derivatives of the Dirac delta function at $\xi = 1$. Hence

$$\text{F.p.}(F(e^{i\theta})) = F_+(e^{i\theta}) + \sum_{j=0}^{m} \alpha_j \, \delta^{(j)}(\theta), \tag{4.4}$$

where $\alpha_0, \cdots, \alpha_m$ are constants. Let $\sum_{n=-\infty}^{\infty} a_n e^{in\theta}$ be the Fourier series of $\text{F.p.}(F(e^{i\theta}))$. Observe that the Fourier series of F_+ contains only terms with positive index, i.e., it is of the form $\sum_{n=0}^{\infty} a_n^+ \, e^{in\theta}$. Since

$$\delta^{(j)}(\theta) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} (in)^j \, e^{in\theta}, \tag{4.5}$$

it follows that

$$a_n = \frac{1}{2\pi} \sum_{j=0}^{m} \alpha_j \, (in)^j, \; n < 0, \tag{4.6}$$

But the Fourier coefficients of a distribution with distributional point value at a point have been characterized [3]. A simple corollary of that characterization is that $a_n \to 0$ as $n \to -\infty$ in the Cesàro sense, $\lim_{n \to -\infty} a_n = 0\ (C)$. But the limit of (4.6) is ∞ unless $\alpha_0 = \cdots = \alpha_m = 0$. Therefore (4.3) follows.

Remark. The same type of argument shows that if F_+ exists distributionally and if $f(\xi) = \text{F.p.}\{F(\xi))$ has lateral distributional point values $f(\xi_0^+)$ and $f(\xi_0^-)$ as ξ approaches ξ_0 from the initial or the final part of the contour, then both lateral point values coincide and (4.3) holds.

Example. Consider the function

$$F(z) = \int_0^{z^{-1}} e^{-\omega^2} d\omega,\ z \in \mathbb{C} \setminus \{0\}.$$

Along the real axis, $f(x) = \text{F.p.}(F(x))$ has a jump discontinuity at $x = 0$, while $f'(x)$ consists of an ordinary function for $x \neq 0$, with just a jump discontinuity, and a Dirac delta function at the origin. Looking at the proof of the lemma and the following remark, such behavior is not possible if F has distributional boundary values from the upper or lower half-planes: naturally the distributional limits F_+ and F_- do not exist in $\mathcal{D}'(\mathbb{R})$.

5. The main result

We can now prove our generalized residue formula.

Theorem 1. *Let F be analytic in a region $\Omega \setminus \mathsf{G}$, where G is a finite set. Let C be a simple smooth contour contained in Ω. Let $b_1, \cdots, b_n$ be the singularities inside C and let $a_1, \cdots, a_m$ be the singularities of F on C. Suppose each a_j is either a pole of F or, if an isolated essential singularity, has a well-defined singular part along C. Then the finite part integral*

$$\text{F.p.} \oint_C F(z)\, dz \tag{5.1}$$

exists.

If also the distributional limit of F from inside C exists, then

$$\text{F.p.} \oint_C F(z)\, dz = 2\pi i \sum_{j=1}^{n} \text{Res}_{z=b_j} F(z) + \pi i \sum_{j=1}^{m} \text{Res}_{z=a_j;C} F(z). \tag{5.2}$$

Proof. Consider the singular parts $S_{b_1}, \cdots, S_{b_n}, S_{a_1}, \cdots, S_{a_m}$. If we write

$$F = S_{b_1} + \cdots + S_{b_n} + S_{a_1} + \cdots + S_{a_m} + G, \tag{5.3}$$

then G is analytic inside C and across $C \setminus \{a_1, \cdots, a_m\}$ and the distributional point values $G(a_1), \cdots, G(a_m)$ along C exist. Therefore the finite part integral

$$\text{F.p.} \oint_C G(z)\,dz \tag{5.4}$$

exists. Since

$$\text{F.p.} \oint_C S_{b_j}(z)\,dz = 2\pi i \ \text{Res}_{z=b_j} F(z), \tag{5.5}$$

while

$$\text{F.p.} \oint_C S_{a_j}(z)\,dz = \pi i \ \text{Res}_{z=a_j} F(z), \tag{5.6}$$

it follows that (5.1) exists.

Suppose now that the distributional limit of F from inside C exists. Then, because of (5.5) and (5.6), the generalized residue formula (5.2) would follow if we show that $F.p.\oint_C G(z)dz = 0$. But G has distributional limits from the inside, as follows from (5.3) since the singular parts do. Therefore using the lemma, if $\phi \in \mathsf{D}(C)$

$$\langle G, \phi \rangle = \lim_{\varepsilon \to 0} \int_a^b G(z(\varepsilon,t))\phi(z(0,t)) \frac{dz(\varepsilon,t)}{dt}\, dt, \tag{5.7}$$

where the curves C_ε given by $z(\varepsilon,t)$, $a \le t \le b$, are interior to C as $\varepsilon \to 0$. In particular, if $\phi = 1$ we obtain

$$\begin{aligned} \text{F.p.} \oint_C G(z)\,dz &= \langle G, 1 \rangle \\ &= \lim_{\varepsilon \to 0} \int_a^b G(z(\varepsilon,t)) \frac{dz(\varepsilon,t)}{dt} dt \\ &= \lim_{\varepsilon \to 0} \oint_{C_\varepsilon} G(z)\,dz \\ &= 0, \end{aligned} \tag{5.8}$$

as required.

Let us consider some illustrations.

Example. Consider the integral (1.3), namely,

$$\oint_{|z|=1} e^{1/(z-1)}\, dz.$$

Here $e^{1/(z-1)}$ is bounded in $|z| \le 1$ and thus has boundary values from the inside of $C : |z| = 1$ [4]. The only singularity is $z = 1$, but since the function is continuous along *C*, we have $\operatorname{Res}_{z=1;C}\ e^{1/(z-1)} = 0$. Therefore (5.2) yields

$$\oint_{|z|=1} e^{1/(z-1)}\, dz = 0. \tag{5.9}$$

Notice that the ordinary residue $\operatorname{Res}_{z=1}\ e^{1/(z-1)} = 1$ cannot be used in the generalized residue formula.

Example. Let us consider the integral (1.4), namely,

$$\int_{-\infty}^{\infty} \sin\left(\frac{1}{x}\right) \frac{dx}{x-z}.$$

One may try to evaluate the integral in the usual way, namely by observing that

$$\int_{-\infty}^{\infty} \sin\left(\frac{1}{x}\right) \frac{dx}{x-z} = \lim_{R\to\infty} \oint_{C_R} \sin\left(\frac{1}{\omega}\right)\frac{d\omega}{\omega - z}, \tag{5.10}$$

where C_R is a contour from $-R$ to R along the real axis and from R to $-R$ along a semicircle on the upper half plane, since the integral over the semicircle tends to 0. Let us try to apply the residue formula (5.2). The integrand,

$$F(\omega) = \sin\left(\frac{1}{\omega}\right) \frac{1}{\omega - z},$$

has two singularities; one is an essential singularity at $\omega = 0$, with $\operatorname{Res}_{\omega=0;R}\ F(\omega) = 0$, the other is a pole at $\omega = z$, with residue $\operatorname{Res}_{\omega=z}\ F(\omega) = \sin(1/z)$. This second singularity contributes to the integral only if $\Im m\, z \ge 0$. Thus one would obtain from the right side of (5.2) the following

$$\text{"result"} = \begin{cases} 2\pi i \ \sin\left(\frac{1}{z}\right) & , \quad \Im m\, z > 0, \\ \pi i \sin\left(\frac{1}{z}\right) & , \quad \Im m\, z = 0, \\ 0 & , \quad \Im m\, z < 0. \end{cases} \tag{5.11}$$

This is not the right result, however!. Indeed [6, example 10],

$$\int_{-\infty}^{\infty} \sin\left(\frac{1}{x}\right) \frac{dx}{x-z} = \begin{cases} \pi(1-e^{-i/z}) & , \quad \Im m\, z > 0 , \\ \pi(1-e^{i/z}) & , \quad \Im m\, z < 0 , \end{cases} \tag{5.12}$$

while

$$\text{p.v.}\int_{-\infty}^{\infty} \sin\left(\frac{1}{x}\right) \frac{dx}{x-z} = \pi\left(1 - \cos\left(\frac{1}{z}\right)\right) \quad , \ \Im m\, z = 0. \tag{5.13}$$

Therefore the generalized residue formula (5.2) does not apply in this case. Why? Because the function $\sin(1/z)$ does not have distributional boundary values from the upper half plane, since there are no constants *M* and β such that

$$\left|F(x + iy)\right| \leq \frac{M}{y^{\beta}} ,$$

for $0 < y < 1$, $\left|x + iy\right| < 1$.

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