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Learning and Teaching Mathematics with a Graphic Calculator

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Abstract. The past three decades has seen many mathematics departments adopting plans for the appropriate use of instructional technology tools in the learning and teaching of mathematics, and in the assessment of mathematics learning.

A device that can be utilized to facilitate learning of mathematics is the graphic calculator, whose technology and levels of sophistication have grown significantly over the last few years. Compared to the United States, Europe, and the Oceania region, the use of hand-held technology in Malaysia is very much in its infancy.

The School of Mathematical Sciences at Universiti Sains Malaysia (USM) offers a Special Topic course on the integration of hand-held technology into the teaching and learning of mathematics beginning the 2001/2002 academic year. The course is taught in an inquiry-based format that highlights explorations and applications of mathematics in a data rich modeling environment. In addition, the course addresses issues related to the effective integration of such technologies into the mathematics and science curriculum. This paper discusses pedagogical and assessment strategies that have been implemented in the course and summarizes student reactions to the innovative learning mode.

1. Introduction

Technology has fundamentally transformed offices, factories, and retail establishments over the past several decades. However, its impact within the classrooms has generally been quite modest. In responding to concerns raised regarding the capacity of the educational system to meet the challenge of the information technology era, mathematics departments of many universities have begun to adopt plans for the appropriate use of instructional technology tools in the learning and teaching of mathematics, and in the assessment of mathematics learning.

According to Higgins and Muijs (1999), educators need to ask pertinent questions about the latest technology; for example,

- What does the new technology offer and how easy is it to use?
- Does it present mathematical concepts in a way that will support student's understanding?
- Can you teach the same content without the technology?
- Is the technology approach really more effective?

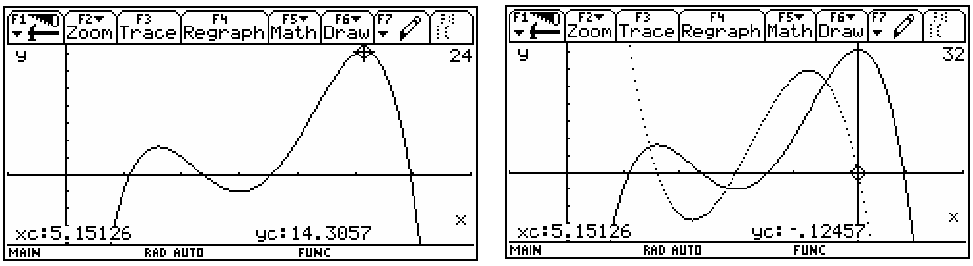
Efforts should be focused on the use of technology to enhance learning, and not only to learn about technology. Although both are worthy of attention, it is important to distinguish between technology as a subject area and the use of technology to facilitate learning about any subject area. Thus technology should be integrated throughout the curriculum, and not simply used to impart technology-related knowledge and skills.

The School of Mathematical Sciences at USM has developed from the ground up a course based on the new learning modes made feasible, indeed imperative, by the graphic calculator enhanced with computer algebra systems (CAS). Students are introduced to the capabilities of graphic calculator as an instructional tool. In addition, the course includes several seminars addressing issues driven by the integration of the new technologies into the classroom, e.g., pedagogical and curricular changes. In many countries, including Austria, France, Australia, USA, and Canada, it is taken for granted that many of the students will have access to a graphic calculator much of the time, if not at all times. In fact, beginning in the year 2001, neighboring Singapore has allowed the use of graphic calculators, without CAS, in the national exams *Further Mathematics* paper. The advent of technology has put at issue teaching pedagogy and strategies. This paper elucidates our development and testing of materials that incorporate the graphic calculators to enhance the understanding of concepts in the mathematics classroom.

2. The graphic calculators

The technology and levels of sophistication of graphic calculators have grown significantly over the last few years. The graphic calculators can plot graphs, visualize 3D surfaces, and are programmable. The CAS tools automate the execution of algebraic and calculus computations. CAS can simplify expressions, evaluate derivatives and integrals symbolically or numerically, perform matrix operations, and solve differential equations. CAS automates most of the calculation skills we teach in mathematics.

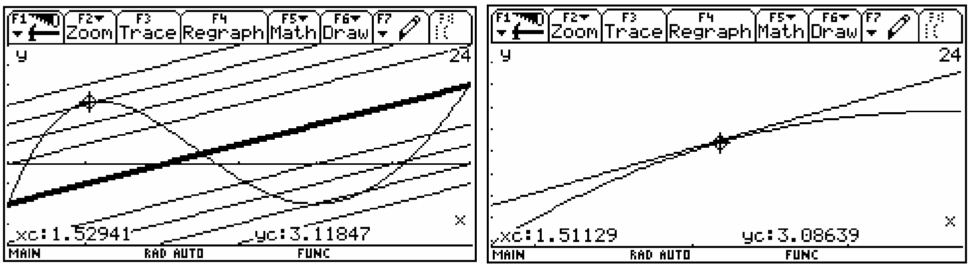
The illustrations below of the calculator as an instructional tool in calculus were drawn from explorations developed for the course. The graphing feature of the calculator readily displays the graph of a function $f(x) = -x^4 + 13x^3 - 57x^2 + 99x - 56$ (Figure 1(a)), in addition it can display the derivative $f'(x) = -4x^3 + 39x^2 - 118x + 99$ along with a vertical line through a zero of $f'(x)$ (Figure 1(b)). The second graph is a visualization of properties of the derivatives; when zeros of the derivative of a function coincide with extrema of the function. Typically a student will require a significant amount of labor to reproduce the same graph using pencil and paper.



(a) Graph of $y = f(x)$ (b) Graphs of f and f'

Figure 1. TI-92Plus plot of a function and its derivative

The next figure is a visualization of the *mean value theorem* of calculus. A geometric view of the mean value theorem is that of all the lines parallel to the secant line from $(a, f(a))$ to $(b, f(b))$, at least one of the lines will also be tangent to f. Figure 2(a) illustrates f along with a family of lines parallel to the secant line (the line drawn with squares) and Figure 2(b) illustrates the use of the zoom feature to better estimate graphically the *x*-coordinate of a point that satisfies the mean value theorem.



(a) Sequence of parallel lines (b) Zoom feature at a point

Figure 2. Interpretation of the mean value theorem

An illustration of the CAS capability is provided by the following example in linear algebra, i.e., to find the eigenvalues of the matrix $A = \begin{bmatrix} 4 & 4 & 6 \\ 7 & 14 & 7 \\ 4 & -4 & 2 \end{bmatrix}$. The graphic calculator explicity determines the characteristic polynomial $f(x) = |A - xI| = -x^3 + 20x^2 - 68x - 224$, and readily finds the zeros of *f*, which yield the eigenvalues -2, 8, and 14.

In addition, a calculator can manage data, calculate standard statistical measures, perform all standard statistical tests and confidence intervals for means, proportions, chi-square analysis and regression. It can also generate a range of statistical graphics such as scatter plots, histograms, box plots with outliers, normal probability plots and residual plots.

The following is an excerpt of a statistics laboratory exploration that was done in class. Based on the tabulated bivariate data, students were asked to get descriptive measures of both variables in the sample, to get a relationship between the two, if one exists, and to predict the blood pressure of a person aged at a certain *x* years.

Table 1. Bivariate data of blood pressure and age

Age, *x*	43	48	56	61	67	70
Blood pressure, *y*	128	120	135	143	142	152

Typed-in data can be displayed and checked for typing errors. Descriptive measures of each variable can readily be obtained, as shown below:

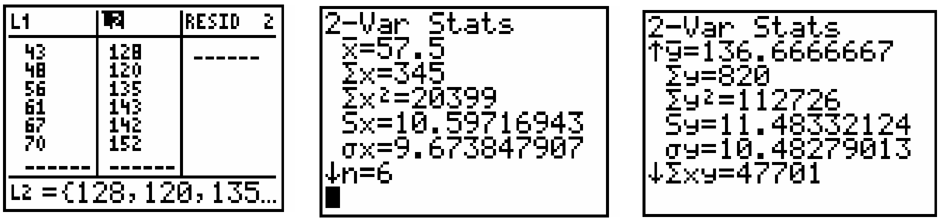


Figure 3. TI-83Plus displays of descriptive statistics

A scatter plot of the data points indicates the existence of a relationship and students are able to see whether the relationship is linear or not. A least square regression procedure produces an equation of a straight line, $y = .981x + 80.242$, that best fit the plotted data. A correlation coefficient of $r = .90557$ confirms a strong linear relationship between the two variables. To aid students in seeing how close the “best straight line” fits the data, the regression line is graphed on top of the scatter plot.

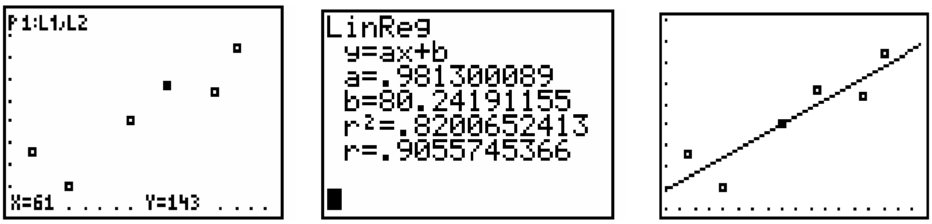


Figure 4. Scatter plot and linear regression plot

To predict the blood pressure of a person aged 65 years, the value 65 is entered in the *CALCULATE* function and the result is obtained as follows:

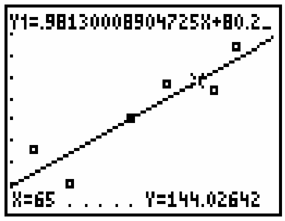


Figure 5. Predicted values

As can be seen from the screen, a 65-year old person is predicted to have a blood pressure of 144.03.

The graphic calculator with its accessibility, portability, cost-effectiveness, powerful built-in functions, and varying CAS capabilities is increasingly seen as a significant tool for the integration of technology in mathematics. The graphic calculator also makes available a wide range of techniques to students to solve problems. It is clear that the graphic calculators do many of the menial tasks for the student; this frees students to work with mathematics at a higher cognitive level. It is very paradoxical that schools are encouraged to use IT for teaching and are provided with expensive and powerful computers and software but the use of a relatively inexpensive tool like graphic calculators is not capitalized on.

Since 1998, the School of Mathematical Sciences at USM has incorporated the use of graphic calculators to illustrate and illuminate several mathematical concepts in a software laboratory course. Details on its use can be found in Ali *et al*. (2000, 1998). The School, in collaboration with colleagues from the School of Educational Studies, has taken a new initiative in the use of graphic calculators through teaching a calculator based laboratory course beginning the 2001/2002 academic year.

3. Course features

The Special Topic course in graphic calculators seeks to explore the impact of such instructional devices and the perspectives they provide. The course is developed for pre-service teachers and students in mathematics. The course objectives are:

- To acquaint students with the CAS calculators and its capabilities.
- To understand the relevance of calculator technology in the teaching and learning of mathematics and sciences.
- To familiarize students with the issues involved in the use of calculator technology in the classroom.
- To model the effective integration of technology into the mathematics curriculum.
- To teach the development of data rich technology explorations that is designed around the capabilities of the calculators.

The course content includes topics from calculus, linear algebra, differential equations, and statistics so a prerequisite is to have completed a first course in these subjects. The TI-92Plus graphic calculator was used throughout the course for calculus, linear algebra, and differential equations, while the TI-83Plus was used for statistics. Students were not required to purchase graphing calculators; each student had a calculator checked out for the duration of the course. In addition, the CBL units, which are data collection units linking to the TI-83s, were incorporated to collect real data in a hands-on environment. There were 24 class meetings of about two hours. The primary teaching mode was alternately an interactive lecture mode and in-class exploration activities. Class activities were supported with laboratory assignments that the students completed and turned in for assessment. The course was team-taught by faculty members from the School of Mathematical Sciences in collaboration with colleagues from the School of Educational Studies.

4. Alternative teaching strategies and content

A graphic calculator is a powerful tool that can carry out complicated mathematical tasks, thus allowing students to spend more time on the understanding of concepts. When used effectively, it becomes a tool to help students actively construct their own knowledge bases and skill sets.

A report by the Panel on Educational Technology (1997) outlines the following "constructivist" paradigm on the potential of technology to support certain fundamental changes in the pedagogic models underlying our traditional approach to the educational enterprise:

- Greater attention is given to the acquisition of higher-order thinking and problem-solving skills, with less emphasis on the assimilation of a large body of isolated facts.
- Basic skills are learned not in isolation, but in the course of undertaking (often on a collaborative basis) higher-level "real-world" tasks whose execution requires the integration of a number of such skills.
- Information resources are made available to be accessed by the student at that point in time when they actually become useful in executing the particular task at hand.
- Fewer topics may be covered than is the case within the typical traditional curriculum, but these topics are often explored in greater depth.
- The student assumes a central role as the active architect of his or her own knowledge and skills, rather than passively absorbing information proffered by the teacher.

Many of the strategies have been incorporated into the Special Topic course on hand-held technology at USM. In what follows, we will elucidate how several of the indicated outcomes of the above strategies have come to fruition when the course was implemented.

There are three basic components to consider in the development of new lesson plans: statements of learning objectives, implementation, and evaluation or expected results. Learning objectives are classroom expectations of behavioral change at the end of the class or series of lectures. The objective statements normally make use of quantifiable verbs as the criteria for evaluation purposes. For example, in an exploration wherein the mean value theorem and derivative test were investigated the learning objectives are: "At the end of the exploration the students will be able to: (a) illustrate the mean value theorem geometrically, (b) describe the relationship between critical points, a function, and its first derivative, (c) describe the relationship between inflection points, a function, and its second derivative, (d) apply the appropriate first or second derivative test from the graphs of a function and its derivatives, and (e) apply the concept of derivative in real life situations." Instead, if the statement is written in the following form, "At the end of the lesson students will be able to *understand* the concept of derivative", then it is not acceptable because 'to understand' is not quantifiable.

The next step is the implementation of learning objectives, which relates to pedagogy and student activities. An instructor must determine appropriate teaching modes such as "a lecture method" or "an inquiry method", relevant teaching aids, and associated student activities. The role of a graphic calculator should match the objectives of a teaching lesson. For example, suppose an objective of a statistics class is "at the end of several teaching lessons students should be able to interpret statistical data by using concepts of variance and standard deviation." For a class without a graphic calculator, the lessons could be taught by asking students to manually compute standard deviation based on a simple data set and the instructor to explain its meaning to the students. The objectives could therefore be achieved without the assistance of a graphic calculator or other statistical tools. Instead, for a class with graphic calculators, the instructor has the option of asking the students to use the tool to do the computations based on a large and real data set. In addition, the interpretation of answers within context will be more meaningful since the graphic calculator could generate varied patterns or results from manipulation of input data.

Our course is developed around the capabilities of the technology to enhance the understanding and learning of mathematical concepts and theories through scientific visualization and laboratory type explorations. Thus, particular attention was given to exploring the potential role of graphic calculators in achieving the learning outcomes through the use of innovative pedagogic methods based on a more active, student-centered approach to learning that emphasizes the development of higher-order reasoning and problem-solving skills. The graphic calculator allows student learning to occur at a higher cognitive level and serves to facilitate inquiries, explorations, and problem-solving activities. It is used as

- a tool for the symbolic manipulation or graphical display of mathematical functions and equations,
- a facility for the collection, examination and analysis of data,
- a tool to foster collaborative learning and teach students to work as a team,

- º a flexible laboratory instrument supporting the collection of scientific data using various physical sensors that allow for the immediate and flexible manipulation of this data,
- º a tool to aid in solving realistic problems that enables the student to concentrate on problem aspects and interpretation rather than computational aspects, and
- º a tool to discover, visualize, or investigate mathematical theories.

Examples of exploration activities in calculus and differential equations may be found in Appendices A and C respectively.

In Ali *et al.* (2001), a discussion of the pedagogical issues and theories underlying the development of materials and the implementation of teaching strategies for the graphic calculator course is presented. Additional references to many articles that address the integration of the graphic calculators into the classroom can be found in Laughbaum (2000).

5. Assessment

A final component to consider in the development of new lesson plans is assessment. In particular, if a student learns differently from a changed pedagogy, then assessment must be done differently.

Perhaps an important assessment issue in a CAS environment is validity of test items. Validity simply implies that a test, or task in the case of a project or a class assignment, should measure what it is supposed to measure. Basically validity is about evidences generated by test scores that support inferences. There are several components of evidence available to support inferences. One such component is *content*, which relates to representativeness of test items in terms of topics to be tested as well as cognitive levels. Another component is *substantiveness*, which concerns with the alignment of the test towards stated objectives.

The leading criterion that is normally employed in matching test items with instructional contents is cognitive levels of learning objectives. One of the most widely used taxonomies of cognitive levels of learning objectives is Bloom's Taxonomy. It contains hierarchical levels of the learning objectives namely: knowledge level, comprehension level, application, analysis level, synthesis level, and evaluation level. In selecting appropriate test items to ensure a valid test, all aspects of a test development plan have to be considered, which include determining cognitive levels of learning objectives and choice on types of items.

For an undergraduate level course, the emphasis should be of higher thinking order items. Knowledge and comprehension items should be included only as initial or starting items preceding the core items. The characteristics of a core item should reflect on the objectives of the lesson and also on how the related topics were taught. The core item should ask the students to use their graphic calculator to *apply* the concepts in a given problem or situation, and to *analyze* new data sets.

After determining the cognitive levels, an instructor then decides on the types of item: multiple-choice, open-ended, or short answers. Every type has its own advantages and limitations. Ideally, an instructor should prepare a test blueprint in the form of a specification table. This table has two dimensions: topics taught and level of thinking skills involved, and is intended to secure the validity of the two evidence components of content and substantiveness.

The Appendices A and B of this paper contain a TI-92Plus exploration on limits and limiting behavior and an associated specification table for the assessment of the exploration, respectively. The activity is designed to explore the concept of limits from calculus and contains real data on fish growth. The exploration was developed around the capabilities of the calculator to help students visualize concepts of limits and to provide visual and numerical evidence of results. The example and problem using real data allow students the opportunity to attach meaning to limits in the world around them. The specification table identifies objectives for each learning level and possible test items that could effectively measure the achievement of the objectives.

In our course, laboratory exercises were collected and assessed; students also had the opportunity to make corrections to their laboratory assignments. In-class examinations were given at the end of each topic that comprised problems requiring them to apply skills learned in completing their laboratory explorations, and their ability to make observations and interpretations from fresh data sets.

6. Student projects

The course culminates with a group project designed in part to facilitate student-initiated inquiries. For their terminal course project, students worked in groups to locate a fresh data set and to develop their own graphics calculator laboratory exploration. It is anticipated that some of the explorations developed by the students will be used in the course in the future. Performance expectations for the terminal projects include the cultivation of higher-order cognitive and problem-solving skills. This is the aspect of the course implementation that most strongly addresses the student's need to work at higher cognitive learning levels.

The projects are designed to foster students' knowledge and critical understanding of principles in mathematics and statistics. They are expected to apply the methods and techniques that they have learned to consolidate those underlying concepts and principles. They are also expected to demonstrate the ability to deploy appropriate approaches to solving problems, and to make use of scholarly sources as and when needed. Such group activities offer students of varying ability levels, and having different interests and prior experience, the opportunity to teach each other.

7. Survey summary

To monitor the impact of graphic calculators in the course, a topic-specific semi-structured survey was prepared and implemented upon completion of each component, i.e., calculus, linear algebra, differential equations, and statistics. This survey requested information on students' perception of their understanding and impression of each topic before taking the course, and also sought their views on the educational value of integrating the graphic calculator into the subject. The students in the class were pre-service teachers and mathematics students but the surveys did not make any distinction between these two groups. The tables below report the findings from student responses to the following items (for the other surveys, verbiage specific to calculus was changed to reflect the appropriate topics):

1. Before taking this course, did you feel that you had a good understanding of the theories and concepts of calculus?
2. Before taking this course, did you feel that calculus was (a) applicable in the real world, (b) interesting, or (c) could be fun to learn?
3. After attending this course for the last 4 weeks, your understanding or appreciation of calculus has changed.
4. The graphic calculator has enhanced your understanding of calculus.
5. Your experience working with the graphic calculator during the past 4 weeks has given you an insight into the value or limitations of integrating technology into the classroom.

Table 2. Percentage respondents to calculus survey
Number of respondents: 21 out of 25 (84%).

Item	1 (%)	2a (%)	2b (%)	2c (%)	3 (%)	4 (%)	5 (%)
SA / Agree	28.6	52.4	61.9	30.8	71.4	95.2	95.2
Not Sure	0.0	4.8	0.0	0.0	14.3	0.0	4.8
SD / Disagree	71.4	42.9	38.1	69.2	14.3	4.8	0.0

SA – Strongly Agree, SD – Strongly Disagree

The results suggest that 71.4% of the students had broadened and deepened their understanding and appreciation of calculus, and 95.2% felt that the graphic calculator has enhanced this understanding. Additionally, students were asked: *Write any new experience or insight that you have gained after attending this course for the last 4 weeks.* We may infer from their comments the following prevailing views:

º Students, who previously had seen calculus as not applicable or difficult to comprehend, now find calculus interesting, illuminating and exciting. The interaction and sharing of ideas creates an enjoyable environment that is more conducive to learning.

º The graphical and tabular approaches to the investigation of topics in calculus yield very encouraging results. Through this methodology, students can now visualize and appreciate important concepts in calculus, such as the limiting behavior of functions, the relationship between a function and its first and second derivatives, and finally the geometric interpretation of the mean value theorem.

- The use of real data promotes an appreciation of the applicability of calculus.
- The CAS graphic calculator reduces time spent on computational and manipulative procedures.
- The first course in calculus should have utilized graphic calculators.

Table 3. Percentage respondents to linear algebra survey
Number of respondents: 23 out of 25 (92%)

Item	1 (%)	2a (%)	2b (%)	2c (%)	3 (%)	4 (%)	5 (%)
SA / Agree	65.2	43.5	65.2	82.6	95.7	91.3	82.6
Not Sure	17.4	21.7	17.4	4.3	0.0	0.0	8.7
SD / Disagree	17.4	34.8	17.4	13.0	4.3	8.7	8.7

The results show that 95.7% of the students agreed that they better understood linear algebra, and 91.3% felt that the graphic calculator has enhanced this understanding. Other prevailing views are:

- Students were amazed at how the graphic calculator can speed up their calculation in basic linear algebra problems such as finding the row-reduced form of a matrix and using the graph to get its eigenvalues.
- The exploration on secret coding of messages with matrix multiplication was a good application of linear algebra in information technology and that the exercise would have been difficult without the aid of the graphic calculator.
- The introduction of least squares method with fresh data provides new experience in using linear algebra to create models for real-life experiments.
- Having the graphic calculator to help in balancing chemical equations improved their understanding in this topic of chemistry, which they have learnt previously.

Table 4. Percentage respondents to differential equations survey
Number of respondents: 22 out of 25 (88%)

Item	1 (%)	2a (%)	2b (%)	2c (%)	3 (%)	4 (%)	5 (%)
SA / Agree	45.5	54.5	0.5	54.5	90.9	81.8	81.8
Not Sure	18.2	13.6	27.3	13.6	4.55	9.09	13.6
SD / Disagree	36.4	31.8	22.7	31.8	4.55	9.09	4.55

The results suggest that 90.9% of the students had broadened and deepened their understanding and appreciation of differential equations, and 81.8% felt that the graphic calculator has enhanced this understanding. Other prevailing views are:

- The students felt that the graphic calculators help them develop a better understanding of differential equations and their applicability.
- Students, who previously found the Existence and Uniqueness Theorem vague, now appreciate it better through the use of the slope field facility. Students can plot a family of solutions of a differential equation easily which provides visual representation that facilitates a deeper understanding of the theorem.
- Students do not have to engage in a great deal of tedious calculations and computations to obtain solutions of a differential equation. The time saved can be used to engage in learning activities to develop better understanding of the concepts of differential equations.
- The activities that involve discrete data of real life problems provide useful and exciting experiences in enhancing mathematics learning.
- Students find that obtaining real data from the Internet and modeling the acquired data is an enjoyable and exciting learning experience.

Table 5. Percentage of respondents to statistics survey
Number of respondents: 21 out of 25 (84%)

Item	1 (%)	2a (%)	2b (%)	2c (%)	3 (%)	4 (%)	5 (%)
SA / Agree	42.8	57.15	57.1	52.4	80.9	66.6	85.7
Not Sure	52.4	33.33	38.1	38.1	4.8	28.6	14.3
SD / Disagree	4.8	9.52	4.8	9.5	14.3	4.8	0.0

The results show that 80.9% of the students agreed that their understanding of statistics had somewhat increased, and 66.6% indicated an enhancement in understanding. Other prevailing views are:

- The use of graphic calculators saves them the trouble of memorizing formulae, speeds up calculations and graphing of data sets.
- Graphical displays, box plots and histograms in particular, are helpful in understanding the idea of the distribution of a data set; its central tendency, spread and shape.
- Statistical testing and the construction of confidence interval become easier. Nevertheless, they still need to know the appropriate test to use.

The surveys also noted several concerns and reservations in the use of graphic calculators:

- Students' over dependence on the calculator in solving problems.
- The cost of the calculator may present an equity issue.

- º Students need to be equipped with theoretical and conceptual understanding of the topics before the use of graphic calculators.
- º The graphics calculator takes a relatively long time to plot slope fields of differential equations.

8. Concluding remarks

We are very encouraged from the experience in running this course. While it is clear that there is a need to continue investigating the impact of the graphic calculator as an instructional technology, the survey suggests a positive student attitude and interest in the graphic calculator. The students' performance exceeded our expectations; the underlying goals of the course were attained early in the semester.

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Appendix A

Laboratory Explorations in Calculus with the TI-92Plus
Limits and Limiting Behavior

Learning objectives

- To visualize the nature of limits from graphical and tabular features.
- To recognize points where functional values or limits are indeterminate.
- To identify the limiting behavior of functions at the point of infinity.
- To understand the physical interpretation of limits for real data.

Example 1. The function $f(x) = \frac{\sin(x)}{x}$ is not defined at $x = 0$. However, the limit of f as x approaches 0 does exist. To see this phenomenon graphically, f is entered as a function in the TI-92Plus as follows:

> press ∞ #, scroll to an unused function and type $\sin(x)/x$, press ∞ % ∞ ∃ and choose the xmin – xres options (eg., xmin = –20, xmax = 20, xscl = 4, ymin = –0.5, ymax = 1.5, yscl = 0.2, xres = 2.) , press ∞ % □ (*trace*).

From the graph it seems apparent that $f(0) = 1$, thus that $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$. However, $f(x)$ is not defined at $x = 0$. Thus one should study the behavior of $f(x)$ close to $x = 0$ from the graph. To do so, zoom in on the graph at $x = 0$ as follows:

> press □ and use the arrow pad (A or B) to move the cursor along the curve near to $x = 0$, press □ ♥ ÷ ÷ ∞ ∃ and change ymin to –0.5, press ∞ % □ and move the cursor so that $x \approx 0$.

Now trace on both sides of zero and observe the *y*-values for positive and negative values of *x* near zero. It will help to zoom a couple of times and to change the display style from line to squares (or dots) to better visualize the limiting process. The following sequence resets the style and zooms twice.

> press ∞ # and highlight the function $\sin(x)/x$, press □ ♠ ∞ % □ and move the cursor *x*-value near 0 , press □ ♥ ÷ □ □ ♥ ÷ □

Have the cursor approach 0 from the left and from the right and note the values of y as the *x*-values approach 0 from either side. Note that a *y*-value is not displayed for $x = 0$. Why not? The calculator knows. To see this fact, view the *TABLE* of values the calculator uses for display purposes, enter the following sequence of commands:

> press ∞ ϶ and scroll down to $x = 0$ (note the y -value), press □, type 0.1 in the tblStart box, 0.01 in the (triangle)tbl box, press ÷ ÷

Scroll to $x = 0$ and observe the values of y as x approaches 0. Also, scroll to the other side of $x = 0$ and observe the values of y as x approaches 0 from the other side. Observe the table value of y (or non value) associated with $x = 0$. It has been demonstrated, graphically, that $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$. The table of values illustrates the result computationally. It is very important to note that the above efforts are tantamount to an illustration of a result, not a proof. The graphical device gives us confidence that the answer is correct, however, the result must be proven rigorously before we may infer it to be true. Note also that the limit exists at $x = 0$ even though f is not defined at $x = 0$.

Example 2. Consider

$$f(x) = \begin{cases} -x, & x < 1.5 \\ 5\cos(x), & x \ge 1.5 \end{cases}$$

The functional value of $f(x)$ does exist at $x = 1.5$, and is given by $f(1.5) = 5\cos(1.5) \approx .353686$. To visualize the limit process for f, enter and plot the function as follows:

> press ∞ # and scroll to a new function, type *when* $(x < 1.5,\ -x,\ 5*\cos(x))$, press ∞ ∃, select $\text{xmin} = -3$, $\text{xmax} = 7$, $\text{ymin} = -6$, $\text{ymax} = 6$, press ∞ %

From the plot a jump discontinuity can be seen at $x = 1.5$, though the calculator draws a line connecting the two curves. To clearly see the jump, reset the plot style to squares: press ∞ # and highlight $f(x)$, press □ ♠ ∞ %. The jump should be quite clear now. Next, zoom in on the plot around $x = 1.5$ to get a feel for the value of the limit of $f(x)$ at $x = 1.5$.

press and move the cursor so x is close to 1.5, press ∞ ♥ ÷ ∞ ∃ and reset the window to view both plots together (e.g., $\text{ymin} = -1.5$), press and have the cursor approach $x = 1.5$ from the left and from the right.

As the cursor approaches the break from the right (the cosine curve) the y values (for this window setting), approach $y = .322246$. As the cursor approaches the break from the left (the linear curve) the y values approach $y = -1.48529$. An estimate of the limiting values at $x = 1.5$, to the nearest hundredth, may be achieved with an application of the TABLE option as follows:

press ∞ ∋ and fill in values tblStart = 1.5 and tbl = 0.01 (type 1.5 press Δ, type .01 press ÷ ÷), now use the up X and down Δ arrows to scroll before and after $x = 1.5$ and note the values of y as x approaches 1.5

Note that at $x = 1.5$, y is defined as $.35369\,(\approx 5\cos(1.5))$; as x approaches 1.5 from the right, y approaches .35369; as x approaches 1.5 from the left, y approaches $-1.49 \approx -1.5$ (to the nearest hundredth). For better approximations the table feature can be used with a more precise *tbl* setting. Compiling the above observations gives

$$\lim_{x\to 1.5-} f(x) = -1.5 \neq .35369 = \lim_{x\to 1.5+} f(x)$$

Thus we conclude that the limit does not exist since the left and right limits are not equal. Again, rigor is needed to validate this conjecture.

Example 3. The following data was collected for the growth rate of the bullhead from Bere Stream. The problem is to plot the data, make observations on growth trends of the fish, and identify any long-term growth trends over time. The data in the table represents monthly average lengths for a period of two years.

Mean lengths (cm) of bullheads in Bere Sream

J	J	A	S	O	N	D	J	F	M	A	M
1.5	2.8	3.4	3.9	4.2	4.4	4.4	4.5	4.5	4.5	4.6	4.9
5.2	5.9	6.4	6.5	6.7	7.8	7.0	7.2	7.2	7.2	7.3	7.5

Data provided by: H.A. Al-Rabai'ah & H.L. Koh, School of Mathematical Sciences, Universiti Sains Malaysia, 11800 Penang, MALAYSIA

The *Data/Matrix Editor* is used in the TI-92Plus to work with data sets, the user can enter data, plot data, and perform statistical analyses. The Data editor may be accessed through the [APPS] key as follows:

> press [APPS], scroll to *Data/Matrix Editor*, press ⊙, scroll to New and press [ENTER] (or press [APPS] [6] [3]), select *Type: Data* and press down arrow (if you press [ENTER] here after you have made your selection you will have to start over), select *Folder: Main* and press down arrow, type *fish1* in the *Variable:* box (you may choose any name you wish), press [ENTER] [ENTER]

Now enter the data. Enter months in column 1 (C1), month 1 is June, and fish length in column 2 (C2).

> R1c1 should be displayed at the bottom of the screen, type 1, press [ENTER], type 2, press [ENTER], type 3, and so on until the months have been entered through 24, scroll up and over to the first cell in C2, r1c2= should be displayed at the bottom of the screen, type 0.95, press [ENTER], type 0.94, press [ENTER], type 1.68, press [ENTER], and so on until you have entered the fish length for the 24th month

The data may now be plotted. Data plots must be done from the *Data/Matrix Editor* screen.

> press [F2] and scroll to an unused Plot number, press [F1], select *Plot Type...Scatter* and press ⊙, select *Mark...Box*, press ⊙, type *C1* in the *x*-box, press ⊙, type *C2* in the *y* -box, press ⊙, select *NO* for Use ..., press [ENTER] [♦] [GRAPH]

A data plot should be visible, if not, reset the plotting WINDOW appropriately. Some observations:

- The fish length is increasing as time (months) increases.
- The fish grows faster when it is young.
- After a quick growth spurt, the fish stops growing for a time and has another period of fast growth before beginning to slow again.
- The rate of fish growth becomes slower as time increases.
- The fish length eventually appears to be approaching a limiting value.

- The fish is small, 1.5 cm, in month one.
- Within 5 months the expected fish length increases rapidly to over 4 cm.
- During months 6-11 the fish grows only another 0.5 cm.
- During months 12-15 the fish growth is fast again, another 2 cm.
- During the last four months the fish grow is only 0.3 cm.
- Fish length seems to be approaching a long time limiting size of about 8 cm.
- The data has an apparent bad value of 7.8 at month 18 (November year two). Fish growth is slow and fish will not shrink, so the bad data value should be taken into consideration, possibly eliminated, in further analysis.

The observations above imply that the long time behavior of a fish growth function, the behavior at the point at infinity, is likely to approach 8, that is, one would not expect the average length of a bullhead in Bere stream to exceed 8 cm. In addition, the slower growth rates occur during months of cooler or winter climates in the northern hemisphere. The data might suggest that this fish will grow slower in cooler temperatures. Such conjectures could be studied through further research and experimentation. The next step in our investigation of the fish problem would be to try to determine a function that models the fish growth and closely fits the data. This is the job of the mathematicians and biologists that do ecological modeling. We will tackle this step in a later exploration.

Appendix B

A Specification Table for Student Assessment

Topics/ Objectives	Knowledge	Comprehension	Application	Analysis
Limits	(a) Ability to recall basic concepts and theorems on limits.	(c) Ability to visualize the nature of limits by using GC at specific points, $\pm\infty$, and indeterminate forms.	(d) Ability to make observations and interpretations from real data using GC.	(e) Ability to manage a fresh new data set.
	Test Items	***Test Items***	***Test Items***	***Test Items***
	Open-ended or MCQ on limit theorems, such as sum of limits, sandwich theorem, and L'Hopital rule.	Students' facility in using GC features, such as zoom, table, and plot features, in determining existence of limits.	Management of data; plotting; observation of specific math features of the graph; physical interpretations.	Take home assignment.
	(b) Ability to compute limits.			(f) Ability to develop a mathematical model from a data set.
	Test Items			***Test Items***
	Finding limits at specific points or at ∞.			Group presentation.

Appendix C

Laboratory Explorations in Differential Equations with the TI-92Plus: Population Models From Differential Data

Learning objectives

- To apply the concept of derivative in real life situation.
- To determine mathematical models from population differential data.
- To use model equations to make population predictions.
- To study population models.

The following data was collected for the population growth rate in Malaysia. We wish to use discrete differential data to determine equations that model population growth. The model equations will be used to estimate (and/or verify) growth trends of the Malaysian people. The process is to find one or more equations that model the rate of population growth, that is, equations that model the 1st derivative of the population function. This amounts to the determination of a differential equation that models the change in population. The differential equation will then be solved to determine an equation that models the population. Whereas the differential equation models population rate, the solution of the DE will model population. In this example, four different DE models will be determined to illustrate the paucity of resolutions for typical modeling problems. We will work with the relative rates of change in population over time.

The data for the population in Malaysia from 1951-2001 is given below:

Table 1. Population of Malaysia from 1951-2001

1950	6,433,799	1966	9,899,803
1951	6,581,839	1967	10,154,878
1952	6,748,378	1968	10,409,339
1953	6,928,942	1969	10,662,303
1954	7,117,564	1970	10,910,216
1955	7,311,720	1971	11,171,333
1956	7,519,663	1972	11,441,462
1957	7,739,235	1973	11,711,866
1958	7,965,930	1974	11,986,260
1959	8,195,711	1975	12,267,303
1960	8,428,493	1976	12,553,963
1961	8,663,401	1977	12,845,381
1962	8,906,385	1978	13,138,530
1963	9,148,451	1979	13,443,844
1964	9,397,464	1980	13,764,352
1965	9,647,654	1981	14,096,663

1982	14,441,916	1992	18,319,502
1983	14,793,099	1993	18,747,901
1984	15,157,328	1994	19,180,324
1985	15,545,028	1995	19,611,116
1986	15,941,178	1996	20,044,560
1987	16,331,785	1997	20,476,091
1988	16,729,187	1998	20,911,977
1989	17,117,834	1999	21,354,459
1990	17,503,607	2000	21,793,293
1991	17,906,485	2001	22,229,040

Data obtained from the website http://www.census.gov/ftp/pub/ipc/www/idbnew.html.

Enter/Compute the discrete differentials and plot

The Data/Matrix editor is used in the TI-92Plus to work with data sets. This function allows the user to enter data, plot data, and perform statistical analyses on the data. Enter the data into a new table in the data editor and produce the scatter plot of the population

Press O and then select 6:Data/Matrix Editor. Select 3: New. Choose Type: Matrix, Folder:main, and Variable: people. Enter. Press □ and scroll to a Plot number that is unused, press □, select *Plot Type…Scatter, Mark…Plus,* type *C1* in the x-box, press Δ, type *C2* in the y-box, press ÷, press ∞ GRAPH (select an appropriate graphing window): try $xmin = 0$, $xmax = 52$, $ymin = 6000000$, $ymax = 23000000$, $yscl = 5000$, $xres = 2$ (Figure 1).

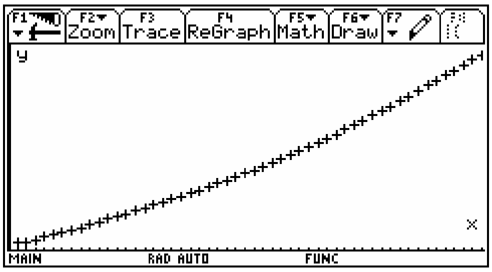

Figure 1. Scatter plot of the population data

Note: We have learnt how to enter and plot data in the Lab DE explorations. In order to save time and effort, only one student needs to enter the population data into his/her graphics calculator. The other students can transmit the data input by linking his/her graphics calculator to this particular student using a TI-92Plus cable.

Approximations of the 1st derivatives of $f(x)$, *discrete* derivatives $\tilde{f}'(x)$, are determined as follows:

$$\tilde{f}'(x_n) = \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}.$$

The discrete derivatives will be computed in the table environment by shifting down the x and y value columns c1 and c2, storing the results in c3 and c4, respectively, subtracting associated columns and dividing. So, c5 will contain $\tilde{f}'(x)$, eg., $\text{r8c5} = (\text{r8c2} - \text{r8c4})/(\text{r8c1} - \text{r8c3})$.

> In the TABLE window: highlight c3 (the label for column three), press ÷ (the entry line should display c3=, with the cursor active), type shift(c1), press ÷. In c3, element r1c3 should be *undef* and other elements should be those of c1 shifted down one row.

The following will store the shifted c2 in c4 and $\tilde{f}'(x)$ in c5.

> Highlight c4 (the label for column four), press ÷ (the entry line should display c4= with the cursor active), type shift(c2), press ÷, highlight c5, press ÷, type shift ((c2-c4)/(c1-c3), 1), press ÷.

Now plot the discrete 1st derivative of $f(x)$.

press □ and scroll to a Plot number that is unused, press □, select *Plot Type…Scatter, Mark…Plus,* type *C1* in the x-box, press Δ, type *C5* in the y-box, press ÷, ∞ % (select an appropriate graphing window if necessary: try $\text{xmin} = 0$, $\text{xmax} = 52$, $\text{ymin} = 140000$, $\text{ymax} = 440000$, $\text{yscl} = 5000$) (Figure 2).

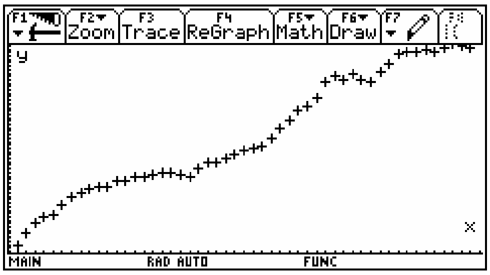

Figure 2. Scatter plot of the 1st derivative of $f(x)$

Determine equations that model the differential data

Linear model

Step 1. *Determine the model equation*

We have the TI-92Plus determine a linear least squares model for the differential people data. This also means that the population function will be a quadratic model equation. Open the data file containing your differential data and complete the following steps to determine the model differential equation.

> *To open the file*: press O, { (Data/Matrix Editor), 2 (Open), select *Type: Data*, press Δ, select *Folder: Main* and press Δ, press B and scroll down to highlight the data base with population difference data, press ÷, ÷.

> *To determine the linear model*: press □, choose *Calculation Type .. LinReg*, press B, ζ, Δ, type c1 (or the column containing x-values, here years), press Δ, type c5 (or the column containing discrete derivative values), choose *Store RegEQ to …. y1(x)*, press B, highlight y1(x), press ÷, ÷

A STAT VARS box (Figure 3) should appear providing values of *a*, the slope, and *b*, the *y*-intercept, of the linear regression equation for the data. After pressing ÷ the equation will be stored in the equation editor as y1(x) (or whatever function you chose). Press ∞ # to see the function. Go to the graph window and you should see the regression line plotted along with your data (Figure 4).

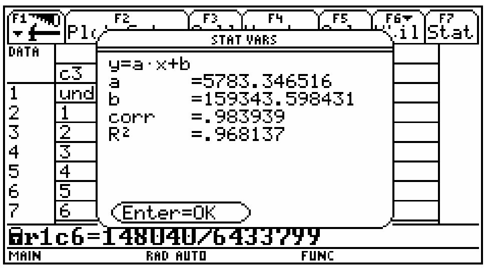

Figure 3. STAT VARS box for the linear model

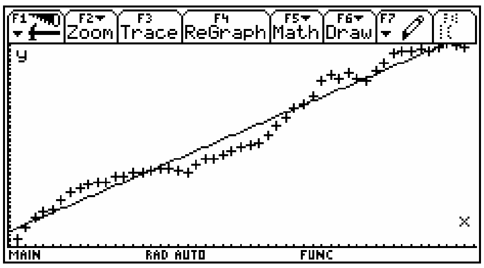

Figure 4. Regression line plotted along with the discrete 1st derivative of $f(x)$

Step 2. *Determine the DE model*

The linear least squares line is a model for the differential data. In order to turn our equation into a proper ordinary differential equation, that can be uniquely solved, an initial condition must be identified. In this instance, the appropriate choice is the first data pair, (1, 6,581,839), that is, in year one (1951), the population of Malaysia is 6,581,839.

$$\frac{df}{dx} = 5783.3465\cdots * x + 159343.5984\cdots 7$$

$$f(1) = 6{,}581{,}839$$

Step 3. *Solve the DE to determine the equation that models the solution*

The TI-92Plus may be used to solve the differential equation as follows. The approach here is to copy the linear equation from the equation editor (instead of typing the whole equation by hand) store the differential equation into the variable *ode*, and solve the problem for the above initial condition.

> Press ∞ #, highlight y1(x) (the linear DE model), press □, ζ, ∞ ∀, press M, type *y* 2 ∪, ℑ, □, {, press ♣, type *ode*, ÷, *press* □, © (or type *deSolve(*), type *ode and y(1951)= 6581839,x,y)*, press ÷

The solution to the DE should be visible on the screen. The solution is:

$$y_2(x) = 2891.67x^2 + 159344x + 6.4196E6$$

Step 4. *Plot and use the solution to make estimates*

Plot the model equation along with the original population data so we can compare to see how well our modeling approach is doing. After checking out the plot, the equation will be used to make some predictions of the population. To plot:

> Press X to highlight the solution equation, □, ζ, ∞ #, highlight y2, ÷, □, {, press 2 A and delete *y*=, press ÷, O, {, ♦, □, select the c1 vs. c2 plot or define a new plot if necessary, select an appropriate viewing window and plot (Figure 5).

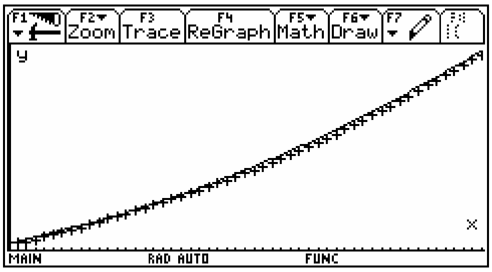

Figure 5. Plot of the solution equation along with the original population data

A logistic model

Step 1. *Determine the model*

$$\frac{dP}{dt} = kP(M - P).$$

The logistic equation to study population growth, is the equation where k and M are constants, subject to the condition $P(0) = P_{0.}$ This equation can be solved by separation of variables to obtain the solution

$$P(t) = \frac{MP_0}{P_0 + (M - P_0)e^{-kMt}}.$$

Notice that $\lim_{t \to \infty} P(t) = M$.

Step 2. *Determine the DE model*

The problem now is to fit a logistic model to the given data. Thus, we want to determine the numerical constants c and d so that the solution $P(t)$ of the initial value problem

$$\frac{dP}{dt} = cP + dP^2, \quad P(0) = P_0 \tag{1}$$

approximates the population data.

Rewriting Eq. (1), in the form

$$\frac{1}{P}\frac{dP}{dt} = c + dP, \tag{2}$$

we can see that the RHS of Eq. (2) defines a linear function with *y*-intercept *c* and slope *d*. Hence, we can plot the points $(P_i, \frac{P_i'}{P_i})$ for $i = 1,2,\cdots,n$, and apply a linear model to approximate these points. Then, determine its y-intercept and slope.

> Highlight c6 (the label for column six), press ÷ (the entry line should display c6= with the cursor active), type c5/c2, press ÷.

Hence, we can plot the points $(P_i, \frac{P_i'}{P_i})$ for $i = 1,2,\ldots,n$, and apply a linear model to approximate these points.

To obtain the points $(P_i, \frac{P_i'}{P_i})$

> Press □ and scroll to a Plot number that is unused, press □, select Plot Type…Scatter, Mark…Plus, type C2 in the *x*-box, press Δ, type C6 in the *y*-box, press ÷

To determine the linear model:

> press F5, choose Calculation Type…LinReg (press B, 5), ÷, press Δ, ÷ c2 (or the column population), press Δ, enter c6 (or the column containing $\frac{P_i'}{P_i}$), choose Store RegEQ to y7(x) press B, highlight y7(x), press ÷, press ÷

A STAT VARS box (Figure 6) should appear giving you the values of the slope a and y-intercept b of the linear regression equation for the data. After pressing ÷ the equation will be stored in the equation editor as y7(x) (or whatever function you choose). Press ∞ Y= to see the function. Go to the graph window and you should see the regression line plotted along with your data.

Solve Eq. (1) with the resulting numerical parameters c and d (b and a from the STATS VAR box respectively.

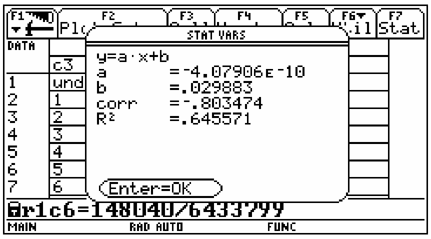

Figure 6. STAT VARS box giving the values of slope a and y-intercept b

The DE modeling our data, with the same initial condition is

$$\frac{dP}{dx} = 0.02988\cdots 1P + -4.07906\cdots 4E - 10P^2$$

$$P(1) = 6{,}581{,}839$$

Step 3. *Solve the DE to determine the equation that models the solution*

The TI-92Plus may be used to solve the differential equation as follows. Store the differential equation into the variable ode, and solve the initial value problem.

> Press ∞ Y=, highlight y7(x) (the logistic DE model), press □, ζ, ∞ HOME, press M, type y 2 ∪, ℑ, □, {, change x to y^2 and insert a y after the y-intercept (c-value). Press ♣, type ***ode***, press ÷ press □, © (or type *deSolve*( ), type ***ode and y(1)=6581839,x,y)***, press ÷

The solution to the DE should be visible on the screen. The solution is:

$$y_8 = \frac{7.32584E7(1.03033)^x}{(1.03033)^x + 10.4377}$$

Step 4. *Plot and use the solution to make estimates*

Plot the model equation along with the original population data and compare the model to the real data, that is, investigate the validity of the modeling approach. The model equation may then be used to make some predictions of future population. To plot:

Press up arrow to highlight the solution equation, □, ζ, ∞ Y=, highlight y6, ÷, □, {, press 2 Aand delete *y*=, ÷, press ÷, O, {, ♦, □, select the c1 vs. c2 plot or define a new plot if necessary, select an appropriate viewing window and plot (Figure 7).

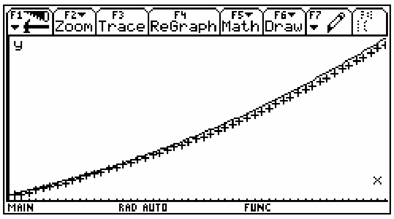

Figure 7. Plot of the solution equation along with the original population data

(**Authors' comment:** The details of the determination of the remaining two models have been suppressed.)

Comparison of DE models

A first step in comparing the equations modeling population from the different DE models is to evaluate each of the models at several values of *x*, the number of months of population. The following table indicates some of these values.

Model	$x = 0$	$x = 24$	$x = 48$	$x = 200$
Quadratic y_2 (linear DE)	6,419,600	11,909,500	20,730,500	51,270,700
5th order, y_4 (quartic DE)	6,438,570	11,929,300	20,784,400	–93,316,100
Exponential, y_E (exp model)	6,400,430	11,889,500	20,626,900	61,471,000
Logistic, y_L	6,405,010	12,019,600	21,009,300	48,013,100

The US Census Bureau predicts a Malaysian population of 43,122,397 in the year 2050. What do your models predict?

From the table of values and graphs of the solution models, several observations may be made. List down the observations.

- Each of the models do a reasonable job in predicting the population in the domain of the function, that is, from $x = 1$ to $x = 52$.
- Each of the models fit the data better, within the data domain for earlier x, where the growth is slow and steady.
- The 5th order polynomial seems to stray from the data more than the other models at $x = 24$ and $x = 48$.
- The predictions by each model for the population in the year 1949 $(x = 0)$ are not unreasonable when compared to the population for the year 1950 $(x = 1)$ which is 6,433,799.
- The 5th order model fails badly at $x = 100$, it tells us that the population will be –93,316,100 after 100 years (2050) which is unreasonable because population is not supposed to assume a negative value. This is a good illustration of the fact that most model equations do not do a good job in making predictions away from the basic data value domain.
- The logistic model at $x = 100$ (year 2050) is closer to the population predicted by the US Census Bureau, which is 43,122,397 than all the other models.

Tech Note: Table values were computed for each function using the vector format:

In the home window, type *y2({t_1,t_2,t_3,t_4})*, press ÷

The function $y2(x)$ from the function editor is evaluated at each of the t values and the vector of y values is displayed on the home screen.

Exercises

Phase I. Graph the population and differences (discrete derivatives) over a certain period of time interval.

Go to the *population* site. Enter into a table of the TI-92Plus the population. In your write-up of the project, include the country and period for which data was collected. Compute the discrete differences of the data using the spreadsheet/table features of the TI-92Plus. Plot a scatter plot of the difference data on the TI-92Plus.

Access the *population* web site.

http://www.census.gov/ftp/pub/ipc/www/idbnew.html.

Select the country and period.

Phase II. Determine a model equation, DE, for the differences.

Repeat the investigation on the Malaysian population. You may use other regression models available in the TI-92Plus. The logistic model has to be included in your investigation. Choose one of the original data pairs to help identify an initial condition.

Phase III. Solve the DE to determine a modeling equation for the population of the chosen country and period. Solve the differential equation using the TI-92Plus capability. For your model DE, the solution of the DE is an equation that models the original population data. Plot the model function (solution of the DE) and the original data on the same plot. Use your model to make predictions of future population, eg., the year 2010, 2020, 2050 etc. How do your predictions compare with the predictions made by the U.S. Census Bureau?