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A Combinatorial Problem in Infinite Groups

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Abstract. Let w be a word in the free group of rank $n \in IN$ and let $V(w)$ be the variety of groups defined by the law $w = 1$. Define $V(w^*)$ to be the class of all groups G in which for any n infinite subsets $X_1, \cdots, X_n$ there exist $X_i \in X_i, 1 \leq i \leq n$, such that $w(x_1, \cdots, x_n) = 1$. Clearly, $V(w) \cup \mathsf{F} \subseteq V(w^*)$; F being the class of finite groups. In this paper, we investigate some words w and some certain classes P of groups for which the equality $(V(w) \cup \mathsf{F}) \cap \mathsf{P} = V(w^*) \cap \mathsf{P}$ holds.

1. Introduction and results

Let w be a word in the free group of rank $n \in IN$ and let $V(w)$ be the variety of groups defined by the law $w = w(x_1, \cdots, x_n) = 1$. Longobardi *et al.* [29] defined $V(w^*)$ to be the class of all groups G in which for any n infinite subsets $X_1, \cdots, X_n$ there exist $x_i \in X_i$, $1 \leq i \leq n$, such that $w(x_1, \cdots, x_n) = 1$ and raised the question of whether $V(w) \cup \mathsf{F} = V(w^*)$ is true; F being the class of finite groups. There is no example, so far, of an infinite group in $V(w) \setminus V(w^*)$. In fact the origin of this problem is the following observation:

Let G be an infinite group such that in every two infinite subsets of G there exist two commuting elements, then G is abelian. This is an immediate consequence of the answer of B.H. Neumman to a question of P. Erdös; B.H. Neumman proved that an infinite group G is centre-by-finite if and only if every infinite subset of G contains two distinct commuting elements [37]. Since this first paper, problems of a similar nature have been the object of several articles (for example [2], [3], [5], [9], [11], [12], [15], [24], [27], [28], [39]).

As far as we know, the equality $V(w) \cup \mathsf{F} = V(w^*)$ is known for the following words: $w = x^m$, $w = [x_1, \cdots, x_n]$ [29], $w = [x, y]^2$ [26], $w = [x, y, y]$ [41], $w = [x, y, y, y]$ [42], $w = (xy)^{-3} x^3 y^3$ [1], $w = x_1^{\alpha_1} \cdots x_m^{\alpha_m}$ where $\alpha_1, \cdots, \alpha_m$ are non-zero integers [4], $w = (xy)^2 (yx)^{-2}$ or $w = [x^m, y]$ where

$m \in \{3, 6\} \cup \{2^k \mid k \in IN\}$ [6], $w = [x^n, y][x, y^n]^{-1}$ where $n \in \{\pm 2, 3\}$ [43] and $w = [x^m, y^m]$ or $w = (x_1{}^m \cdots x_n{}^m)^2$ where $m \in \{2^k \mid k \in IN\}$ [8].

In [38], P. Puglisi and L.S. Spiezia proved that every infinite locally finite group (or locally soluble group) in $V([x,_k y]^*)$ is a k-Engel group; (recall that $[x,_k y]$ is defined inductively by $[x,_0 y] = x$ and $[x,_k y] = [[x,_{k-1} y], y]$ for $k \in IN$). In [10], C. Delizia proved the equality $V(w) \cup \mathsf{F} = V(w^*)$ on the classes of hyperabelian, locally soluble and locally finite groups where $w = [x_1, \cdots, x_k, x_1]$ and k is an integer greater than 2. Later G. Endimioni generalized these results by proving that every infinite locally finite or locally soluble group in $V(w^*)$ belongs to the variety $V(w)$, where w is a word in a free group such that finitely generated soluble groups in $V(w)$ are nilpotent (see Theorem 3 of [14]) (recall that the variety $V([x_1, \cdots, x_k, x_1])$ $(k > 2)$ is exactly the variety of nilpotent groups of nilpotency class at most k [35] and every finitely generated soluble Engel group is nilpotent [17].)

We say that a group G is locally graded if and only if every finitely generated non-trivial subgroup of G has a non-trivial finite quotient. We proved in Theorem 4 of [3] that an infinite locally graded group in $V([x_1,_k x_2]^*)$ is a k-Engel group. We generalize this result as Theorem A, below. In order to state our first result we need the following definition. Following [20] we say that a group G is restrained if and only if $\langle x \rangle^{\langle y \rangle} = \langle x^{y^i} \mid i \in \mathbf{Z} \rangle$ is finitely generated for all $x, y \in G$. We show by Proposition 1 below, why the following theorem improves the above mentioned results.

Theorem A. *Let w be a word in a free group such that every finitely generated residually finite group in $V(w)$ is polycyclic-by-finite. Then every infinite finitely generated locally graded restrained group in $V(w^*)$ belongs to the variety $V(w)$.*

G. Endimioni proved that every infinite locally nilpotent group in $V(w^*)$ belongs to the variety $V(w)$, where w is a word in a free group (see Theorem 1 of [14]). Note that it is easy to see, any quotient G/N is in $V(w)$ if G is a group in $V(w^*)$ and N is an infinite normal subgroup of G. The following theorem generalizes Theorem 1 of [14].

Theorem B. *Let w be a word in a free group and let P be a class of groups satisfying the following conditions:*

(1) the class P is closed under taking subgroups.
(2) every P-group is soluble.
(3) every infinite finitely generated (P-by-finite)-group in $V(w^*)$ belongs to the variety $V(w)$.

Then every infinite residually [(locally P*)-by-finite] group in* $V(w^*)$ *belongs to* $V(w)$.

For example, the classes of nilpotent groups, polycyclic groups, abelian-by-nilpotent groups and soluble residually finite groups satisfy the assumptions of Theorem B.

Here we also obtain some reductions in investigation of the equality $V(w) \cup \mathsf{F} = V(w^*)$ on certain classes of groups and certain words w. For example

Theorem C. *Let* w *be a non-trivial word in a free group. Then every non-linear simple locally finite group does not belong to the class* $V(w^*)$.

In [14], G. Endimioni proved that if w be a word in a free group such that finitely generated soluble groups in $V(w)$ are polycyclic, then every finitely generated soluble group in $V(w^*)$ belongs to the variety $V(w)$. Before stating our next result, we need a notation (see [16]). Let α be a non-zero element of some field of characteristic p. Denote the group generated by the matrices

$$\left\{\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}, \begin{bmatrix}\alpha & 0\\ 0 & 1\end{bmatrix}\right\} \text{ by } M(\alpha, p).$$

Theorem D. *Let* w *be a word in a free group such that every infinitely presented* $M(\alpha, p) \notin V(w)$ *for all* $p \geq 0$ *or* $C_q wr C_\infty \notin V(w)$ *for all primes* q. *Then every infinite locally soluble group in* $V(w^*)$ *belongs to the variety* $V(w)$.

We note that the group $M(\alpha, p)$ is finitely presented if and only if

(i) $p \neq 0$ and α is algebraic over the prime field, or

(ii) $p = 0$ and at least one of α or α^{-1} is an algebraic integer (see Lemma 11 of [16]).

Theorem D generalizes Theorems 2 and 3 of [14], since we note that if V is a variety of groups in which every finitely generated soluble group in V is polycyclic then V contains no infinitely presented $M(\alpha, p)$ since $M(\alpha, p)$ is finitely generated metabelian; the subgroup $C_q^{C_\infty}$ of $C_q wr C_\infty$ is not finitely generated and, $C_q wr C_\infty$ is not polycyclic for any prime q.

2. Proofs

We start the proof of Theorem A.

Proof of Theorem A. Let G be an infinite finitely generated locally graded restrained group in $V(w^*)$ and let R be the finite residual of G. Then G/R is a finitely generated residually finite group in $V(w^*)$ and so, by Lemma 1 of [14], it belongs to $V(w)$. Thus by hypothesis, G/R is polycyclic-by-finite. Therefore by repeated use of Lemma 3 of [20], R is finitely generated. If R is finite then G is residually finite and so by Lemma 1 of [14], G belongs to the variety $V(w)$. Now suppose, for a contradiction, that R is infinite. By hypothesis, R has a normal proper subgroup of finite index in R, then the finite residual subgroup T of R is proper in R. Therefore R/T is a residullay finite group in $V(w)$ and so G/T is polycylic-by-finite. Thus G/T is residually finite and $R \subseteq T$, a contradiction. This completes the proof.

The following proposition generalizes the result of [7].

Proposition 1. *Finitely generated residually finite groups in a variety V in which every finite group is nilpotent, are nilpotent.*

Proof. We first prove that there exists a positive integer k depending only on the variety V such that for all primes p, $C_p wr C_{p^k} \notin V$. By the Lemma of [13], there exists an integer t depending only on V such that every 2-generated metabelian group in V is nilpotent of class at most t. Now suppose that $C_p wr C_{p^m} \in V$ for some prime p and positive integer m. Since $C_p wr C_{p^m}$ is a 2-generated metabelian group then it is nilpotent of class at most t. But the nilpotency class of $C_p wr C_{p^m}$ is exactly p^m, by a result of Liebeck (see [25] or Theorem 2.5 in page 76 of [36]) and so $p^m \leq t$. Now the same argument as in Theorem 2 of [44] completes the proof.

Theorem A improves Theorem 3 of [14] since by the result of [7], in a variety, all finite groups are nilpotent if and only if all finitely generated soluble groups are nilpotent. Therefore by Proposition 1, every variety in which all finitely generated soluble groups are nilpotent is contained in a variety in which all finitely generated residually finite groups are polycyclic-by-finite.

Corollary 2. *Let w be a word in a free group such that finitely generated soluble groups in $V(w)$ are nilpotent. Then every infinite locally graded restrained group in $V(w^*)$ belongs to the variety $V(w)$.*

Proof. As noticed before, every finitely generated residually finite group in $V(w)$ is polycyclic-by-finite. Let G be an infinite locally graded restrained group in $V(w^*)$ and assume that w is a word in the free group of rank $n \in IN$. Let $x_1, \cdots, x_n \in G$, we must prove that $w(x_1, \cdots, x_n) = 1$. Assume that there exists an infinite finitely generated subgroup H of G which contains $x_1, \cdots, x_n$. Then by Theorem A, $H \in V(w)$. Now, we may assume that every finitely generated subgroup of G containing $x_1, \cdots, x_n$ is finite. Thus there exists an infinite locally finite subgroup L which contains $x_1, \cdots, x_n$ and so by Theorem 3 of [14], L belongs to the variety $V(w)$. This completes the proof.

In the following lemmas we use some notion: we say that a word $w \neq 1$ in a free group is a semigroup word if w is of the form uv^{-1}, where u and v are words in a free semigroup and we say, following [30], that a group G has no free subsemigroups if and only if for every pair (a, b) of elements of G, the subsemigroup generated by a, b has a relation of the form

$$a^{r_1} b^{s_1} \cdots a^{r_j} b^{s_j} = b^{m_1} a^{n_1} \cdots b^{m_k} a^{n_k} \tag{1}$$

where r_i, s_i, m_i and n_i are all non-negative and r_1 and m_1 are positive integers. If (a, b) is a pair of elements in G satsfying a relation of type (1), then we call $j + k$ the width of the relation and the sum $r_1 + \cdots + r_j + n_1 + \cdots + n_k$ the exponent of a (denoted $\exp(a)$) in the relation.

We say that a word w in a free group F generated by $x_1, \cdots, x_n$, is a commutator word whenever w belongs to the derived subgroup of F. In the following we study infinite groups in $V(w^*)$ where w is not a commutator word. We note that if w is not a commutator word then there is a positive integer e depending only on w such that every group in the variety $V(w)$ is of exponent dividing e; for let G be a group in the variety generated by a non-commutator word w, since w is not a commuatator word, for some i the sum of the exponents of x_i in w is non-zero: let this sum be r and let $g \in G$. If we replace x_i by g and x_j by 1 when $j \neq i$, then w assumes the value g^r. Thus g has a finite order r and G is of finite exponent.

Lemma 3. *Let w be a semigroup word in the free group of rank* 2. *Then every group in $V(w^*)$ has no free subsemigroups, and there exist positive integers M and N depending only on w such that for all pairs (a, b) of elements in G there is a relation of the form* (1) *whose width and* $\exp(a)$ *is at most M and N, respectively.*

Proof. Let a, b be in G. If b is of finite order m then $ab^m = b^m a$, $\exp(a) = 2$ and the width is 2. Now, assume that b is of infinite order and consider two sets $X = \{a^{b^n} \mid n \in IN\}$ and $Y = \{b^m \mid m \in IN\}$. If X is finite then the centre of $H = \langle a, b \rangle$ is infinite and so by Lemma 3 of [14], H belongs to the variety $V(w)$. Therefore $w(a,b) = w(b,a) = 1$ and so the pair (a, b) satisfies a relation of the form (1) whose width and $\exp(a)$ is at most M_1 and N_1, respectively, where M_1 and N_1 are positive fixed integers depending only on w. Now we may assume that X is infinite, then by the property $V(w^*)$, there exists a relation of the form

$$\left(a^{b^t}\right)^{r_1} b^{s_1} \cdots \left(a^{b^t}\right)^{r_j} b^{s_j} = b^{m_1}\left(a^{b^t}\right)^{n_1} \cdots b^{m_k}\left(a^{b^t}\right)^{n_k}$$

where r_i, s_i, m_i and n_i are non-negative integers and r_1, m_1 and t are positive integers; also the sum $r_1 + \cdots + r_j + n_1 + \cdots + n_k$ is the same N_1 and $j + k = M_1$. Therefore the pair (a, b) satisfies a relation of the form (1) whose width is at most $M := \max\{2, M_1\}$ and $\exp(a)$ is at most $N := \max\{2, N_1\}$.

Recall that a group G is right orderable if there exists a total order relation $\leq$ on G such that for all a,b,g in G, $a \leq b$ implies $ag \leq bg$, equivalenty, if there exists a subset P in G such that $PP = P$, $P \cup P^{-1} = G$, and $P \cap P^{-1} = 1$.

Proposition 4. *Let w be a semigroup word in the free group of rank* 2. *Then every right orderable group in $V(w^*)$, belongs to the variety $V(w)$.*

Proof. By Theorem 5 of [30] and Lemma 3, G is locally nilpotent-by-finite. Let $x_1, \cdots, x_n \in G$. Since G is right orderable, G is torsion-free. Thus every finitely generated subgroup of G is an infinite finitely generated nilpotent-by-finite group and so residually finite. Therefore by Lemma 1 of [14], G belongs to the variety $V(w)$.

Lemma 5. *Let w be a semigroup word in a free group. Then every group in $V(w^*)$ is restrained.*

Proof. Let G be a group in $V(w^*)$ and let x, y in G. We must prove that $H = \langle x \rangle^{\langle y \rangle}$ is finitely generated. We may assume that y is of infinite order. Suppose that w is in the free group of rank $n > 0$. Consider a partition of the set $X = \{xy^{-1}, xy^{-2}, \cdots\}$ in n infinite subsets $X_1, X_2, \cdots, X_n$. Then by the property $V(w^*)$, there exist negative integers $t_1, \cdots, t_n$ such that

$$xy^{t_{f(1)}} \cdots xy^{t_{f(m)}} = xy^{t_{g(1)}} \cdots xy^{t_{g(s)}}$$

for some functions f from $\{1, 2, \cdots, m\}$ to $\{1, 2, \cdots, n\}$ and g from $\{1, 2, \cdots, s\}$ to $\{1, 2, \cdots, n\}$, where m and s depend only on w. Now, arguing as in Lemma 1 (ii) of [20], H is finitely generated. This completes the proof.

Lemma 6. *Let w be a word in a free group such that w is not a commutator word. Then every group in $V(w^*)$ is torsion. In particular, G is restrained.*

Proof. Let G be a group. Suppose, for a contradiction, that G has an element a of infinite order, then, by Lemma 3 of [14], $\langle a \rangle$ belongs to the variety $V(w)$ and so a is of finite order, a contradiction.

We note that Theorem A can be applied for the following words w in a free group: by Proposition 1 and the result of [7], any word w such that every finitely generated soluble group in the variety $V(w)$ is nilpotent; by Zelmanov's positive solution to the restricted Burnside problem (see [46] and [47]), any non-commutator word w and by Theorem A of [20], every semigroup word w.

By Theorem A and Lemmas 5 and 6 and the above remarks we have

Corollary 7. *Let w be a non-commutator word or a semigroup word in a free group. Then every infinite finitely generated locally graded group in $V(w^*)$ belongs to the variety $V(w)$.*

Lemma 8. *Let G be an infinite group in $V(w^*)$ and H be a finite subgroup of G. If G has an infinite normal locally soluble subgroup, then H belongs to $V(w)$.*

Proof. Let S be a normal locally soluble infinite subgroup of G. If S is Černikov, then S has an infinite normal characteristic abelian subgroup (see [40] Vol. I page 68) so G has an infinite normal abelian subgroup whence G belongs to $V(w)$ by Lemma 3 of [14].

Therefore, we may assume that S is not Černikov. By a result of Zaicev (see [45]), there is an infinite abelian subgroup B of S such that H normalizes B. Hence B is an infinite normal subgroup of the group BH and so again by Lemma 3 of [14], H belongs to $V(w)$.

Proof of Theorem B. It suffices to prove that an infinite [(locally P)-by-finite] group in $V(w^*)$ belongs to the variety $V(w)$. Let H be a normal locally P-subgroup of G of finite index. If G is torsion, then G is locally finite and H is a locally soluble infinite normal subgroup of G, so by Lemma 8, $G \in V(w)$. Therefore we may assume that G

has an element a of infinite order. Let $x_1, \ldots, x_n$ be arbitrary elements of G. Then $K = < a, x_1, \ldots, x_n >$ is a finitely generated P-by-finite infinite group and so by condition (3), $K \in V(w)$.

Corollary 9. *Let G be an infinite locally finite $V(w^*)$ group. If G satisfies one of the following conditions, then G belongs to the variety $V(w)$.*

(1) G has an infinite locally soluble normal subgroup.
(2) G contains an element with finite centralizer.
(3) G contains an element of prime power order with Černikov centralizer in G.

Proof. Let $x_1, \ldots, x_n$ be arbitrary elements of G, we must prove that $w(x_1, \ldots, x_n) = 1$. Since G is locally finite, $H = \langle x_1, \ldots, x_n \rangle$ is finite. If G has an infinite locally soluble normal subgroup, then, by Theorem B, $H \in V(w)$. If G satisfies the conditions (2) or (3) then by Hartley's results of [33] and [31] G is (locally soluble)-by-finite and so by part (1), the proof is complete.

Let w be a word in a free group. Now we state some reductions in investigation of the equality $V(w) \cup \mathsf{F} = V(w^*)$ on the class of locally soluble groups and locally finite groups.

Let G be an infinite locally soluble group in $V(w^*)$. If G is torsion then by Corollary 9 (1), G belongs to the variety $V(w)$. Therefore we may assume that G has an element g of infinite order and so in order to prove that $G \in V(w)$ it suffices to show that for all $x_1, \ldots, x_n$, the infinite finitely generated soluble subgroup $< x_1, \ldots, x_n, g >$ belongs to the variety $V(w)$. Therefore we have

Remark 10. Let w be a word in a free group. Then the following are equivalent:

(1) any infinite locally soluble group in $V(w^*)$ belongs to the variety $V(w)$.
(2) any infinite finitely generated soluble group in $V(w^*)$ belongs to $V(w)$.

We note that, by Lemma 6, every finitely generated soluble group in $V(w^*)$ where w is not a commutator word, is finite.

Let G be an infinite locally finite group in $V(w^*)$. In order to prove that $G \in V(w)$, we must show that $\langle x_1, \ldots, x_n \rangle \in V(w)$ for all $x_1, \ldots, x_n \in G$, therefore we may assume that G is countable. Fix $x_1, \ldots, x_n \in G$ and let $H = \langle x_1, \ldots, x_n \rangle$. If $C_G(H)$ is infinite, then there is an infinite abelian subgroup A in $C_G(H)$, as G is locally finite (see Theorem 3.43 of [40]). Therefore the centre of $K = \langle A, H \rangle$ is infinite and so by Lemma 3 of [14], $K \in V(w)$. Thus we may assume that $C_G(H)$ is finite. Also, by Lemma 4 of [14] and Corollary 9 we may assume that H is not supersoluble and the centralizer of any element in G is infinite and the centralizer of every element of prime power order is not Černikov. These conditions on a locally finite group lead us to the following defenitions.

We say that a group G is an L-group whenever G is an infinite countable locally finite group and there exists a finite subgroup H of G such that

(1) H is not supersoluble and $C_G(H)$ is finite.

(2) $C_G(x)$ is infinite for all $x \in G$.

(3) $C_G(g)$ is not Černikov for all elements $g \in G$ of prime power order.

(4) the largest normal locally soluble subgroup of G is finite.

In this case, we say that G is an L-group with respect to H. Also, we say that G is an L^*-group with respect to H whenever every infinite subgroup of G which contains H, is an L-group with respect to H. By these discussions we have

Remark 11. Let w be a word in a free group. Then the following are equivalent:

(1) any infinite locally finite group in $V(w^*)$, belongs to the variety $V(w)$.

(2) any infinite L^*- group in $V(w^*)$, belongs to the variety $V(w)$.

We use Remark 11 for the study of an infinite locally finite group G in $V(w^*)$ where w is not a commutator word in the free group of rank $n > 0$, and obtain another condition on such groups G. We prove that G is of finite exponent dividing e, where e is a positive integer depending only on w such that every group in the variety $V(w)$ is of exponent dividing e.

For, let a be an element of G, then $C_G(a)$ is infinite and by Theorem 3.43 of [40] there exists an infinite abelian subgroup A in $C_G(a)$. By Lemma 3 of [14], $A \in V(w)$. Consider infinite subsets $X_1 = \cdots = X_n = aA$. Therefore, by the property $V(w^*)$, there exist $a_1, \ldots, a_n \in A$ such that $w(aa_1, \ldots, aa_n) = 1$. Thus $w(a, \ldots, a)w(a_1, \ldots, a_n) = 1$. But $w(a_1, \ldots, a_n) = 1$ and so $w(a, \ldots, a)$ and $a^e = 1$. Therefore we have:

Remark 12. Let w be a non-commutator word in a free group and e be a positive integer depending only on w such that every group in the variety $V(w)$ is of exponent dividing e. Then the following are equivalent:

(1) any infinite locally finite group in $V(w^*)$ belongs to the variety $V(w)$.

(2) any infinite L^*-group of exponent dividing e belongs to the variety $V(w)$.

A natural question which arises is the following:

Is there an infinite L^*-group of finite exponent? We only know that such a group is not simple. For by a result of L.G. Kovács [22], any infinite, simple, locally finite group G involves infinitely many non-isomorphic non-abelian finite simple groups; hence, if G satisfies non-trivial laws, then according to a result of G.A. Jones (see Theorem of [18]), the variety generated by infinitely many finite simple groups is the variety of all groups. But the variety generated by G is a proper variety, a contradiction.

Now we study infinite simple locally finite groups in $V(w^*)$ where w is a non-trivial word in a free group. As we have seen earlier, there is no infinite simple locally finite group which satisfies a non-trivial identity. Call a simple locally finite group an *S*-group. The *S*-groups fall into two classes with widely different properties---the linear groups and the non-linear groups. Every linear *S*-group is a group of Lie type over an infinite locally finite field (see [34]).

Proof of Theorem C. Suppose, for a contradiction, there exists a non-linear *S*-group G in $V(w^*)$. By a result of Hartley [32], there exists a section C/D of G such that C/D is a direct product of finite alternating groups of unbounded orders. Thus C/D is an infinite residually finite group in $V(w^*)$ and so C/D belongs to the variety $V(w)$. Since C/D is a direct product of finite alternating groups of unbounded orders, the variety $V(w)$ contains infinitely many non-isomorphic finite alternating groups. Therefore, by Theorem of [18], $V(w)$ is the variety of all groups and so w is the trivial word, a contradiction. This completes the proof.

P.S. Kim in [19] studied $V(w_2{}^*)$ on the class of locally soluble groups, where $w_2 = [[x_1, x_2], [x_3, x_4]]$. For this word the variety $V(w_2)$ is the variety of metabelian groups. It is proved in [19], that every infinite locally soluble group in $V(w_2{}^*)$ is metabelian and also it is proved that any infinite group belonging to $V(w_2{}^*)$ is metabelian if and only if there is no infinite simple group in $V(w_2{}^*)$. We study $V(w^*)$ on the class of locally finite groups, where w is a soluble word that is $w = w_d$ for some

$d \in IN$ where $w_0 = x$, $w_i = [w_{i-1}, w_{i-1}]$ and w_{i-1} is the word on 2^{i-1} distinct letters which has been defined inductively, for all $i \in IN$.

Corollary 13. *Let w be a soluble word and let G be an infinite locally finite $V(w^*)$-group. Then the following are equivalent:*

(1) $G \in V(w)$.

(2) G *has no infinite linear simple locally finite section.*

Proof. Suppose that (1) is true. Then G is soluble and (2) is clear. Now suppose that (2) is true and $w = w_d$ for some positive integer d. Suppose, for a contradiction, that $G \notin V(w)$. Thus G is not soluble of derived length at most d. Suppose, if possible, that $K = G^{(d+1)}$ is finite. Then $H = G^{(d)}$ is an FC-group and so H is soluble by applying suitably Lemma 1 of [4]. Thus G is a torsion soluble group and so by Theorem B, G is soluble of derived length at most d, a contradiction. Hence K is infinite and so G/K is a soluble group of derived length d. Therefore $G^{(d+1)} = G^{(d)}$ that is $H = H'$, which implies that H is a perfect group. Suppose that H has an infinite proper normal subgroup N, then H/N is soluble of derived length at most d, this implies $H = H^{(d)} \leq N$ since H is perfect, a contradiction. Let N be a finite normal subgroup of H, then $C_H(N)$ has finite index in H. Since H has no infinite normal proper subgroups, $C_H(N) = H$. Hence the centre Z of H is the unique maximal normal subgroup of H so that $S = H/Z$ is simple. By Theorem C, S is an infinite linear simple locally finite group, which is a contradiction.

Now, we start proving Theorem D, for this we need the following lemma:

Lemma 14. *Every infinite locally soluble group of finite rank in $V(w^*)$ belongs to the variety $V(w)$.*

Proof. Let G be an infinite locally soluble group of finite rank in $V(w^*)$. By Remark 10, we may assume that G is finitely generated. Therefore G is a minimax group, and so by Theorem 10.33 of [40], the finite residual of G is the direct product of finitely many quasicyclic subgroups of G, thus G is residually finite or G has an infinite normal abelian subgroup, then, by Lemma 1 or Lemma 3 of [14] respectively, the proof is complete.

Proof of Theorem D. Let G be an infinite locally soluble group in $V(w^*)$. By Remark 10, we may assume that G is a finitely generated infinite soluble group. Firstly, suppose that w is a word such that $C_p wr C_\infty \notin V(w)$ for all primes p. We prove that G is a minimax group and so G is of finite rank, then Lemma 14 completes the proof.

By a deep result of Kropholler (see [23]), which asserts that every finitely generated soluble group having no sections of type $C_p wr C_\infty$ is minimax, it suffices to show that if $C_p wr C_\infty \in V(w^*)$ then $C_p wr C_\infty \in V(w)$. But $C_p wr C_\infty$ has an infinite normal abelian subgroup, therefore by Lemma 3 of [14], $C_p wr C_\infty \in V(w)$, which is a contradiction.

Now, suppose that w is a word such that every infinitely presented $M(\alpha, p) \notin V(w)$. If G is not semi-polycyclic group (see 16), then there exists a subgroup of a quotient group of G which is isomorphic to an infinitely presented $M(\alpha, p)$. But $M(\alpha, p)$ is an infinite residually finite group in $V(w^*)$ and so $M(\alpha, p) \in V(w)$, a contradiction. Therefore G is semi-polycyclic and so is of finite rank (see [16]). Thus, by Lemma 14, $G \in V(w)$. This completes the proof.

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